Chapter 6

Classification of semisimple Lie algebras

When we studied $\mathrm{sl}_2(\mathbb{C})$, we discovered that it is spanned by elements e, f and h fulfilling the relations:

$$[e, h] = -2e, \qquad [f, h] = 2f \qquad \text{and} \qquad [e, f] = h.$$

Furthermore h was diagonalisable in every irreducible representation and $H := \mathrm{Span}(h)$ is obviously an abelian subalgebra. Note that $h = h + 0$ is the abstract Jordan decomposition of h, that $H = C_L(H)$ is the weight space of H, acting on L with the adjoint action, corresponding to the weight $0 \in H^*$. Likewise, $\mathrm{Span}(e)$ is the weight space for the weight $c \cdot h \mapsto -2c$ for $c \in \mathbb{C}$, and $\mathrm{Span}(f)$ is the weight space for the weight $c \cdot h \mapsto 2c$ for $c \in \mathbb{C}$.
This approach can be generalised. Our big plan will be:

1. Find a maximal abelian subalgebra H consisting of elements that are diagonalisable in every representation.
2. Restrict the adjoint representation of L to H and show that L is the direct sum of weight spaces with respect to H ("root space decomposition").
3. Prove general results about the set of weights ("root systems").
4. Show that the isomorphism type of L is completely determined by its root system.
5. Classify such root systems.

The rest of the course will be more expository than before.

14 Maximal toral subalgebras

Definition 14.1 (Semisimple elements)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. An element $x \in L$ is called **semisimple**, if its abstract Jordan decomposition is $x = x + 0$, that is, the nilpotent part is equal to zero (see Theorem 13.1). This means, that x acts diagonalisably on every L-module (see Theorem 13.2).

Definition 14.2 (Maximal toral subalgebras)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. A **toral subalgebra** T is a subalgebra consisting of semisimple elements. A toral subalgebra $T \leq L$ is called a **maximal toral subalgebra** if L has no toral subalgebra properly containing T. It is clear that every finite-dimensional semisimple Lie algebra over $\mathbb{C}$ has a maximal toral subalgebra. All these are non-zero since L contains semisimple elements (because of Theorem 13.1, note that if all elements of L were equal to their nilpotent part in the abstract Jordan decomposition, then they would in particular be ad-nilpotent and thus L would be nilpotent, a contradiction to being semisimple).

Lemma 14.3 (Maximal toral subalgebras are abelian)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. Every maximal toral subalgebra T of L is abelian.

Proof. Omitted. ■

Definition 14.4 (Cartan subalgebra)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. A maximal abelian toral subalgebra is called **Cartan subalgebra**. By Lemma 14.3 every such L has a Cartan subalgebra since every maximal toral subalgebra is abelian.

Theorem 14.5 (Cartan subalgebras are self-centralising)
Let H be a Cartan subalgebra of a finite-dimensional semisimple Lie algebra L over $\mathbb{C}$. Then $C_L(H) = H$.

Proof. Omitted. ■

Theorem 14.6 (Simultaneous diagonalisation)
Let $T_1, T_2, \ldots, T_k \in \mathrm{End}(V)$ be endomorphisms of a finite-dimensional $\mathbb{F}$-vector space V. Suppose that all T_i are diagonalisable and that $T_i T_j = T_j T_i$ for all $1 \leq i < j \leq k$. Then there is a basis $\mathcal{B}$ of V such that the matrices of all T_i with respect to $\mathcal{B}$ are diagonal.

Proof. Omitted here, see Exercise 3 of tutorial sheet 3 or a text on Linear Algebra. ■

For the rest of the chapter L will always be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and H a Cartan subalgebra. We denote the Killing form by κ.

