Chapter 2

Fundamental definitions

4 Lie algebras

Definition 4.1 (Lie algebra)
A **Lie algebra** is a vector space L over a field $\mathbb{F}$ together with a multiplication

$$L \times L \to L, (x, y) \mapsto [x, y],$$

satisfying the following axioms:

(L1) $[x + y, z] = [x, z] + [y, z]$ and $[x, y + z] = [x, y] + [x, z]$,

(L2) $[\lambda x, y] = [x, \lambda y] = \lambda[x, y]$,

(L3) $[x, x] = 0$, and

(L4) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$,

whenever $x, y, z \in L$ and $\lambda \in \mathbb{F}$. Axiom (L4) is called the **Jacobi identity**.
Note that $[[x, y], z]$ is **not necessarily** equal to $[x, [y, z]]$, we do **not** have associativity!

Remark: In this course, we will mostly study Lie algebras over the complex field $\mathbb{C}$.

Lemma 4.2 (First properties)
Let L be a Lie algebra over a field $\mathbb{F}$. Then $[x, y] = -[y, x]$ for all $x, y \in L$. The Lie multiplication is **anticommutative**.

Proof. We have $0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$. ■

Example 4.3 (Abelian Lie algebras)
Every $\mathbb{F}$-vector space L with $[x, y] = 0$ for all $x, y \in L$ is a Lie algebra. Such a Lie algebra is called **abelian**. Abelian Lie algebras are somewhat boring.

Example 4.4 (Lie($\mathcal{A}$)**, the Lie algebra of an associative algebra)**
Let $\mathcal{A}$ be an associative algebra over a field $\mathbb{F}$. That is, $\mathcal{A}$ is a ring with identity together with a ring homomorphism $\iota : \mathbb{F} \to Z(\mathcal{A})$ where $Z(\mathcal{A}) := \{x \in \mathcal{A} \mid xy = yx \text{ for all } y \in \mathcal{A}\}$ is the centre of $\mathcal{A}$, the set of elements of $\mathcal{A}$ that commute with every other element. Such an $\mathcal{A}$ is then automatically an $\mathbb{F}$-vector space by setting $\lambda \cdot a := \iota(\lambda) \cdot a$ for $\lambda \in \mathbb{F}$ and $a \in \mathcal{A}$. In particular, the multiplication of $\mathcal{A}$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
If you do not remember this structure, simply think of $\mathcal{A} = \mathbb{C}^{n \times n}$, the set of all $n \times n$-matrices with componentwise addition and matrix multiplication. The map ι here is the embedding of $\mathbb{C}$ into the scalar multiples of the identity matrix.

Every associative algebra $\mathcal{A}$ becomes a Lie algebra by defining the Lie product in this way:

$$[x, y] := x \cdot y - y \cdot x \qquad \text{for all } x, y \in \mathcal{A}.$$

We check the axioms:

(L1) $[x + y, z] = (x + y) \cdot z - z \cdot (x + y) = x \cdot z + y \cdot z = [x, z] + [y, z]$.

(L2) $[\lambda x, y] = (\lambda x) \cdot y - y \cdot (\lambda x) = x \cdot (\lambda y) - (\lambda y) \cdot x = [x, \lambda y]$ and this is equal to $\lambda(x \cdot y - y \cdot x)$.

(L3) $[x, x] = x \cdot x - x \cdot x = 0$.

(L4) $[[x, y], z] = [xy - yx, z] = xyz - yxz + zxy - zyx$, permuting cyclically and adding up everything shows the Jacobi identity.

For a $\mathbb{C}$-vector space V, the set of endomorphisms $\mathrm{End}(V)$ (linear maps of V into itself) is an associative algebra with composition as multiplication. The map ι here is the embedding of $\mathbb{C}$ into the scalar multiples of the identity map.
We set $\mathrm{gl}(V) := \mathrm{Lie}(\mathrm{End}(V))$. By choosing a basis of V this is the same as $\mathrm{Lie}(\mathbb{C}^{n\times n})$ if $\dim_{\mathbb{C}}(V) = n$ (see 4.8 below). Thus, we can compute in $\mathrm{gl}(\mathbb{C}^{1\times 2})$

$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\right] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

using the standard basis.

Example 4.5 (Vector product)
Let $L := \mathbb{R}^{1\times 3}$ be the 3-dimensional real row space with the following product:

$$[[a, b, c], [x, y, z]] := [a, b, c] \times [x, y, z] := [bz - cy, cx - az, ay - bx].$$

This is a Lie algebra over the field $\mathbb{R}$ of real numbers.
Note: The vector $v := [a, b, c] \times [x, y, z]$ is zero if and only if the vectors $[a, b, c]$ and $[x, y, z]$ are parallel. Otherwise v is orthogonal to both $[a, b, c]$ and $[x, y, z]$ and its length is equal to the area of the parallelogram spanned by $[a, b, c]$ and $[x, y, z]$.
Exercise: Check the Jacobi identity for this Lie algebra.

