Chapter 4

Engel's and Lie's Theorems

9 Engel's Theorem on nilpotent Lie algebras

Definition 9.1 (Nilpotent elements)
Let V be a vector space and $T \in \mathrm{End}(V)$ an endomorphism. Then T is called **nilpotent**, if there is a $k \in \mathbb{N}$ such that $T^k = 0$ (the zero map).
Let L be a Lie algebra and $x \in L$. Then x is called **ad-nilpotent**, if $x^{\mathrm{ad}} \in \mathrm{End}(L)$ is nilpotent.
Note that this means that $(x^{\mathrm{ad}})^k = 0$ for some $k \in \mathbb{N}$ and this uses the regular composition of maps rather than the Lie product!

Proposition 9.2 (Eigenvalues of nilpotent elements)
Let V be a finite-dimensional vector space over $\mathbb{F}$ and $T \in \mathrm{End}(V)$ be nilpotent. Then 0 is the only eigenvalue of T.

Proof. Let λ be an eigenvalue with eigenvector $0 \neq v \in V$ and let $k \in \mathbb{N}$ with $T^k = 0$. Then $0 = vT^k = \lambda^k v$ so $\lambda^k = 0$ and thus $\lambda = 0$ since $\mathbb{F}$ is a field. However, 0 is an eigenvalue since T is not invertible. ■

In this section we want to prove the following theorem:

Theorem 9.3 (Engel)
Let L be a finite-dimensional Lie algebra over a field $\mathbb{F}$. Then L is nilpotent if and only if every element x of L is ad-nilpotent.

We only prove the "only-if"-part here, the "if"-part is proved in the rest of this section.

Proof. If L is nilpotent, then there is a k such that $L^k = 0$. This means in particular that every expression

$$[[\cdots[[x_0, x_1], x_2], \cdots], x_k] = 0$$

for arbitrary elements $x_0, x_1, \ldots, x_k \in L$. This implies immediately that

$$x_1^{\mathrm{ad}} \cdot x_2^{\mathrm{ad}} \cdot \cdots \cdot x_k^{\mathrm{ad}} = 0 \in \mathrm{End}(L)$$

and in particular that $(x^{\mathrm{ad}})^k = 0$ for all $x \in L$. So every element x of L is ad-nilpotent. ■

We first prove some helper results:

Lemma 9.4 (Quotient modules)
Let L be a Lie algebra and V an L-module with a submodule $0 < W < V$. Then the quotient space $V/W = \{v + W \mid v \in V\}$ is an L module with the induced action

$$(v + W)x := vx + W.$$

Proof. Details omitted, but routine verification. Check well-definedness first, the module actions are directly inherited from V. ■

Lemma 9.5 (ad-quotients)
Let L be a Lie algebra and H a subalgebra. Then we can restrict $\mathrm{ad} : L \to \mathrm{Lie}(\mathrm{End}(L))$ to H and thus get a representation $\mathrm{ad}|_H : H \to \mathrm{Lie}(\mathrm{End}(L))$. This makes L into an H-module and H itself is an H-submodule of L. Thus the quotient space L/H is an H-module as well. If $y \in H$ is ad-nilpotent, then it acts as a nilpotent endomorphism on L/H as well.

Proof. It is clear that $\mathrm{ad}|_H$ is a Lie algebra homomorphism and thus that L is an H-module. Since H is a subalgebra (i.e. $[H, H] \le H$), it follows that H is an H-submodule of L. By Lemma 9.4, the quotient space L/H (which is **not a Lie algebra!**) is an H-module as well with action $(x + H)h := xh^{\mathrm{ad}} + H = [x, h] + H$ for all $x \in L$ and all $h \in H$. If $(h^{\mathrm{ad}})^k = 0$ for some k, then $(x + H)(h^{\mathrm{ad}})^k = x(h^{\mathrm{ad}})^k + H = 0 + H$ for all $x \in L$. ■

Lemma 9.6 (ad-nilpotency)
Let L be a Lie subalgebra of $\mathrm{gl}(V)$ for some finite-dimensional vector space V over $\mathbb{F}$ and suppose that L consists of nilpotent endomorphisms of V. Then for all $x \in L$ the endomorphism $x^{\mathrm{ad}} \in \mathrm{End}(L)$ is nilpotent.

Proof. If $k \in \mathbb{N}$ such that $x^k = 0$, then

$$\underbrace{[\cdots[[y, x], x], \ldots], x]}_{2k \text{ times}} = \sum_{i=0}^{2k} c_i x^i y x^{2k-i}$$

for some numbers $c_i \in \mathbb{F}$. Since for every summand in this sum there are at least k factors of x on at least one side of y, the whole sum is equal to 0. As this holds for all $y \in L$, we have proved that $(x^{\mathrm{ad}})^{2k} = 0$. ■

Proposition 9.7 (Helper for Engel)
Let V be a finite-dimensional vector space over $\mathbb{F}$ and L a Lie subalgebra of $\mathrm{gl}(V)$ consisting of nilpotent endomorphisms. Then there is a non-zero $v \in V$ with $vx = 0$ for all $x \in L$.