Definition/Proposition 14.7 (Root space decomposition)
In this situation, L is an H-module by the adjoint action of H on L: The map

$$\begin{array}{rccc} \mathrm{ad}|_H & : & H & \to & \mathrm{Lie}(\mathrm{End}(L)) \\ & & h & \mapsto & h^{\mathrm{ad}} \end{array}$$

is a representation of H. We consider all its weight spaces (see Definition 10.1). Let $\Phi \subseteq H^*$ be the set of non-zero weights, note that the zero map $(h \mapsto 0)$ is a weight and that $L_0 = H$ by Theorem 14.5.
The space L is the direct sum of the weight spaces for H:

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

This decomposition is called the **root space decomposition of L with respect to H**. As defined in Definition 10.1, we have

$$L_\alpha = \{x \in L \mid [x, h] = (h\alpha) \cdot x \text{ for all } h \in H\}.$$

The set Φ is called the **set of roots** of L with respect to H and the L_α for $\alpha \in \Phi \cup \{0\}$ are called the **root spaces**. Note that we immediately conclude from the finite dimension of L that Φ is finite!

Proof. Let $h_1, \ldots, h_k$ be a basis of H. Since H is abelian, the endomorphisms $h_1^{\mathrm{ad}}, \ldots, h_k^{\mathrm{ad}} \in \mathrm{End}(L)$ fulfill the hypothesis of Theorem 14.6. Thus L has a basis $\mathcal{B}$ of simultaneous eigenvectors of the h_i^{ad}. Since every element of $\mathcal{B}$ is contained in a root space, L is the sum of the weight spaces. The intersection of two root spaces L_α and L_β for $\alpha \neq \beta$ is equal to the zero space, since if $h\alpha \neq h\beta$, then $x \in L_\alpha \cap L_\beta$ implies $(h\alpha)x = xh = (h\beta)x$ and thus $x = 0$. A short inductive argument shows that the sum of all root spaces in the root space decomposition is in fact direct (just add in one root space at a time). ■

In the sequel we will study the set Φ of roots.

Lemma 14.8 (Properties of Φ)
Suppose that $\alpha, \beta \in \Phi \cap \{0\}$. Then:

(i) $[L_\alpha, L_\beta] \leq L_{\alpha+\beta}$.

(ii) If $\alpha + \beta \neq 0$, then $\kappa(L_\alpha, L_\beta) = \{0\}$.

(iii) The restriction of κ to L_0 is non-degenerate.

Proof. Let $x \in L_\alpha$ and $y \in L_\beta$. Then

$$[[x, y], h] = [[x, h], y] + [x, [y, h]] = (h\alpha)[x, y] + (h\beta)[x, y] = (h(\alpha + \beta))[x, y],$$

thus $[x, y] \in L_{\alpha+\beta}$ which proves (i).
For (ii), we conclude from $\alpha + \beta \neq 0$ that there is some $h \in H$ with $h(\alpha + \beta) \neq 0$. Then

$$(h\alpha)\kappa(x, y) = \kappa([x, h], y) = \kappa(x, [h, y]) = -(h\beta)\kappa(x, y),$$

and thus $(h(\alpha + \beta) \cdot \kappa(x, y) = 0$. Therefore, $\kappa(x, y) = 0$.
For (iii), suppose that $z \in L_0$ and $\kappa(z, x_0) = 0$ for all $x_0 \in L_0$. Since every $x \in L$ can be written as

$$x = x_0 + \sum_{\alpha \in \Phi} x_\alpha$$

with $x_\alpha \in L_\alpha$, we immediately get $\kappa(z, x) = 0$ for all $x \in L$ from (ii) contradicting the non-degeneracy of κ on L. ∎

Quite surprisingly, every semisimple Lie algebra over $\mathbb{C}$ contains lots of copies of $\mathrm{sl}_2(\mathbb{C})$:

Theorem 14.9 (Copies of $\mathrm{sl}_2(\mathbb{C})$ in L)
Let $\alpha \in \Phi$ and $0 \neq e \in L_\alpha$. Then $-\alpha$ is a root and there exists $f \in L_{-\alpha}$ such that $\mathrm{Span}(e, f, h)$ with $h := [e, f]$ is a Lie subalgebra of L with

$$[e, h] = -2e \qquad \text{and} \qquad [f, h] = 2f.$$

Thus, it is isomorphic to sl_2.

Note that we can replace (e, f, h) by $(\lambda e, f/\lambda, h)$ for some $0 \neq \lambda \in \mathbb{C}$ without changing the relations. However, h and $\mathrm{Span}(e, f, h)$ remains always the same!