Example 4.6 (sl_2)
Let sl_2 be the subspace of $\mathbb{C}^{2\times 2}$ containing all matrices of trace 0:

$$\mathrm{sl}_2 := \left\{ M \in \mathbb{C}^{2\times 2} \mid \mathrm{Tr}(M) = 0 \right\}$$

(remember, the trace $\mathrm{Tr}(M)$ of a square matrix M is the sum of the main diagonal entries).
Then sl_2 with

$$[A, B] := A \cdot B - B \cdot A \qquad \text{for all } A, B \in \mathrm{sl}_2$$

as Lie product is a Lie algebra, since $\mathrm{Tr}(A \cdot B) = \mathrm{Tr}(B \cdot A)$ for arbitrary square matrices A and B. This Lie algebra will play a **major role** in this whole theory! It is somehow the **smallest interesting building block**.

Definition 4.7 (Homomorphisms, isomorphisms)
Let L_1 and L_2 be Lie algebras over the same field $\mathbb{F}$. A **homomorphism of Lie algebras** from L_1 to L_2 is a linear map $\varphi : L_1 \to L_2$, such that

$$[x, y]\varphi = [x\varphi, y\varphi] \qquad \text{for all } x, y \in L_1.$$

If φ is bijective, then it is called an **isomorphism of Lie algebras**.

Example 4.8 ($\mathrm{gl}(\mathbb{C}^{1\times n})$ **and** $\mathrm{Lie}(\mathbb{C}^{n\times n})$ **are isomorphic)**
Choosing a basis $(v_1, \ldots, v_n)$ of the $\mathbb{C}$-vector space $\mathbb{C}^{1\times n}$ gives rise to an isomorphism of Lie algebras $\mathrm{gl}(\mathbb{C}^{1\times n}) \cong \mathrm{Lie}(\mathbb{C}^{n\times n})$ by mapping a linear map $\varphi : \mathbb{C}^{1\times n} \to \mathbb{C}^{1\times n}$ to its matrix with respect to the basis $(v_1, \ldots, v_n)$ as in Theorem 2.9.

Definition 4.9 (Subalgebras and ideals)
Let L be a Lie algebra over a field $\mathbb{F}$ and let H and K be subspaces of L. We then set

$$[H, K] := \mathrm{Span}\left(\left\{[h, k] \in L \;\middle|\; h \in H, k \in K\right\}\right).$$

Note $[H, K] = [K, H]$ and that we have to use Span here to ensure that this is a subspace of L.
A **Lie subalgebra** or short **subalgebra** of L is a subspace H with $[H, H] \le H$.
A **Lie ideal** or short **ideal** of L is a subspace K with $[K, L] \le K$.
Obviously, every ideal is a subalgebra.

Example 4.10 (Centre and derived subalgebra)
Let L be a Lie algebra over a field $\mathbb{F}$. Then the **centre** $Z(L) := \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$ and the **derived algebra** $[L, L] := \mathrm{Span}(\{[x, y] \mid x, y \in L\})$ are ideals in L.

Definition 4.11 (Normaliser and centraliser)
Let L be a Lie algebra over a field $\mathbb{F}$ and let H be a subspace of L (not necessarily a subalgebra!). We then define the **normaliser** $N_L(H)$ of H in L to be the space

$$N_L(H) := \{x \in L \mid [x, H] \subseteq H\}.$$

We define the **centraliser** $C_L(H)$ of H in L to be the space

$$C_L(H) := \{x \in L \mid [x, H] = 0\}.$$

Exercise: Use the Jacobi identity to show that both the normaliser and the centraliser are Lie subalgebras of L.

Proposition 4.12 (Properties of subalgebras)
Let H and K be subspaces of a Lie algebra L over $\mathbb{F}$ and let $H + K := \{h + k \mid h \in H, k \in K\}$ be their sum as subspaces.

(i) If H and K are subalgebras, then $H \cap K$ is.

(ii) If H and K are ideals, then $H \cap K$ is.

(iii) If H is an ideal and K is a subalgebra, then $H + K$ is a subalgebra of L.

