Disturbance Decoupling for Polynomial Systems – Example

Let $\mathcal{P} = \mathbb{R}[X_1, X_2, X_3, X_4]$,

$$f = \begin{bmatrix} X_2 \\ X_1 \\ X_1X_3 \\ X_1X_4 \end{bmatrix}, \quad g = \begin{bmatrix} X_4 & X_2X_4 \\ 1 & X_2 \\ X_3 & 0 \\ 0 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} X_3 \\ X_4 \\ 0 \\ 0 \end{bmatrix}, \quad h = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

and $\nu = 2$. First, we compute Ω^* by Algorithm 2 and the related remark. Set

$$W_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

then we have

$$W_0 \cdot g = \begin{bmatrix} X_4 & X_2X_4 \\ 1 & X_2 \end{bmatrix}, \quad A_0 = [-1,\ X_4], \quad B_0 = [-1,\ X_4,\ 0,\ 0].$$

Thus, $\Omega_0 = \mathrm{im}(\cdot W_0)$ and $\Omega_0 \cap \ker(\cdot g) = \mathrm{im}(\cdot B_0)$.
Further, it is

$$\begin{aligned} L_f B_0 &= \begin{bmatrix} X_4, & X_1X_4 - 1, & 0, & 0 \end{bmatrix}, \\ L_{g_1} B_0 &= \begin{bmatrix} 0, & 0, & 0, & -1 \end{bmatrix}, \\ L_{g_2} B_0 &= \begin{bmatrix} 0, & 2X_4 + 1, & 0, & X_2 \end{bmatrix}, \end{aligned}$$

which results in $\Omega_0 \subsetneq \Omega_1 = \mathrm{im}(\cdot W_1)$ with

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As above, we derive

$$W_1 \cdot g = \begin{bmatrix} X_4 & X_2X_4 \\ 1 & X_2 \\ 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -1, & X_4, & 0 \end{bmatrix}, \quad B_1 = B_0.$$

Hence, $\Omega_2 = \Omega_1$ and the algorithm terminates with $\Omega^* = \Omega_1$, which means

$$\mathrm{syz}(\Omega^*) \;=\; \ker(W_1 \cdot) \;=\; \mathrm{im}(\begin{bmatrix} 0, & 0, & 1, & 0 \end{bmatrix}^T \cdot).$$

However, this implies $d \notin \operatorname{syz}(\Omega^*)$, by which the output is not decouplable from disturbances for the above choice of d. To answer the question whether or not the output is decouplable from disturbances for all $d \in \operatorname{syz}(\Omega^*)$, we use Theorem 22. For $\lambda_1 = X_1, \lambda_2 = X_2, \lambda_3 = X_4$, we have

$$\Omega^* \;=\; \operatorname{im}(\cdot W_1) \;=\; \langle \tfrac{\partial}{\partial X}\lambda_1, \tfrac{\partial}{\partial X}\lambda_2, \tfrac{\partial}{\partial X}\lambda_3 \rangle,$$

$\langle \frac{\partial}{\partial X}\lambda_2, \frac{\partial}{\partial X}\lambda_3 \rangle \cap \ker(\cdot g) = \{0\}$, $\frac{\partial}{\partial X}\lambda_1 - X_4 \cdot \frac{\partial}{\partial X}\lambda_2 \in \ker(\cdot g)$ and

$$\det(A) \;=\; \det(\begin{bmatrix} L_{g_1}\lambda_2 & L_{g_2}\lambda_2 \\ L_{g_1}\lambda_3 & L_{g_2}\lambda_3 \end{bmatrix}) \;=\; \det(\begin{bmatrix} 1 & X_2 \\ 0 & 1 \end{bmatrix}) \;=\; 1.$$

Thus, Ω^* is controlled invariant for (f, g) by the pair

$$\beta := \begin{bmatrix} 1 & -X_2 \\ 0 & 1 \end{bmatrix}, \quad \alpha := -\beta \cdot \begin{bmatrix} L_f\lambda_2 \\ L_f\lambda_3 \end{bmatrix} = \begin{bmatrix} X_1X_2X_4 - X_1 \\ -X_1X_4 \end{bmatrix}.$$

For illustration, consider the system caused by (α, β):

$$f + g\alpha = \begin{bmatrix} X_2 - X_1X_4 \\ 0 \\ X_1X_2X_3X_4 \\ 0 \end{bmatrix}, \quad g\beta_1 = \begin{bmatrix} X_4 \\ 1 \\ X_3 \\ 0 \end{bmatrix}, \quad g\beta_2 = \begin{bmatrix} 0 \\ 0 \\ -X_2X_3 \\ 1 \end{bmatrix}.$$

Both first components of the state x_1, x_2 on which the output depends are independent of the third state component x_3. Thus, by the form of Ω^* the output is invariant under disturbances.