Proof. This proof was not be presented in the class but is contained in the notes for the sake of completeness.
Since κ is non-degenerate, there is an $x \in L$ with $\kappa(e, x) \neq 0$. Write $x = \sum_{\alpha \in \Phi \cup \{0\}} x_\alpha$ with $x_\alpha \in L_\alpha$. By Lemma 14.8.(ii) we conclude that $x_{-\alpha} \neq 0$ and $\kappa(e, x_{-\alpha}) \neq 0$. Therefore, $-\alpha$ is a root. Set $\tilde{f} := x_{-\alpha}$. Since $\alpha \neq 0$ there is a $t \in h$ with $t\alpha \neq 0$. Thus

$$\kappa([e, \tilde{f}], t) = \kappa(e, [\tilde{f}, t]) = -(t\alpha) \cdot \kappa(e, \tilde{f}) \neq 0$$

showing that $\tilde{h} := [e, \tilde{f}] \neq 0$. Note that $\tilde{f} \in H = L_0$ by Lemma 14.8.(i).
We claim that $\tilde{h}\alpha \neq 0$. Namely, if $\tilde{h}\alpha$ were equal to 0, then $[e, \tilde{h}] = (\tilde{h}\alpha)e = 0$ and $[\tilde{f}, \tilde{h}] = -(\tilde{h}\alpha)\tilde{f} = 0$. Therefore by Proposition 14.10 $\tilde{h}^{\mathrm{ad}}$ would be nilpotent. However, $\tilde{h}$ is semisimple, and the only semisimple and nilpotent element is 0. We can now set $f := -2\tilde{f}/(\tilde{h}\alpha)$ and $h := [e, f] = -2\tilde{h}/(\tilde{h}\alpha)$ to get the relations in the theorem. ∎

Note that by this L is an $\mathrm{sl}_2(\mathbb{C})$-module in many ways! This allows us to use our results about the representations of $\mathrm{sl}_2(\mathbb{C})$ for every $\alpha \in \Phi$ separately!
We have used:

Proposition 14.10
Let $x, y \in \mathrm{End}(V)$ be endomorphism of the finite-dimensional complex vector space V. Suppose that both x and y commute with $[x, y] = xy - yx$. Then $[x, y]$ is a nilpotent map.

15 Root systems

We keep our general hypothesis that L is a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra H and corresponding set of roots Φ.
For this section, let E be a finite-dimensional vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form $(-|-) : E \times E \to \mathbb{R}$ ("positive definite" means $(x|x) > 0$ if and only if $x \neq 0$).

Definition 15.1 (Reflections)
For $v \in E$, the map

$$\begin{array}{rccc} s_v & : & E & \to & E \\ & & x & \mapsto & x - \frac{2(x|v)}{(v|v)} v \end{array}$$

is called the **reflection along** v. It is linear, interchanges v and $-v$ and fixes the hyperplane orthogonal to v. As an abbreviation, we use $\langle x|v\rangle := \frac{2(x|v)}{(v|v)}$ for $x, v \in E$, note that $\langle -|-\rangle$ is only linear in the first component. We have $xs_v = x - \langle x|v\rangle\, v$.

Definition 15.2 (Root system)
A subset $R \subseteq E$ is called a **root system**, if

(R1) R is finite, $\mathrm{Span}(R) = E$ and $0 \notin R$.

(R2) If $\alpha \in R$, then the only scalar multiples of α in R are α and $-\alpha$.

(R3) If $\alpha \in R$, then s_α permutes the elements of R.

(R4) If $\alpha, \beta \in R$, then $\langle \alpha|\beta\rangle \in \mathbb{Z}$.

Theorem 15.3 (Φ is a root system)
Then Φ is a root system if we take E to be the $\mathbb{R}$-span of Φ with the bilinear form induced by the Killing form κ.