Proof. Left as an exercise for the reader. ■

Example 4.13 ($\mathrm{Lie}(\mathbb{C}^{n\times n})$ **revisited)**
Let $L = \mathbb{C}^{n\times n}$ with $n \ge 2$.
The subspace H of matrices with trace 0 is an ideal since $\mathrm{Tr}(A \cdot B) = \mathrm{Tr}(B \cdot A)$ for arbitrary matrices A and B and thus $\mathrm{Tr}([A, B]) = 0$ for all $A \in H$ and $B \in L$.
The subspace K of skew-symmetric matrices, i.e. $\left\{A \in \mathbb{C}^{n\times n} \mid A^t = -A\right\}$ where A^t is the transposed matrix of A, is a subalgebra but not an ideal: If $A, B \in K$, then

$$[A, B]^t = (A \cdot B - B \cdot A)^t = B^t \cdot A^t - A^t \cdot B^t = (-B) \cdot (-A) - (-A) \cdot (-B) = [B, A] = -[A, B].$$

Exercise: Show that K is not an ideal.

Lemma 4.14 (Kernels of homomorphisms are ideals and images are subalgebras)
Let L and H be Lie algebras over a field $\mathbb{F}$ and $\varphi : L \to H$. Then the image $I := \mathrm{im}\,(\varphi)$ is a Lie subalgebra of H and the kernel $K := \ker\,(\varphi)$ is a Lie ideal of L.

Proof. Since φ is $\mathbb{F}$-linear, both I and K are subspaces.
If $x, y \in I$, there are $\tilde{x}, \tilde{y} \in L$ with $\tilde{x}\varphi = x$ and $\tilde{y}\varphi = y$. Then $[x, y] = [\tilde{x}, \tilde{y}]\varphi \in I$ as well.
If $x \in K$, that is, $x\varphi = 0$, then for $y \in L$ we have $[x, y]\varphi = [x\varphi, y\varphi] = [0, y\varphi] = 0$ and thus $[x, y] \in K$ as well. ∎

Note: In fact, Lie ideals are exactly the kernels of Lie algebra homomorphisms, as we will see next.

Definition 4.15 (Quotient Lie algebra)
Let L be a Lie algebra over a field $\mathbb{F}$ and K an ideal of L. Then the **quotient space**

$$L/K := \{x + K \mid x \in L\}$$

consisting of the **cosets** $x + K := \{x + k \mid k \in K\}$ for $x \in L$, is a Lie algebra by defining

$$(x+K)+(y+K) := (x+y)+K \text{ and } \lambda\cdot(x+K) := (\lambda\cdot x)+K \text{ and } [x+K, y+K] := [x, y]+K$$

for all $x, y \in L$ and all $\lambda \in \mathbb{F}$. This is well-defined because K is an ideal, and it inherits all the axioms directly from L. There is a surjective homomorphism $\pi : L \to L/K, x \mapsto x + K$ of Lie algebras called the **canonical map**.

Proof. Lots of little details have to be checked here. Most of it is just the standard construction of the quotient vector space, which can be found in every book on linear algebra and we do not repeat them here. The most important additional one is the well-definedness of the Lie product: Assume $x + K = \tilde{x} + K$ and $y + K = \tilde{y} + K$, that is, $\tilde{x} = x + k_1$ and $\tilde{y} = y + k_2$ for some $k_1, k_2 \in K$. Then

$$[x + k_1, y + k_2] = [x, y] + \underbrace{[x, k_2] + [k_1, y] + [k_1, k_2]}_{\in K}$$

but all three latter products lie in K because K is an ideal. All statements about π are routine verifications. ∎

Theorem 4.16 (First Isomorphism Theorem)
Let $\varphi : L \to H$ a homomorphism of Lie algebras over a field $\mathbb{F}$ and $K := \ker\,(\varphi)$. Then

$$\begin{array}{rccc} \psi: & L/K & \to & \mathrm{im}\,(\varphi) \\ & x + K & \mapsto & x\varphi \end{array}$$

is an isomorphism of Lie algebras.

Proof. The map ψ is well-defined since $x + K = y + K$ is equivalent to $x - y \in K = \ker\,(\varphi)$ and thus $x\varphi = y\varphi$. This also proves that ψ is injective, and surjectivity to the image of φ is obvious. The map ψ is clearly $\mathbb{F}$-linear and a homomorphism of Lie algebras because φ is. ∎

Theorem 4.17 (Second Isomorphism Theorem)
Let L be a Lie algebra, K an ideal and H a subalgebra. Then $H \cap K$ is an ideal of H and the map

$$\begin{array}{rccc} \psi: & H/(H \cap K) & \to & (H + K)/K \\ & h + (H \cap K) & \mapsto & h + K \end{array}$$

is an isomorphism of Lie algebras.

