Chapter 3

Representations of sl_2

For the whole chapter let sl_2 from Example 4.6, which is the $\mathbb{C}$-span of the three elements

$$e := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with the usual commutator $[a, b] := a \cdot b - b \cdot a$ as Lie product. We know that it is a simple Lie algebra and the following relations hold (see Example 5.2):

$$[e, f] = h \qquad \text{and} \qquad [h, e] = 2e = -[e, h] \qquad \text{and} \qquad [h, f] = -2f = -[f, h].$$

We want to classify all its finite-dimensional modules. Since sl_2 is simple, it is semisimple (see Example 5.11). Thus by Weyl's Theorem 6.14 it is enough to classify the irreducible modules, because all others are direct sums of irreducible ones.

7 The irreducible sl_2-modules introduced

Proposition 7.1 (The modules V_d)
Let $d \in \mathbb{N} \cup \{0\}$ and let $\mathbb{C}[X, Y]$ be the polynomial ring over $\mathbb{C}$ in two indeterminates X and Y. Let

$$V_d := \mathrm{Span}(X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d),$$

this is a $\mathbb{C}$-vector space of dimension $d + 1$, actually, V_d is the set of homogeneous polynomials of total degree d. For $d = 0$, the vector space V_0 consists of the constant polynomials and $\dim(V_0) = 1$. The following equations together with linear extension make V_d into an sl_2-module:

$$\begin{aligned} (X^a Y^b)e &:= Y \cdot \frac{\partial}{\partial X}(X^a Y^b) = a \cdot X^{a-1} Y^{b+1}, \\ (X^a Y^b)f &:= X \cdot \frac{\partial}{\partial Y}(X^a Y^b) = b \cdot X^{a+1} Y^{b-1}, \\ (X^a Y^b)h &:= (a - b) \cdot X^a Y^b \end{aligned}$$

all for $a + b = d$ and $0 \le a, b \le d$.

Proof. Since we can prescribe a linear map from V_d into itself arbitrarily on a basis, this defines endomorphisms for e, f and h uniquely. Linear extension gives us a $\mathbb{C}$-linear map

$$\varphi : \mathrm{sl}_2 \to \mathrm{Lie}(\mathrm{End}(V_d)).$$

To check that this is a representation of Lie algebras we only have to check that it respects the Lie product, that is:

$$v([x, y]\varphi) = (v(x\varphi))(y\varphi) - (v(y\varphi))(x\varphi)$$

for all $v \in V_d$ and all $x, y \in \mathrm{sl}_2$. Since φ is $\mathbb{C}$-linear and all $(x\varphi)$ are $\mathbb{C}$-linear it is enough to check all this for basis elements, that is, we have to check

$$\begin{aligned}
(X^aY^b)[e, f] &= ((X^aY^b)e)f - ((X^aY^b)f)e \quad \text{and} \\
(X^aY^b)[h, e] &= ((X^aY^b)h)e - ((X^aY^b)e)h \quad \text{and} \\
(X^aY^b)[h, f] &= ((X^aY^b)h)f - ((X^aY^b)f)h
\end{aligned}$$

for all $0 \le a, b \le d$ with $a + b = d$. This is left as an exercise for the reader. ■

Illustration 7.2 (The action on V_d)
Pictorially, this means:

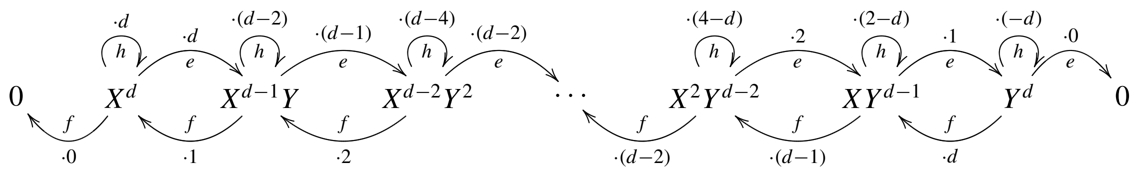

Illustration 7.3 (The action as matrices)
If we express the action of e, f and h by matrices with respect to the monomial basis

$$(X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d)$$

in row convention, we get:

$$e \leftrightarrow \begin{bmatrix} 0 & d & 0 & \cdots & 0 \\ 0 & 0 & d-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

$$f \leftrightarrow \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & d-1 & 0 & 0 \\ 0 & \cdots & 0 & d & 0 \end{bmatrix}$$

$$h \leftrightarrow \begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & d-2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -d \end{bmatrix}$$

Proposition 7.4 (All V_d are irreducible)
For all $d \in \mathbb{N} \cup \{0\}$, the module V_d is irreducible.

