# Minimal Classes and maximal Subgroups

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# Contents

## Notation

- $K$ imaginary quadratic number field
- $\mathcal{O}_K$ ring of integers of $K$
- $\mathcal{C}\ell_K$ ideal class group of $K$, $|\mathcal{C}\ell_K| =: h_K$
- $\{\mathfrak{a}_1, \dots, \mathfrak{a}_{h_K}\}$ set of representatives of the ideal classes, chosen to be integral and of minimal norm

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- $\{\mathfrak{a}_1, \dots, \mathfrak{a}_{h_K}\}$ set of representatives of the ideal classes, chosen to be integral and of minimal norm
- $\mathcal{H}_n$ set of Hermitian $n \times n$-matrices
- $\mathcal{H}_n^+$ cone of positive definite Hermitian matrices
- $\mathcal{A} \in \mathcal{H}_n^+,\ x \in K^n$: $\mathcal{A}[x] := x\mathcal{A}x^*$

### Definition

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## Theorem (Steinitz)

1. There are fractional ideals $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ of $K$ s.t. $L \cong \bigoplus_{i=1}^n \mathfrak{b}_i$.
2. $\mathrm{St}(L) := [\mathfrak{b}_1 \cdot \ldots \cdot \mathfrak{b}_n] \in \mathcal{C}\ell_K$, Steinitz class of $L$.
3. Two lattices $L$, $M$ are isomorphic iff $\mathrm{St}(L) = \mathrm{St}(M)$.

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## Remark

$\mathfrak{a} \oplus \mathfrak{b} \cong \mathcal{O}_K \oplus \mathfrak{ab} \Longrightarrow \mathcal{O}_K^{n-1} \oplus \mathfrak{a}$, $\mathfrak{a} \in \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{h_K}\}$ is a set of representatives of the isomorphism classes of $n$-dimensional $\mathcal{O}_K$-lattices.

$L = \mathcal{O}_K^{n-1} \oplus \mathfrak{a}$

## Lemma

$$\mathrm{End}_{\mathcal{O}_K}(L) \cong \left(\begin{array}{ccc|c} & & & \mathfrak{a} \\ & \mathcal{O}_K^{(n-1)\times(n-1)} & & \vdots \\ & & & \mathfrak{a} \\ \hline \mathfrak{a}^{-1} & \dots & \mathfrak{a}^{-1} & \mathcal{O}_K \end{array}\right)$$

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## Theorem

$$GL(L) := (\mathrm{End}_{\mathcal{O}_K}(L))^* = \{A \in \mathrm{End}_{\mathcal{O}_K}(L) \mid \det(A) \in \mathcal{O}_K^*\}.$$

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## Problem

For which lattices $L_1, L_2$ do we have $\mathrm{GL}(L_1) \cong \mathrm{GL}(L_2)$?

## Partial Answer

$\mathrm{St}(L_1) \in \mathrm{Gal}(L/K)\ \mathrm{St}(L_2)\ \mathcal{C}\ell_K^n \Longrightarrow \mathsf{GL}(L_1) \cong \mathsf{GL}(L_2)$

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## General strategy

- Determine a set of representatives of conjugacy classes of maximal finite subgroups.

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## General strategy

- Determine a set of representatives of conjugacy classes of maximal finite subgroups.
- Use Voronoi's algorithm and "minimal classes" - find maximal finite subgroups as stabilizers of a suitable group action

# Contents

$L := \mathcal{O}_K^{n-1} \oplus \mathfrak{a}$, $\mathcal{A} \in \mathcal{H}_n^+$

## Definition: Minimum and Perfection

- $(x_1, ..., x_n) \in L$: $\mathfrak{a}_x := x_1\mathcal{O}_K + ... + x_{n-1}\mathcal{O}_K + x_n\mathfrak{a}^{-1} \subseteq \mathcal{O}_K$, $N(\mathfrak{a}_x) = |\mathcal{O}_K/\mathfrak{a}_x|$

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**Theorem (G. Voronoi (1908), P. Humbert (1949))**

Up to the natural action of $\mathrm{GL}(L)$ and multiplication by $\mathbb{R}_{>0}$ there are only finitely many perfect $\mathcal{A} \in \mathcal{H}_n^+$.