Proof. The ideal K of L is automatically an ideal of the subalgebra $H + K$ (see Proposition 4.12.(iii)). Define a map $\tilde{\psi} : H \to (H + K)/K$ by setting $h\tilde{\psi} := h + K$. This is clearly linear and a homomorphism of Lie algebras. Its image is all of $(H + K)/K$ since every coset in there has a representative in H. The kernel of $\tilde{\psi}$ is exactly $H \cap K$ and it follows that this is an ideal in H. The First Isomorphism Theorem 4.16 then does the rest.
Alternative proof (to get more familiar with quotient arguments): The subspace $H + K$ is a subalgebra by Proposition 4.12.(iii) and K is an ideal in $H + K$ because it is one even in L. The subspace $H \cap K$ is an ideal in H because $[h, l] \in H \cap K$ for all $h \in H$ and $l \in H \cap K$. Thus we can form both quotients.
The map ψ is well-defined, since if $h + (H \cap K) = \tilde{h} + (H \cap K)$, that is, $h - \tilde{h} \in H \cap K$, then in particular $h - \tilde{h} \in K$ and thus $h + K = \tilde{h} + K$. In fact, this reasoning immediately shows that ψ is injective. The map ψ is clearly linear and a homomorphism of Lie algebras by routine verification. It is surjective since every coset in $(H + K)/K$ has a representative in H. ■

5 Nilpotent and soluble Lie algebras

Definition 5.1 (Simple and trivial Lie algebras)
A Lie algebra L is called **simple**, if it is non-abelian (that is, the Lie product is not constant zero) and has no ideals other than 0 and L. A one-dimensional Lie algebra is automatically abelian and is called **the trivial Lie algebra**.

Example 5.2 (sl_2 **revisited)**
The 3-dimensional Lie algebra $L := \mathrm{sl}_2$ from Section 4.6 is simple. Let

$$x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we then have $L = \mathrm{Span}(x, y, h)$ and the relations

$$[x, y] = h \qquad \text{and} \qquad [h, x] = 2x = -[x, h] \qquad \text{and} \qquad [h, y] = -2y = -[y, h].$$

Let $0 \neq K \leq L$ be an ideal of L and $0 \neq z := ax + by + ch \in K$ with $a, b, c \in \mathbb{C}$. Then we have $[[z, x], x] = -2bx$ and $[[z, y], y] = 2ay$ and thus, if either a or b is non-zero, then $K = L$. If otherwise $a = b = 0$, then $c \neq 0$ since $z \neq 0$ and thus $K = L$ as well because $[z, x] = 2cx$ and $[z, y] = -2cy$. Thus the only ideals of L are 0 and L itself.

Definition 5.3 (Lower central series, nilpotent Lie algebra)
Let L be a Lie algebra over a field $\mathbb{F}$. We define the **lower central series** as $L^0 := L$ and $L^i := [L^{i-1}, L]$ for $i \geq 1$. This gives a descending sequence of ideals

$$L = L^0 \supseteq L^1 = [L, L] \supseteq L^2 \supseteq \cdots$$

The Lie algebra L is called **nilpotent**, if there is an $n \in \mathbb{N}$ with $L^n = 0$.
Exercise: Convince yourself that all L^i are in fact ideals of L. Remember that $[L, L]$ is the **span** of all bracket expressions $[x, y]$ for $x, y \in L$.

Example 5.4 (Strictly lower triangular matrices)
Every abelian Lie algebra L is nilpotent, since $L^1 = [L, L] = 0$. No simple Lie algebra L is nilpotent, since $[L, L]$ is a non-zero ideal and thus equal to L. In particular, sl_2 from 4.6 is not nilpotent.
Let L be the subalgebra of $\mathrm{Lie}(\mathbb{C}^{n\times n})$ of stricly lower triangular matrices (with zeros on the diagonal). Then L has dimension $n(n-1)/2$ and L^i is strictly smaller than L^{i-1} for $i \geq 1$ and $L^n = 0$, thus L is nilpotent. This is proved by proving that

$$L^i = \left\{ \left(m_{j,k}\right) \in \mathbb{C}^{n\times n} \mid m_{j,k} = 0 \text{ if } j \leq k + i \right\}$$

using induction. (**See exercise sheet 1**.)