Proof. Assume $0 < W \le V_d$ is a non-zero subspace that is invariant under the action of sl_2. The endomorphism of W induced by the action of h has an eigenvalue λ with a corresponding eigenvector $0 \ne w \in W$ (see Proposition 6.11). Since h has 1-dimensional eigenspaces spanned by the monomials $X^d, XY^{d-1}, \ldots, Y^d$, the vector w is a scalar multiple of one of these. But then the subspace W contains all such monomials since successive applications of e and f map one to some non-zero scalar multiple of every other one. Thus $W = V_d$ and we have proved that V_d is irreducible. ■

8 Every irreducible sl_2-module is isomorphic to one of the V_d

Lemma 8.1 (Eigenvectors to different eigenvalues are linearly independent)
Let V be an $\mathbb{F}$-vector space and $\varphi \in \mathrm{End}(V)$ an arbitrary endomorphism. Let $(v_1, v_2, \ldots, v_n)$ be a tuple of eigenvectors of φ to pairwise different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Then $(v_1, \ldots, v_n)$ is linearly independent.

Proof. Assume for a contradiction that $(v_1, \ldots, v_n)$ is linearly dependent. Let $k \in \mathbb{N}$ be minimal such that $(v_1, \ldots, v_k)$ is linearly independent and $v_{k+1} \in \mathrm{Span}(v_1, \ldots, v_k)$. We have $k \geq 1$ because eigenvectors are non-zero and $k < n$ because of our assumption. If $v_{k+1} = \sum_{i=1}^{k} \mu_i v_i$ for some $\mu_i \in \mathbb{F}$, then

$$\sum_{i=1}^{k} \lambda_i \cdot \mu_i v_i = \sum_{i=1}^{k} (\mu_i v_i)\varphi = v_{k+1}\varphi = \lambda_{k+1} \cdot v_{k+1} = \sum_{i=1}^{k} \lambda_{k+1} \cdot \mu_i v_i,$$

which is a contradiction since $(v_1, \ldots, v_k)$ is linearly independent and the eigenvalues are pairwise different. ∎

Lemma 8.2 (Eigenvectors in sl_2-modules)
Let V be an sl_2-module over $\mathbb{C}$ and λ be an eigenvalue of h with eigenvector $v \in V$.

- Either $ve = 0$ or ve is an eigenvector of h for the eigenvalue $\lambda - 2$.
- Either $vf = 0$ or vf is an eigenvector of h for the eigenvalue $\lambda + 2$.

Proof. By the module axioms and the relations $[h, e] = 2e$ and $[h, f] = -2f$, we get:

$$\begin{aligned} (ve)h &= (vh)e - v[h, e] = \lambda \cdot (ve) - v \cdot (2e) = (\lambda - 2) \cdot (ve) \\ (vf)h &= (vh)f - v[h, f] = \lambda \cdot (vf) + v \cdot (2f) = (\lambda + 2) \cdot (vf) \end{aligned}$$

This proves the lemma, since eigenvectors have to be non-zero by definition. ∎

Lemma 8.3 (Highest weights)
Let V be a finite-dimensional sl_2-module over $\mathbb{C}$. Then V contains an eigenvector w of h such that $wf = 0$.

Proof. Since we work over the complex numbers $\mathbb{C}$, the endomorphism of V induced by h has an eigenvalue λ with corresponding eigenvector v (see Proposition 6.11). We consider the sequence

$$v, vf, vf^2, \ldots, vf^k, \ldots,$$

where vf^k stands for the vector one gets by acting repeatedly with f altogether k times. By Lemma 8.2 these are all either equal to zero or are eigenvectors of h to different eigenvalues, namely $\lambda, \lambda + 2, \lambda + 4, \ldots$. If they were all non-zero, then they would all be linearly independent by Lemma 8.1, which can not be true since V is finite-dimensional. Thus there is a k with $vf^k \neq 0$ and $vf^{k+1} = 0$, the vector $w := vf^k$ is an eigenvector of h with $wf = 0$. ∎

Definition 8.4 (Highest weight vector)
A vector w as in Lemma 8.3 is called a **highest weight vector** of the sl_2-module V and its corresponding eigenvalue is called a **highest weight**. We shall extend this definition later.

We are now in a position to prove the main result of this chapter:

Theorem 8.5 (Classification of finite-dimensional irreducible sl_2-modules)
Let V be an irreducible sl_2-module of dimension $d + 1$, then V is isomorphic to V_d.