Proposition 15.4 (Moving forms)
The Killing form κ restricted to H is non-degenerate by Lemma 14.8.(iii). Therefore, the linear map

$$\begin{array}{ccc} H & \to & H^* \\ h & \mapsto & (x \mapsto \kappa(h, x)) \end{array}$$

is injective and thus bijective since H and H^* have the same finite dimension. Therefore, for every $\alpha \in H^*$, there is a unique $t_\alpha \in H$ with $x\alpha = \kappa(t_\alpha, x)$ for all $x \in H$. We set $(\alpha|\beta) := \kappa(t_\alpha, t_\beta)$ for all $\alpha, \beta \in H^*$, this defines a non-degenerate bilinear form on H^*, which we call **the bilinear form on H^* induced by κ.**

The proof of Theorem 15.3 works through a series of little results always using all those $\mathrm{sl}_2(\mathbb{C})$-subalgebras and the fact that L is an $\mathrm{sl}_2(\mathbb{C})$-module in different ways. Here we just look at a few of them without proofs:

Lemma 15.5
Let $\alpha \in \Phi$. If $x \in L_{-\alpha}$ and $y \in L_\alpha$, then $[x, y] = \kappa(x, y)t_\alpha$.

Proof. For all $h \in H$, we have

$$\kappa([x, y], h) = \kappa(x, [y, h]) = (h\alpha)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y) = \kappa(\kappa(x, y)t_\alpha, h).$$

Thus $[x, y] - \kappa(x, y)t_\alpha \in H^\perp$ and is therefore equal to 0, since κ is non-degenerate on H. ∎

Lemma 15.6
Let $\alpha \in \Phi$ and $0 \neq e \in L_\alpha$ and $\mathrm{sl}_\alpha := \mathrm{Span}(e, f, h)$ as in Theorem 14.9. If M is an sl_α-submodule of L, then the eigenvalues of h on M are integers.

Proof. Follows immediately from Weyl's Theorem and our classification of sl_2-modules. ■

Lemma 15.7
Let $\alpha \in \Phi$. The root spaces L_α and $L_{-\alpha}$ are 1-dimensional. Moreover, the only scalar multiples of α that are in Φ are α itself and $-\alpha$.

Note that it follows that this now identifies the copy of $\mathrm{sl}_2(\mathbb{C})$ sitting in $L_\alpha \oplus H \oplus L_{-\alpha}$ uniquely since we only have a choice for $e \in L_\alpha$ up to a scalar. All these choices give us the same sl_α. It even identifies a unique $h_\alpha \in H$!

Lemma 15.8
Suppose that $\alpha, \beta \in \Phi$ and $\beta \notin \{\alpha, -\alpha\}$. Then:

(i) $h_\alpha \beta = \frac{2(\beta|\alpha)}{(\alpha|\alpha)} = \langle \beta|\alpha \rangle \in \mathbb{Z}$.

(ii) There are integers $r, q \geq 0$ such that for all $k \in \mathbb{Z}$, we have $\beta + k\alpha \in \Phi$ if and only if $-r \leq k \leq q$. Moreover, $r - q = h_\alpha \beta$.

(iii) $\beta - (h_\alpha \beta) \cdot \alpha = \beta - \langle \beta|\alpha \rangle \alpha = \beta s_\alpha \in \Phi$.

(iv) $\mathrm{Span}(\Phi) = H^*$.

Lemma 15.9
If α and β are roots, then $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$ and $(\alpha|\beta) = \kappa(t_\alpha, t_\beta) \in \mathbb{Q}$.
It follows, that if $\{\alpha_1, \dots, \alpha_n\} \subseteq \Phi$ is a basis of H^* and $\beta \in \Phi$, then β is a linear combination of the α_i with coefficients in $\mathbb{Q}$.

Proposition 15.10
The bilinear form defined by $(\alpha|\beta) := \kappa(t_\alpha, t_\beta)$ is a positive definite symmetric bilinear form on the real span E of Φ.

16 Dynkin diagrams

In this section we will classify all possible root systems, we will only use the axioms in Definition 15.2.