Definition 5.5 (Derived series, soluble Lie algebra)
Let L be a Lie algebra over a field $\mathbb{F}$. We then define the **derived series** as $L^{(0)} := L$ and $L^{(i)} := [L^{(i-1)}, L^{(i-1)}]$ for $i \geq 1$. This gives a descending sequence of ideals

$$L = L^{(0)} \supseteq L^{(1)} = [L, L] \supseteq L^{(2)} \supseteq \cdots$$

The Lie algebra L is called **soluble**, if there is an $n \in \mathbb{N}$ with $L^{(n)} = 0$.
Exercise: Convince yourself that all $L^{(i)}$ are in fact ideals of L. Remember that $[L, L]$ is the **span** of all bracket expressions $[x, y]$ for $x, y \in L$.

Example 5.6 (Lower triangular matrices)
Every abelian Lie algebra L is soluble, since $L^{(1)} = [L, L] = 0$. No simple Lie algebra L is soluble, since $[L, L]$ is a non-zero ideal and thus equal to L. In particular, sl_2 from 4.6 is not soluble.
Let L be the subalgebra of sl_2 (see 4.6) of lower triangular matrices

$$L := \left\{ \begin{bmatrix} a & 0 \\ b & -a \end{bmatrix} \;\middle|\; a, b \in \mathbb{C} \right\}.$$

Then $L^{(1)} = [L, L]$ are the subset of matrices with zeros on the diagonal and $L^{(2)} = 0$, thus L is soluble. Note that it is not nilpotent.

Theorem 5.7 (Nilpotent and soluble Lie algebras)
Let L be a Lie algebra H a subalgebra and K an ideal. Then the following hold:

(i) If L is abelian, nilpotent or soluble, then the same is true for H and L/K.

(ii) If K and L/K are soluble then L is soluble, too.

(iii) If $L/Z(L)$ is nilpotent, then so is L.

(iv) $[L^k, L^m] \leq L^{k+m}$ for all $k, m \in \mathbb{N}$.

(v) $L^{(m)} \leq L^{2^{m-1}}$ for all $m \in \mathbb{N}$.

(vi) Every nilpotent Lie algebra is soluble.

(vii) If L is nilpotent and $L \neq 0$, then $Z(L) \neq 0$.

Proof. If L is abelian, then clearly all Lie products in H and L/K are zero as well which proves (i) for "abelian". For nilpotent and soluble the inclusions

$$H^i \subseteq L^i \qquad \text{and} \qquad H^{(i)} \subseteq L^{(i)}$$

and the equations

$$(L/K)^i = \{x + K \mid x \in L^i\} \qquad \text{and} \qquad (L/K)^{(i)} = \{x + K \mid x \in L^{(i)}\}$$

immediately imply (i).
For (ii) assume that both K and L/K are soluble, that is, there are $m, k \in \mathbb{N}$ such that $(L/K)^{(m)} = \{0 + K\}$ and $K^{(k)} = 0$. But the former directly implies $L^{(m)} \subseteq K$ and thus $L^{(m+k)} = 0$ which shows that L is soluble. Note that the same proof for nilpotent does not work!
To prove (iii) assume $L^n \subseteq Z(L)$, then $L^{n+1} = [L^n, L] \subseteq [Z(L), L] = 0$.

We now consider (iv). The statement holds for $m = 1$ by definition of $L^{k+1} = [L^k, L]$ since $[L^k, L^1] = [L^k, [L, L]] \leq [L^k, L] = L^{k+1}$. We now use induction on m. Suppose (iv) is true for all $m \leq r$ and all k. Then

$$\begin{aligned} [L^k, L^{r+1}] &= [L^k, [L^r, L]] = [[L^r, L], L^k] \overset{(1)}{\leq} [[L^k, L^r], L] + [[L, L^k], L^r] \\ &\overset{(2)}{\leq} [L^{k+r}, L] + [L^{k+1}, L^r] \overset{(3)}{\leq} L^{k+r+1}, \end{aligned}$$

where the inequality (1) follows from the Jacobi identity $[[x, y], z] = -[[z, x], y] - [[y, z], x]$ for all $x, y, z \in L$ and (2) and (3) follow by induction.

Statement (v) now follows by induction on m. Namely, we have $L^{(1)} = [L, L] = L^1 = L^{2^0}$ for the induction start. Suppose $L^{(i)} \leq L^{2^{i-1}}$, then $L^{(i+1)} = [L^{(i)}, L^{(i)}] \leq [L^{2^{i-1}}, L^{2^{i-1}}] \leq L^{2^{i-1}+2^{i-1}} = L^{2^i}$ by using (iv).

For statement (vi) suppose that L is nilpotent, that is, there is an n such that $L^n = 0$ and thus all $L^k = 0$ for all $k \geq n$. But then $L^{(n)} \leq L^{2^{n-1}} = 0$ as well since $2^{n-1} \geq n$ for all $n \geq n$. Thus L is soluble.