Proof. We proceed by induction on $\dim(L)$. If $\dim(L) = 1$, then L consists of the scalar multiples of a single nilpotent endomorphism $x \in \mathrm{End}(V)$. By Proposition 9.2 it has 0 as eigenvalue, thus there is an eigenvector $0 \neq v \in V$ with $vx = 0$ and we are done.
Now suppose $\dim(L) > 1$ and the proposition is already proved for nilpotent Lie algebras of smaller dimension. We proceed in two steps:
Step 1: Let H be a maximal subalgebra of L (that is, H is a subalgebra such that there is no subalgebra K of L with $H < K < L$). Such an H exists and is non-zero, since every 1-dimensional subspace of L is a subalgebra and $\dim(L) < \infty$. We claim that $\dim(H) = \dim(L) - 1$ and that H is an ideal in L.
As in Lemma 9.5 we view L as H-module with submodule H and thus L/H as H-module with the action $(x + H)h := xh^{\mathrm{ad}} + H$. This gives us a representation of H on the vector space L/H and thus a homomorphism of Lie algebras $\varphi : H \to \mathrm{Lie}(\mathrm{End}(L/H))$. Since L and thus H consists of nilpotent elements we conclude that $H\varphi$ consists of nilpotent endomorphisms of L/H using Lemma 9.6. Since $\dim(H\varphi) \le \dim(H) < \dim(L)$, we can use the induction hypothesis to conclude that there is a $y \in L \setminus H$ such that $(y + H)h = 0 + H$ for all $h \in H$, that is, $[y, H] \le H$ but $y \notin H$. But then $H + \mathrm{Span}(y)$ is a subalgebra of L that properly contains H. By the maximality of H it follows that $H + \mathrm{Span}(y) = L$ and so $\dim(H) = \dim(L) - 1$ and H is an ideal in L.
Step 2: Now we apply the induction hypothesis to $H \le L \le \mathrm{gl}(V)$. We conclude that there is a $w \in V$ with $wh = 0$ for all $h \in H$. Thus $W := \{v \in V \mid vh = 0 \ \forall h \in H\}$ is a non-zero

subspace of V. It is certainly invariant under H (mapped to 0 by it!) and invariant under y, since $vyh = v[y, h] + vhy = 0$ for all $v \in W$ and all $h \in H$, since $[y, h] \in H$. Since y is nilpotent on V and thus on W, it has an eigenvector $0 \neq v \in W$ with eigenvalue 0 (see Proposition 9.2), that is, $vy = 0$. However, since $vh = 0$ for all $h \in H$ and $L = H + \mathrm{Span}(y)$, it follows that $vx = 0$ for all $x \in L$. ∎

Now we prove a theorem, from which Engel's Theorem 9.3 follows immediately:

Theorem 9.8 (Engel's Theorem in $\mathrm{gl}(V)$**)**
Let K be a Lie subalgebra of $\mathrm{gl}(V)$ for some finite-dimensional vector space V over a field $\mathbb{F}$, such that every element x of K is a nilpotent endomorphism. Then there is a basis $\mathcal{B}$ of V such that every element x of K corresponds to a strictly lower triangular matrix with respect to $\mathcal{B}$. It follows that K is a nilpotent Lie algebra.

Proof. We proceed by induction on $\dim(V)$. If $\dim(V) = 1$ then the dimension of K is either 0 or 1 and in both cases the matrices with respect to any basis $\mathcal{B}$ are all zero because they are nilpotent 1×1-matrices.
Suppose now that $n := \dim(V) \geq 2$ and the statement is proved for all cases with smaller dimension. By Proposition 9.7 there is a vector $0 \neq v_0 \in V$ with $v_0 x = 0$ for all $x \in K$. Obviously, $W := \mathrm{Span}(v_0)$ is a K-submodule of V and thus by Proposition 9.4, the quotient space V/W is a K-module. We denote the Lie subalgebra of $\mathrm{gl}(V/W)$ induced by this action of K by $\bar{K}$. Since $\dim(V/W) = \dim(V) - 1 = n - 1$ and $\bar{K}$ consists of nilpotent endomorphisms, we can use the induction hypothesis to conclude that V/W has a basis $\bar{\mathcal{B}} = (v_1 + W, \ldots, v_{n-1} + W)$ such that every element of $\bar{K}$ corresponds to a strictly lower triangular matrix with respect to $\bar{\mathcal{B}}$. But then every element of K corresponds to a strictly lower triangular matrix with respect to $\mathcal{B} := (v_0, v_1, \ldots, v_{n-1})$. This implies that K is isomorphic to a subalgebra of the Lie algebra of all strictly lower triangular matrices, which was shown to be nilpotent in Example 5.4. Thus K itself is nilpotent as well. ∎

We can now prove the missing implication in Engel's Theorem 9.3.