Lemma 16.1 (Finiteness Lemma)
Let R be a root system in a finite-dimensional real vector space E equipped with a positive-definite symmetric bilinear form $(-|-) : E \times E \to \mathbb{R}$. Let $\alpha, \beta \in R$ with $\beta \notin \{\alpha, -\alpha\}$. Then

$$\langle \alpha|\beta \rangle \cdot \langle \beta|\alpha \rangle \in \{0, 1, 2, 3\}.$$

Proof. By (R4), the product is an integer. We have

$$(x|y)^2 = (x|x) \cdot (y|y) \cdot \cos^2(\theta)$$

if θ is the angle between two non-zero vectors $x, y \in E$. Thus $\langle x|y \rangle \cdot \langle y|x \rangle = 4\cos^2\theta$ and this must be an integer. If $\cos^2\theta = 1$, then θ is an integer multiple of π and so α and β are linearly dependent which is impossible because of our assumptions and (R2). ■

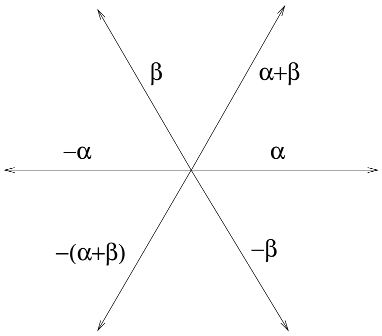
We immediately conclude that there are only very few possibilities for $\langle\alpha|\beta\rangle$, $\langle\beta|\alpha\rangle$, the angle θ and the ratio $(\beta|\beta)/(\alpha|\alpha)$ (without loss of generality we assume $(\beta|\beta) \geq (\alpha|\alpha)$):

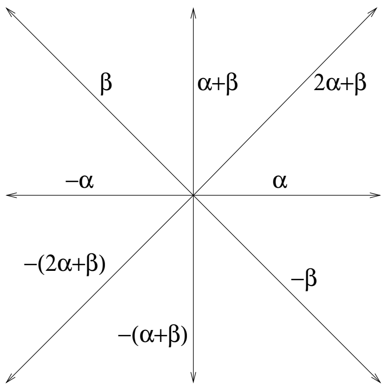
$\langle\alpha\|\beta\rangle$	$\langle\beta\|\alpha\rangle$	θ	$\frac{(\beta\|\beta)}{(\alpha\|\alpha)}$
0	0	$\pi/2$	–
1	1	$\pi/3$	1
−1	−1	$2\pi/3$	1
1	2	$\pi/4$	2
−1	−2	$3\pi/4$	2
1	3	$\pi/6$	3
−1	−3	$5\pi/6$	3

Lemma 16.2
Let R be a root system with E as in Lemma 16.1 and let $\alpha, \beta \in R$ with $(\alpha|\alpha) \leq (\beta|\beta)$. If the angle between α is strictly obtuse, then $\alpha + \beta \in R$. If the angle between α and β is strictly acute, then $\alpha - \beta \in R$.

Proof. Use (R3) saying that $\alpha s_\beta = \alpha - \langle\alpha|\beta\rangle\,\beta \in R$ together with the above table. ■

Example 16.3 (Examples of root systems)
The following are two different root systems in $\mathbb{R}^2$:





Check the axioms yourself. You find examples for most but not all cases in the above table.

Definition 16.4 (Bases for root systems)
Let R be a root system in a real vector space E. A subset $\mathcal{B} \subseteq R$ is called a **base** of R, if

(B1) $\mathcal{B}$ is a vector space basis of E, and

(B2) every $\alpha \in R$ can be written as $\alpha = \sum_{\beta\in\mathcal{B}} k_\beta \beta$ with $k_\beta \in \mathbb{Z}$, such that all the non-zero coefficients k_β are either all positive or all negative.

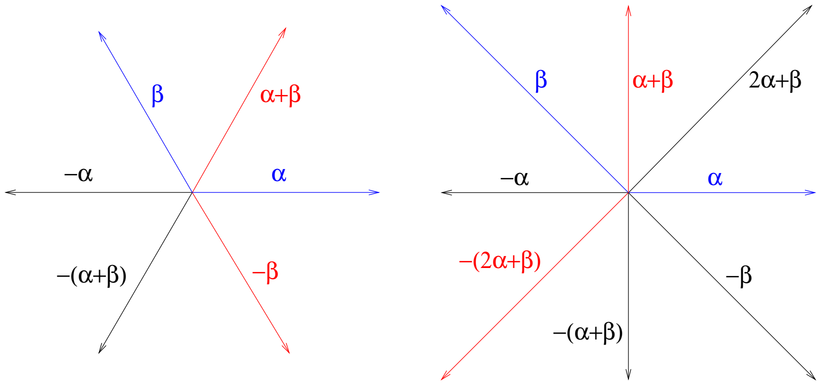
For a fixed base $\mathcal{B}$, we call α **positive** if all its non-zero coefficients with respect to $\mathcal{B}$ are positive and **negative** otherwise. We denote the subset of R of positive roots by R^+ and the subset of negative roots R^-.