Finally for statement (vii) note that the last non-zero term in the lower central series is contained in the centre $Z(L)$. ■

Theorem 5.8 (Radical)
Let L be a Lie algebra and H_1, H_2 soluble ideals of L. Then $H_1 + H_2$ is a soluble ideal of L, too. Furthermore, if L is finite-dimensional, there is a soluble ideal $\mathrm{rad}(L)$ of L that contains every soluble ideal of L. It is called the **radical** or L.

Proof. Suppose H_1 and H_2 are soluble ideals. Then $(H_1 + H_2)/H_1 \cong H_2/(H_1 \cap H_2)$ by the Second Isomorphism Theorem 4.17 and it follows from Proposition 5.7.(i) that this is soluble as quotient of the soluble Lie algebra H_2. But then H_1 is an ideal of $H_1 + H_2$ such that both the quotient $(H_1 + H_2)/H_1$ and the ideal are soluble, so by Proposition 5.7.(ii) the Lie algebra $H_1 + H_2$ is soluble as well.

If L is finite-dimensional, then there is a soluble ideal K of maximal dimension. By the above reasoning and maximality this ideal contains every other soluble ideal and is thus uniquely determined. It is called the radical and denoted by $\mathrm{rad}(L)$. ■

Definition 5.9 (Semisimple Lie algebra)
A Lie algebra L over a field $\mathbb{F}$ is called semisimple if it has no soluble ideals other than 0.

Lemma 5.10 (Radical quotient is semisimple)
For every finite-dimensional Lie algebra L, the quotient Lie algebra $L/\mathrm{rad}(L)$ is semisimple.

Proof. The preimage of any soluble ideal of $L/\mathrm{rad}(L)$ under the canonical map $L \to L/\mathrm{rad}(L)$ would be a soluble ideal of L that properly contains $\mathrm{rad}(L)$ (use Proposition 5.7.(ii) again). ■

Example 5.11 (Direct sums of simple Lie algebras are semisimple)
Every simple Lie algebra L is semisimple, since it contains no ideals other than L and 0 and L is not soluble (see 5.6). The direct sum $L_1 \oplus \cdots \oplus L_k$ of simple Lie algebras $L_1, \ldots, L_k$ is semisimple.

Proof. By the direct sum we mean the direct sum of vector spaces with component-wise Lie product. It is a routine verification that this makes the direct sum into a Lie algebra, such that every summand L_i is an ideal, since $[L_i, L_j] = 0$ for $i \neq j$ in this Lie algebra.

Assume now that K is any ideal of the sum $L_1 \oplus \cdots \oplus L_k$. We claim that for every summand L_i we either have $L_i \subseteq K$ or $L_i \cap K = \{0\}$. This is true, because $L_i \cap K$ is an ideal in L_i and L_i is simple. Thus, K is the (direct) sum of some of the L_i. However, if $K \neq \{0\}$, then K is not soluble, since $[L_i, L_i] = L_i$ for all i and at least one L_i is fully contained in K in this case. ■

In fact, we will prove the following theorem later in the course:

Theorem 5.12 (Characterisation of semisimple Lie algebras)
A Lie algebra L over $\mathbb{C}$ is semisimple if and only if it is the direct sum of minimal ideals which are simple Lie algebras.

Proof. See later. ■

We can now formulate the ultimate goal of this course:

Classify all finite-dimensional, semisimple Lie algebras over $\mathbb{C}$ up to isomorphism.

In view of the promised Theorem 5.12, this amounts to proving this theorem and classifying the simple Lie algebras over $\mathbb{C}$ up to isomorphism.

6 Lie algebra representations

Definition 6.1 (Lie algebra representation)
Let L be a Lie algebra over the field $\mathbb{F}$. A **representation** of L is a Lie algebra homomorphism

$$\rho : L \to \mathrm{Lie}(\mathrm{End}(V))$$

for some $\mathbb{F}$-vector space V of dimension $n \in \mathbb{N}$, which is called the **degree** of ρ. This means nothing but: ρ is a linear map and

$$[x, y]\rho = [x\rho, y\rho] = (x\rho) \cdot (y\rho) - (y\rho) \cdot (x\rho)$$

for all $x, y \in L$.
Two representations $\rho : L \to \mathrm{Lie}(\mathrm{End}(V))$ and $\rho' : L \to \mathrm{Lie}(\mathrm{End}(V'))$ of degree n are called **equivalent**, if there is an invertible linear map $T : V \to V'$ such that $(x\rho) \cdot T = T \cdot (x\rho')$ for all $x \in L$ (the dot $\cdot$ denotes composition of maps).