Proof. Since V is finite-dimensional over $\mathbb{C}$, the endomorphism h of V has an eigenvector w with $wf = 0$ be Lemma 8.3. Let λ be the corresponding eigenvalue. We consider the sequence

$$w, we, we^2, \ldots$$

where we^k stands for the vector one gets by acting repeatedly with e altogether k times. By Lemma 8.2 these are all either equal to 0 or eigenvectors of h with eigenvalues $\lambda, \lambda-2, \lambda-4, \ldots$ respectively. As in the proof of Lemma 8.3 we conclude that there is a k with $we^{k+1} = 0$ and $we^k \neq 0$.

We claim that $W := \mathrm{Span}(w, we, we^2, \ldots, we^k)$ is an sl_2-submodule of V and that

$$\mathcal{B} := (w, we, \ldots, we^k)$$

is a basis. All these vectors are eigenvectors of h, so W is invariant under h. By construction and because of $we^{k+1} = 0$ the space W is invariant under e. Note that $\mathrm{Span}(w, we, \ldots, we^i)e = \mathrm{Span}(w, we, \ldots, we^{i+1})$.

Invariance under f comes from the fact that

$$(we^i)f = (we^{i-1})ef - (we^{i-1})fe + (we^{i-1})fe = (we^{i-1})h + ((we^{i-1})f)e \quad \text{for } 1 \le i \le k$$

and $wf = 0$ using induction by i. We have shown that W is invariant under h, e and f and thus under all elements of sl_2. Since W is non-zero and V is irreducible, we have $W = V$. Since $\mathcal{B} = (w, we, \ldots, we^k)$ is linearly independent by Lemma 8.1, it is a basis of W and thus of V and we conclude $k = d$ because $\dim(V) = d+1$.

With respect to the basis $\mathcal{B}$ the endomorphism induced by h is a diagonal matrix with diagonal entries $\lambda, \lambda-2, \ldots, \lambda-2d$, thus its trace is equal to $\lambda\cdot(d+1) - d(d+1)$ (recall that $\sum_{i=0}^{d} = d(d+1)/2$). But since $h = [e, f]$ this trace is zero, from which follows $\lambda = d$. The eigenvalues of h in its action on V are therefore $d, d-2, d-4, \ldots, 4-d, 2-d, -d$.

We now modify our basis $\mathcal{B}$ of V slightly to show that the action of sl_2 on V is the same as the one on V_d. Let $w_0 := w$ and $w_{i+1} := \frac{1}{d-i}\cdot w_i e$ for $0 \le i < d$, forming a new basis $\mathcal{B}' := (w_0, w_1, \ldots, w_d)$ of V.

With respect to this basis, the endomorphisms induced by the action of h and e are exactly as in Illustration 7.3, since we have

$$w_i h = (d-2i)w_i \quad \text{and} \quad w_i e = (d-i)w_{i+1}$$

for $0 \le i \le d$ where $w_{d+1} := 0$. We claim that the same holds for the endomorphism induced by the action of f. We have $w_0 f = wf = 0$ so the first row is zero. Furthermore, we claim that $w_i f = i w_{i-1}$ for $1 \le i \le d$. This follows by induction using a similar computation as above, we have

$$\begin{aligned} w_{i+1}f &= \frac{1}{d-i}w_i ef = \frac{1}{d-i}(w_i h + (w_i f)e) = \frac{1}{d-i}((d-2i)w_i + i w_{i-1}e) \\ &= \frac{d-2i+i(d+1-i)}{d-i}w_i = \frac{id-i^2+d-i}{d-i}w_i = (i+1)w_i \end{aligned}$$

for $0 \le i < d$ where $w_{-1} := 0$.

Since the action of h, e and f, and thus of all elements of sl_2, are the same with respect to the bases $(X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d)$ of V_d and $\mathcal{B}'$ of V, the linear map $X^{d-i}Y^i \mapsto w_i$ is an isomorphism of V_d onto V, proving the theorem. ■

Because of Weyl's Theorem we have thus proved:

Theorem 8.6 (Representations of $\mathrm{sl}_2(\mathbb{C})$**)**
Let V be a finite-dimensional $\mathrm{sl}_2(\mathbb{C})$-module. Then V has irreducible submodules $W_1, W_2, \ldots, W_k$, for some $k \in \mathbb{N}$, such that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ and there are numbers $d_1, \ldots, d_k \in \mathbb{N} \cup \{0\}$ such that $W_i \cong V_{d_i}$. ■