Note that some coefficients can be equal to zero, only the non-zero ones need to have the same sign! Note furthermore that the definition of R^+ and R^- actually depends on $\mathcal{B}$ and that there are different choices for $\mathcal{B}$ possible! For example, for any base $\mathcal{B}$, the set $-\mathcal{B}$ is also a base!

Theorem 16.5 (Existence of bases for root systems)
Let R be a root system in the real vector space E. Then R has a base $\mathcal{B}$.

Proof. Omitted here. ∎

Example 16.6 (Example of a root system)
In the following two diagrams we have coloured a base of the root system in blue and one in red:



So in the first diagram, both (α, β) and $(\alpha+\beta, -\beta)$ are bases. In the second diagram, both (β, α) and $(\alpha+\beta, -(2\alpha+\beta))$ are bases. These are not all possible choices!

Definition 16.7 (Isomorphism of root systems)
Let $R_1 \subseteq E_1$ and $R_2 \subseteq E_2$ be two root systems. An isomorphism between the two root systems R_1 and R_2 is a bijective $\mathbb{R}$-linear map $\psi : E_1 \to E_2$ such that

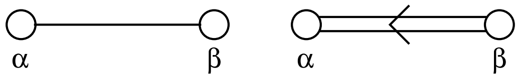
(i) $R_1\psi = R_2$, and

(ii) for any $\alpha, \beta \in R_1$ we have $\langle \alpha|\beta \rangle = \langle \alpha\psi|\beta\psi \rangle$.

Note that condition (ii) basically ensures that the angle θ between $\alpha\psi$ and $\beta\psi$ is the same as the angle between α and β since $4\cos^2\theta = \langle \alpha|\beta \rangle \cdot \langle \beta|\alpha \rangle$.
We can now come up with a graphical way to describe root systems. At first however, it seems that we describe a basis of a root system!

Definition 16.8 (Coxeter graphs and Dynkin diagrams)
Let R be a root system in a real vector space E and let $\mathcal{B} = (b_1, \ldots, b_n)$ be a base of R. The **Coxeter graph** of $\mathcal{B}$ is an undirected graph with n vertices, one for every element b_i and with $\langle b_i|b_j \rangle \cdot \langle b_j|b_i \rangle$ edges between vertex b_i and b_j for all $1 \leq i < j \leq n$. In the **Dynkin diagram**, we add for any pair of vertices $b_i \neq b_j$ for which $(b_i|b_i) \neq (b_j|b_j)$ (which are then necessarily connected) an arrow from the vertex corresponding to the longer root to the one corresponding to the longer root.

Example 16.9 (Dynkin diagrams)
Here are the two Dynkin diagrams for the base (α, β) in each of the two root systems in Example 16.6:



Surprisingly, the information in the Dynkin diagram is sufficient to describe the isomorphism type of the root system:

Proposition 16.10 (Dynkin diagram decides isomorphism type)
Let $R_1 \subseteq E_1$ and $R_2 \subseteq E_2$ be two root systems and let $\mathcal{B}_1$ be a base of R_1 and $\mathcal{B}_2$ one of R_2. If there is a bijection $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ such that ψ maps the Dynkin diagram of $\mathcal{B}_1$ to the one of $\mathcal{B}_2$, then R_1 and R_2 are isomorphic in the sense of Definition 16.7.

More formally, if

$$\langle\alpha|\beta\rangle \cdot \langle\beta|\alpha\rangle = \langle\alpha\psi|\beta\psi\rangle \cdot \langle\beta\psi|\alpha\psi\rangle$$

and $(\alpha|\alpha) < (\beta|\beta)$ if and only if $(\alpha\psi|\alpha\psi) < (\beta\psi|\beta\psi)$ for all $\alpha, \beta \in \mathscr{B}_1$, then the $\mathbb{R}$-linear extension of ψ to an $\mathbb{R}$-linear map from $E_1 \to E_2$ is an isomorphism between the root systems R_1 and R_2.