Definition 6.2 (Lie algebra module)
Let L be a Lie algebra over a field $\mathbb{F}$. An **L-module** is a finite-dimensional $\mathbb{F}$-vector space V together with an action

$$V \times L \to V, (v, l) \mapsto vl$$

such that

- $(v + w)x = vx + wx$ and $(\lambda v)x = \lambda(vx)$ (the action is linear),
- $v(x + y) = vx + vy$, and
- $v[x, y] = (vx)y - (vy)x$.

for all $v, w \in V$ and all $x, y \in L$ and all $\lambda \in \mathbb{F}$ respectively.

Lemma 6.3 (Representations and modules are the same thing)
Let L be a Lie algebra over a field $\mathbb{F}$. A representation $\rho : L \to \mathrm{Lie}(\mathrm{End}(\mathbb{F}^{1\times n}))$ makes the row space $\mathbb{F}^{1\times n}$ into an L-module by setting $vx := v(x\rho)$. Conversely, if V is an L-module then expressing the linear action as endomorphisms defines a representation of L of degree n.
Thus, the two concepts are two aspects of the same thing.

Proof. The first axiom in Definition 6.2 is needed to make the action of elements of L on V into linear maps. The other two axioms are needed to make the map $L \to \mathrm{Lie}(\mathrm{End}(V))$ a Lie algebra homomorphism. The remaining details of this proof are left as an exercise to the reader. ■

Example 6.4 (A representation)
Let L be the Lie subalgebra of $\mathrm{Lie}(\mathbb{C}^{n\times n})$ of lower triangular matrices. The map

$$\begin{array}{rccc} \pi_2 & : & L & \to & \mathrm{End}(\mathbb{C}^{1\times 2}) \\ & & [a_{i,j}]_{1\le i,j\le n} & \mapsto & \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \end{array}$$

(and then viewing the 2×2-matrices as endomorphisms of $\mathbb{C}^{1\times 2}$) is a Lie algebra homomorphism and thus a representation. This makes $\mathbb{C}^{1\times 2}$ into an L-module.
We notice that a representation might not "see all of L". The map π_2 for example has a non-trivial kernel.

Example 6.5 (The adjoint representation)
Let L be any Lie algebra over a field $\mathbb{F}$. The **adjoint representation** of L is its **action on itself**:

$$\begin{array}{rccl} \mathrm{ad}: & L & \to & \mathrm{Lie}(\mathrm{End}(L)) \\ & x & \mapsto & x^{\mathrm{ad}} := (y \mapsto [y,x]) \end{array}.$$

Note that we denote the image of an element $x \in L$ under the map ad by x^{ad} throughout. The map ad is in fact a Lie algebra homomorphism and thus a representation. To verify this, we first check that x^{ad} is a linear map from L to L for every $x \in L$:

$$(y+\lambda z)x^{\mathrm{ad}} = [y+\lambda z, x] = [y,x] + \lambda[z,x] = yx^{\mathrm{ad}} + \lambda(zx^{\mathrm{ad}})$$

for $y, z \in L$ and $\lambda \in \mathbb{F}$. The map ad itself is linear, since

$$z(x+\lambda y)^{\mathrm{ad}} = [z, x+\lambda y] = [z,x] + \lambda[z,y] = zx^{\mathrm{ad}} + z(\lambda y^{\mathrm{ad}})$$

for all $x, y, z \in L$ and all $\lambda \in \mathbb{F}$. Finally, the Jacobi identity shows that ad is a homomorphism of Lie algebras:

$$z[x,y]^{\mathrm{ad}} = [z,[x,y]] = -[x,[y,z]] - [y,[z,x]] = [[z,x],y] - [[z,y],x] = (zx^{\mathrm{ad}})y^{\mathrm{ad}} - (zy^{\mathrm{ad}})x^{\mathrm{ad}}$$

for all $x, y, z \in L$.

Example 6.6 (One-dimensional representation)
A one-dimensional representation of a Lie algebra L over $\mathbb{F}$ is simply a linear map $\rho : L \to \mathbb{F}$ with

$$[x,y]\rho = (x\rho)\cdot(y\rho) - (y\rho)\cdot(x\rho) = 0$$

for all $x, y \in L$ since $\mathbb{F}$ is commutative. So the one-dimensional representations of L are precisely the $\mathbb{F}$-linear maps to $\mathbb{F}$ that vanish on the subspace $L^1 = L^{(1)} = [L, L]$. This shows for example that the simple Lie algebra sl_2 from 4.6 has only one one-dimensional representation which is the zero map:

$$\mathrm{sl}_2 \to \mathbb{C}, x \mapsto 0.$$

Anyway, the kernel of such a representation is an ideal so it can only be 0 or sl_2 because sl_2 is simple. Since $\mathbb{C}$ is one-dimensional, the kernel cannot be 0 because of the dimension formula for linear maps.