Proof. Suppose that L is a finite-dimensional Lie algebra over a field $\mathbb{F}$ such that every element of L is ad-nilpotent. Then $K := L^{\mathrm{ad}}$ is a Lie subalgebra of $\mathrm{Lie}(\mathrm{End}(L))$ fulfilling the hypotheses of Theorem 9.8 and is thus nilpotent. Since ad is a homomorphism of Lie algebras with kernel $Z(L)$ and image K, we have shown that $L/Z(L) \cong K$ is nilpotent, using the First Isomorphism Theorem 4.16. Therefore by Theorem 5.7 the Lie algebra L itself is nilpotent. ∎

Remark 9.9 (A warning)
Not for every nilpotent Lie algebra contained in $\mathrm{gl}(V)$ there is a basis of V such that all elements correspond to strictly lower triangular matrices. For example $L := \mathrm{Span}(\mathrm{id}_V)$ is abelian and thus nilpotent but it contains the identity, which corresponds to the identity matrix with respect to every basis of V.

10 Lie's Theorem on soluble Lie algebras

We want to derive a similar result to Theorem 9.8 for soluble Lie algebras over $\mathbb{C}$.

Definition 10.1 (Dual space and weights)
Let L be any $\mathbb{F}$-vector space. Then we denote the set of $\mathbb{F}$-linear maps from L to $\mathbb{F}$ by L^* and call it the **dual space** of L.
Let L be a Lie algebra over $\mathbb{F}$ and V a finite-dimensional L-module. A **weight of** L **(on** V**)** is an element $\lambda \in L^*$ such that

$$V_\lambda := \{v \in V \mid vx = (x\lambda) \cdot v \text{ for all } x \in L\}$$

is not equal to $\{0\}$. The subspace V_λ for a weight λ is called a **weight space**. It consists of **simultaneous eigenvectors** of all elements of L and the zero vector.

The following lemma is crucial for what we want to do:

Lemma 10.2 (Invariance)
Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic 0, V a finite-dimensional L-module and K an ideal in L. Assume that λ is a weight of K on V, that is, the weight space

$$V_\lambda := \{v \in V \mid vk = (k\lambda)v \text{ for all } k \in K\}$$

is non-zero. Then V_λ is invariant under the action of L.

Proof. Let $0 \neq v \in V_\lambda$ and $x \in L$. Then

$$vxk = v[x,k] + vkx = ([x,k]\lambda) \cdot v + (k\lambda) \cdot vx.$$

Note, that $[x,k] \in K$ since K is an ideal of L. That is, if we could show that $[x,k]\lambda = 0$ for all $k \in K$ and all $x \in L$, we would be done.
To this end, we consider the sequence of vectors

$$v, vx, vx^2, \ldots$$

and let m be the least integer, such that $(v, vx, \ldots, vx^m)$ is linearly dependent. We claim that $U := \mathrm{Span}(v, vx, \ldots, vx^{m-1})$ is invariant under K and that the matrix M_k of the action of any $k \in K$ with respect to the basis $\mathcal{B} := (v, vx, \ldots, vx^{m-1})$ is a lower triangular matrix with all diagonal entries being $k\lambda$:

$$M_k = \begin{bmatrix} k\lambda & 0 & \cdots & 0 \\ * & k\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & k\lambda \end{bmatrix}.$$

Indeed, $vk = (k\lambda)v$ showing that the first row of M_k is $(k\lambda, 0, \ldots, 0)$. We then proceed by induction on the rows showing that $vx^ik = (k\lambda)vx^i + w$ for some $w \in \mathrm{Span}(v, vx, \ldots, vx^{i-1})$ for $1 \leq i < m$ using

$$vx^{i+1}k = vx^i[x,k] + vx^ikx = (k\lambda)vx^{i+1} + u$$

for some $u \in \mathrm{Span}(v, vx, \ldots, vx^i)$ because $[x,k] \in K$ and the induction hypothesis.
We have showed that U is invariant under K and under x, so it is invariant under the whole Lie subalgebra $K + \mathrm{Span}(x)$ of L. For every element $k \in K$, the commutator $[x,k]$ is contained in K, so the matrix $M_{[x,k]}$ of its action on U with respect to the basis $\mathcal{B}$ is lower triangular with $[x,k]\lambda$ on the diagonal. On the other hand, this matrix is the commutator of the matrices M_x and M_k, so in particular its trace is zero. Thus $[x,k]\lambda = 0$ and we have proved the Invariance Lemma.
Note that we have proved at the same time that $U \leq V_\lambda$. ■

We prove a Proposition analogous to Proposition 9.7:

Proposition 10.3 (Helper for Lie)
Let L be a soluble Lie subalgebra of $\mathrm{gl}(V)$ for some finite-dimensional $\mathbb{C}$-vector space V. Then L has a weight λ on V and thus a non-zero weight space.