Proof. Omitted. ■

Proposition 16.11 (Dynkin diagram is property of isomorphism type)
If two root systems are isomorphic then they have the same Dynkin diagram. In particular, the Dynkin diagram does not depend on the choice of base.

Proof. Omitted. ■

So Dynkin diagrams are the same as isomorphism types of root systems.

17 How everything fits together

Definition 17.1 (Irreducible root systems)
A root system R is called **irreducible**, if it cannot be written as the disjoint union $R_1 \cup R_2$ such that $(\alpha|\beta) = 0$ whenever $\alpha \in R_1$ and $\beta \in R_2$.

Lemma 17.2 (Root systems can be decomposed into irreducible ones)
Let R be a root system in the real vector space E. Then R is the disjoint union $R = R_1 \cup \cdots \cup R_k$ of subsets $R_1, \ldots, R_k$ where each R_i is an irreducible root system in $E_i := \mathrm{Span}(R_i)$ and E is an orthogonal direct sum of the subspaces $E_1, \ldots, E_k$.

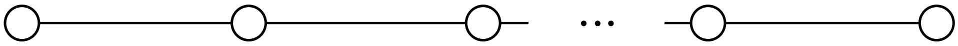
Proof. Omitted here. ■

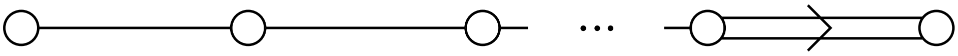
Note that both root systems in Example 16.3 are irreducible.

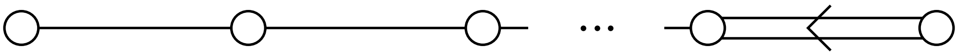
Proposition 17.3 (Irreducibility in the Dynkin diagram)
A root system is irreducible if and only if its Dynkin diagram is connected.

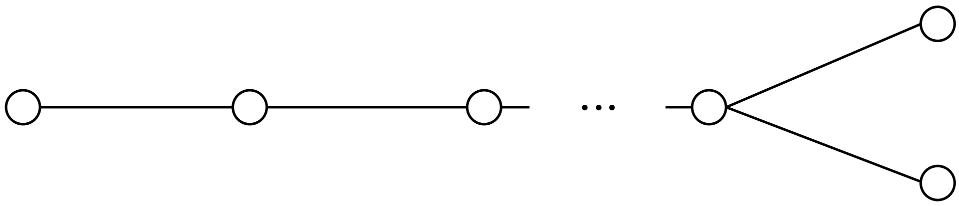
Proof. Follows immediately from the definitions of "irreducible" for root systems and of Dynkin diagrams. ■

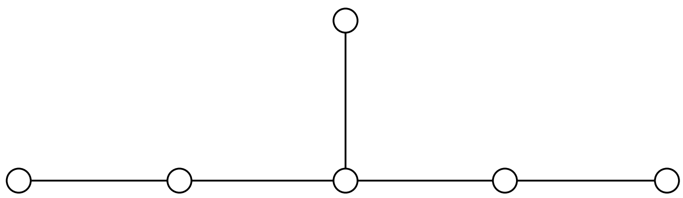
Theorem 17.4 (Classification of irreducible root systems)
Every irreducible root system has one of the following Dynkin diagrams and these diagrams all occur as Dynkin diagrams of a root system:

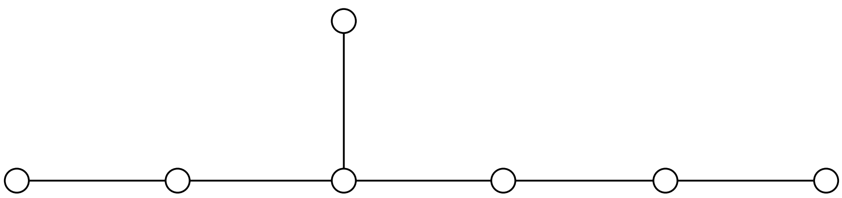
$A_n (n \geq 1)$: 