Definition 6.7 (Submodules, irreducible modules)
Let L be a Lie algebra over a field $\mathbb{F}$ and V be an L-module. A subspace W of V is called a **submodule**, if it is invariant under the action of L:

$$wx \in W \qquad \text{for all } w \in W \text{ and } x \in L.$$

A module V is called **irreducible**, if it has no submodules other than 0 and V itself. A module V is the direct sum $W_1 \oplus \cdots \oplus W_k$ of submodules $W_1, W_2, \ldots, W_k$, if it is the direct vector space direct sum of the W_i. A module V is called **indecomposable** if it is not the direct sum of two non-trivial submodules.

Remark 6.8 (Irreducible implies indecomposable)
An irreducible L-module is clearly indecomposable. However, the reverse implication does **not hold** in general. There are Lie algebras with modules V that have a proper submodule $0 < W < V$, for which there is no other submodule U with $V = W \oplus U$.

Remark 6.9 (Irreducible adjoint representation)
Let $V := L$ be the L-module given by the adjoint representation (see 6.5). A submodule of V is the same as an ideal of L. The module V is irreducible if and only if L is a simple Lie algebra.

Definition 6.10 (Homomorphisms of modules)
Let L be a Lie algebra over a field $\mathbb{F}$. A **homomorphism of L-modules** is an $\mathbb{F}$-linear map

$$T : V \to V'$$

between two L-modules V and V', such that $(vT)x = (xv)T$ for all $v \in V$ and all $x \in L$. It is called an **isomorphism** if there is a homomorphism $S : V' \to V$ of L-modules with $TS = \mathrm{id}_V$ and $ST = \mathrm{id}_{V'}$.

Definition/Proposition 6.11 (Eigenvectors and eigenvalues)
Let V be an $\mathbb{F}$-vector space and $T : V \to V$ a linear map. Then an **eigenvalue** is an element $\lambda \in \mathbb{F}$, for which a vector $v \in V \setminus \{0\}$ exists with

$$vT = \lambda \cdot v.$$

Every such v is called an **eigenvector for the eigenvalue** λ. The set of eigenvectors for the eigenvalue λ together with the zero vector is called the **eigenspace for the eigenvalue** λ. Note that an eigenvector v has to be non-zero, otherwise every $\lambda \in \mathbb{F}$ would be an eigenvalue.
For $\mathbb{F} = \mathbb{C}$, every endomorphism T has an eigenvalue, since the characteristic polynomial of T has a root ($\mathbb{C}$ is algebraically closed).

Proof. See any linear algebra book and use the fundamental theorem of algebra. ∎

Lemma 6.12 (Schur I)
Let V and V' be irreducible L-modules for a Lie algebra L over $\mathbb{F}$ and let $T : V \to V'$ be an L-module homomorphism. Then either T maps every element of V to zero or it is an isomorphism.

Proof. The image $\operatorname{im} T$ and the kernel $\ker T$ of T are submodules of V' and V respectively. Since both V and V' are irreducible, either $\operatorname{im} T = 0$ and $\ker T = V$, or $\operatorname{im} T = V'$ and $\ker T = 0$. ∎

Corollary 6.13 (Schur II)
Let V be an irreducible L-module for a Lie algebra L over $\mathbb{C}$ and $T : V \to V$ be an L-module homomorphism (or shorter L-endomorphism). Then T is a scalar multiple of the identity map (possibly the zero map).

Proof. Let $T : V \to V$ be any L-endomorphism. Then T is in particular a linear map from V to V so it has an eigenvalue λ with corresponding eigenvector $v \in V$ by Proposition 6.11. Thus, the linear map $T - \lambda \cdot \mathrm{id}_V$ has $v \neq 0$ in its kernel, and it is an L-endomorphism, since both T and id_V are. By Lemma 6.13, this linear map $T - \lambda \cdot \mathrm{id}_V$ must be equal to zero and thus $T = \lambda \cdot \mathrm{id}_V$. Note that λ (and thus T) can be equal to 0. ∎

Theorem 6.14 (Weyl)
Let L be a semisimple Lie algebra over $\mathbb{C}$ and V a finite-dimensional L-module. Then V has irreducible submodules $W_1, W_2, \ldots, W_k$, such that $V = W_1 \oplus \cdots \oplus W_k$, for some $k \in \mathbb{N}$. That is, V is the direct sum of irreducible submodules.

Proof. Omitted. ∎