Proof. We need to find a simultaneous eigenvector for all elements of L. We proceed by induction on $\dim(L)$ very similarly to the proof of Proposition 9.7. If $\dim(L) = 1$, then L consists of the scalar multiples of a single non-zero element x. This element has an eigenvalue μ with corresponding eigenvector v by Proposition 6.11 because V is over $\mathbb{C}$. Thus $\lambda : L \to \mathbb{C}, c \cdot x \mapsto c \cdot \mu$ for $c \in \mathbb{C}$ is a weight with weight space V_λ containing at least $\mathrm{Span}(v)$.
Now suppose $n := \dim(L) \geq 2$ and the statement is already proved for all Lie algebras of dimension less than n. Since L is soluble, the space $L^{(1)} = [L, L]$ is a proper ideal of L. Let K be an $(n-1)$-dimensional subspace of L containing $[L, L]$ and $x \in L \setminus K$, such that we have $L = K + \mathrm{Span}(x)$. The subspace K is an ideal of L since every commutator in L is contained in $[L, L]$ and thus in K. Therefore K is in particular a subalgebra of smaller dimension than L and thus by Theorem 5.7 itself soluble. Using the induction hypothesis we conclude that K has a weight $\tilde{\lambda} \in K^*$. Let $W := V_{\tilde{\lambda}}$ be the corresponding weight space.
Using the Invariance Lemma 10.2 we conclude that W is invariant under x and thus under all of L. Since we are working over the complex numbers $\mathbb{C}$, the endomorphism induced by x on W has an eigenvector w with eigenvalue μ, that is, $wx = \mu \cdot w$. But if we now define $\lambda : L \to \mathbb{C}$ setting

$$(k + \nu \cdot x)\lambda := k\tilde{\lambda} + \nu\mu$$

this defines a $\mathbb{C}$-linear map and thus an element $\lambda \in L^*$, such that

$$w(k + \nu \cdot x) = wk + \nu \cdot wx = (k\tilde{\lambda})w + \nu\mu w = (k + \nu \cdot x)\lambda \cdot w$$

for all $k \in K$ and all $\nu \in \mathbb{C}$ showing that λ is a weight of L such that the weight space V_λ contains w. ■

Theorem 10.4 (Lie)
Let L be a soluble Lie algebra over $\mathbb{C}$ and V is a finite-dimensional L-module. Then there is a basis $\mathcal{B}$ of V, such that the matrix the action of every element of L with respect to $\mathcal{B}$ is a lower triangular matrix.

Proof. We proceed by induction on $\dim(V)$. If $\dim(V) = 1$ then the dimension the matrices with respect to any basis $\mathcal{B}$ are lower triangular because they are 1×1-matrices.
Suppose now that $n := \dim(V) \geq 2$ and the statement is proved for all cases with smaller dimension. Being a module, V gives rise to a Lie algebra homomorphism $\varphi : L \to \mathrm{gl}(V)$ and the image $L\varphi$ is soluble using Theorem 5.7 and the First Isomorphism Theorem 4.16. By Proposition 10.3 applied to $L\varphi$ there is weight λ' of $L\varphi$ on V. However, this immediately gives rise to a weight $\lambda := \varphi\lambda'$ of L. In particular, we have a non-zero vector v_0 in the weight space V_λ. That is, $v_0 x = (x\lambda)v_0$ for all $x \in L$. Obviously, $W := \mathrm{Span}(v_0)$ is an L-submodule of V and thus by Proposition 9.4, the quotient space V/W is an L-module of smaller dimension. Since $\dim(V/W) = \dim(V) - 1 = n - 1$ we can use the induction hypothesis to conclude that V/W has a basis $\bar{\mathcal{B}} = (v_1 + W, \ldots, v_{n-1} + W)$ such that every element of L corresponds to a lower triangular matrix with respect to $\bar{\mathcal{B}}$. But then every element of L corresponds to a lower triangular matrix with respect to the basis $\mathcal{B} := (v_0, v_1, \ldots, v_{n-1})$ of V. ■