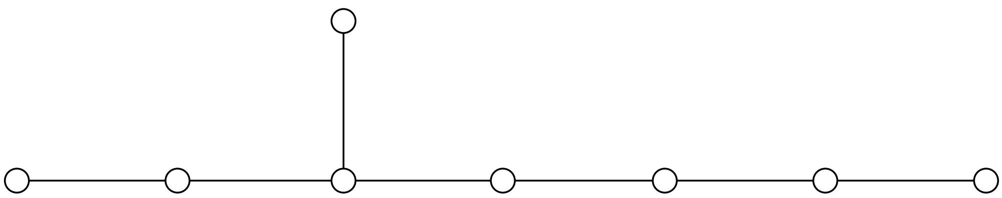
$B_n (n \geq 2)$: 

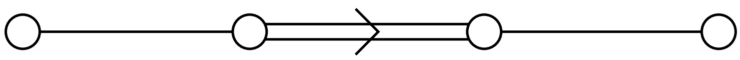
$C_n (n \geq 3)$: 

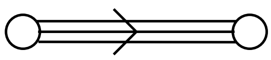
$D_n (n \geq 4)$: 

E_6: 

E_7: 

E_8: 

F_4: 

G_2: 

The first four types A_n to D_n cover each infinitely many cases. Each diagram has n vertices. The restrictions on n are there to avoid duplicates.

Proof. Very nice, but omitted here, unfortunately. ■

We have now done the following:

$$\begin{tikzcd}
\boxed{\text{F.d. semisimple Lie algebra } L \text{ over } \mathbb{C}} \arrow[d, "\text{we choose}"] \\
\boxed{\text{Cartan subalgebra } H} \arrow[d, "\text{which gives}"] \\
\boxed{\text{Root space decomposition } L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha} \arrow[d, "\text{which defines}"] \\
\boxed{\text{Root system } \Phi \subseteq H^*} \arrow[d, "\text{we choose}"] \\
\boxed{\text{Base } \mathcal{B} \text{ of root system } \Phi} \arrow[d, "\text{which defines}"] \\
\boxed{\text{Dynkin diagram}}
\end{tikzcd}$$

The big plan is:

- We know all resulting diagrams that can possibly occur.
- The result does not depend on our choices (we need to show this!).
- Two isomorphic Lie algebras give the same Dynkin diagram.
- Two non-isomorphic Lie algebras give different Dynkin diagrams.
- All Dynkin diagrams actually occur.
- L is simple if and only if the Dynkin diagram is irreducible.

To this end, we would need to prove the following results:

Theorem 17.5
Let L be a finite dimensional semisimple Lie algebra over $\mathbb{C}$ and let H_1 and H_2 be two Cartan subalgebras with associated root systems Φ_1 and Φ_2. Then Φ_1 and Φ_2 are isomorphic as root systems.

Theorem 17.6 (Serre)
Let Φ be an irreducible root system with n vertices and base $\mathcal{B} = (b_1, \ldots, b_n)$ and let $c_{i,j} := \langle b_i | b_j \rangle$ for $1 \le i, j \le n$ (the so-called **Cartan matrix**).
Let L be the Lie algebra over $\mathbb{C}$ generated by generators e_i, f_i and h_i for $1 \le i \le n$ subject to the relations

(S1) $[h_i, h_j] = 0$ for all $1 \le i, j \le n$,
(S2) $[e_i, h_j] = c_{i,j} e_i$ and $[f_i, h_j] = -c_{i,j} f_i$,
(S3) $[e_i, f_i] = h_i$ for all $1 \le i \le n$ and $[e_i, f_j] = 0$ for all $i \ne j$,
(S4) $(e_i)(e_j^{\mathrm{ad}})^{1-c_{i,j}} = 0$ and $(f_i)(f_j^{\mathrm{ad}})^{1-c_{i,j}} = 0$ if $i \ne j$.

Then L is finite dimensional and semisimple, $H := \mathrm{Span}(h_1, \ldots, h_n)$ is a Cartan subalgebra and its root system is isomorphic to Φ.

Theorem 17.7
Let L be a finite dimensional simple Lie algebra over $\mathbb{C}$ with root system Φ. Then Φ is irreducible.