Determining these finitely many perfect forms:

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### Definition: Voronoi-cone

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**Definition: Neighbor / contiguous form**

$\mathcal{A} \in \mathcal{H}_n^+$ perfect, $\mathfrak{S}$ facet of $\mathcal{V}_L(\mathcal{A})$, $R$ facet vector, i.e. $\mathrm{Trace}(RS) = 0 \; \forall \; S \in \mathfrak{S}$, $\mathrm{Trace}(RT) \geq 0 \; \forall \; T \in \mathcal{V}_L(\mathcal{A})$.

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$\exists! \rho > 0$, s.t. $\mathcal{A} + \rho R$ perfect. $\mathcal{A} + \rho R$ is called a neighbor of $\mathcal{A}$ or the contiguous form of $\mathcal{A}$ through $\mathfrak{S}$.

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There exists an implementation of this algorithm in Magma. Testing for isometry is based on *Computing Isometries of Lattices* by W. Plesken and B. Souvignier.

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## Definition: Minimal classes

$\mathcal{A}, \mathcal{B} \in \mathcal{H}_n^+$.
$\mathcal{A}, \mathcal{B}$ **minimally equivalent** if $S_L(\mathcal{A}) = S_L(\mathcal{B})$

$$\mathrm{Cl}_L(\mathcal{A}) := \{F \in \mathcal{H}_n^+ \mid S_L(F) = S_L(\mathcal{A})\},$$

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$C := \mathrm{Cl}_L(\mathcal{A})$, $S_L(C) := S_L(\mathcal{A})$
Call $C$ **well-rounded** if $S_L(C)$ contains a $K$-basis of $K^n$.

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## Remark

$\mathcal{A}$ perfect iff $\mathrm{Cl}_L(\mathcal{A}) = \{\alpha\mathcal{A} \mid \alpha \in \mathbb{R}_{>0}\}$

### Remark

$\mathrm{GL}(L)$ acts on $L \Rightarrow \mathrm{GL}(L)$ acts on the minimal classes.
Minimal classes **equivalent** iff they are in the same orbit.
$\mathrm{Aut}_L(C) := \{g \in \mathrm{GL}(L) \mid S_L(C)g^{-1} = S_L(C)\}$

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$C, C'$ equivalent iff $T_C^{-1}, T_{C'}^{-1}$ $L$-isometric
(i.e. $\exists\, g \in \mathrm{GL}(L) \;:\; T_C^{-1} = T_{C'}^{-1}[g]$)

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## Theorem

*$\mathcal{A} \in \mathcal{H}_n^+$ perfect. Any codimension $k$ face of $\mathcal{V}_L(\mathcal{A})$ naturally corresponds to a minimal class of perfection corank $k$, represented by*

$$F := \mathcal{A} + \frac{1}{2k}\sum_{i=1}^{k} \rho_i R_i = \frac{1}{k}\sum_{i=1}^{k}\left(\mathcal{A} + \frac{\rho_i}{2} R_i\right) \in \mathcal{H}_n^+$$

*with facet vectors $R_i$.*

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$G \leq \mathrm{GL}(L)$ finite.

$$\mathcal{F}(G) := \{F \in \mathcal{H}_n \mid F[g] = F \ \forall\, g \in \mathrm{GL}(L)\}.$$

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## Lemma ([2])

$C$ $G$-invariant minimal class.

$$C \cap \mathcal{F}(G) = \pi_G(C)$$

**Theorem ([2])**

$G \leq \mathrm{GL}(L)$ maximal finite $\Rightarrow G = \mathrm{Aut}_L(C)$, $C$ well-rounded minimal class, s.t. $\dim(\langle C \cap \mathcal{F}(G) \rangle) = 1$.

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**Sketch of proof**

If $C$ is a well-rounded $G$-minimal class (such a class always exists), then $G \leq \mathrm{Aut}_L(C)$ and $\mathrm{Aut}_L(C)$ is finite; we have equality if $G$ is maximal finite.

### Theorem ([2])

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### Remark

This theorem yields a finite set of finite subgroups of $\mathrm{GL}(L)$, containig a set of representatives of conjugacy classes of maximal finite subgroups of $\mathrm{GL}(L)$.

**Remark**

There are algorithmic tests to decide if a finite group is maximal finite and to check if two maximal finite groups are conjugate. They are described in [2].

# $\mathbb{Q}(\sqrt{-6})$

Well-rounded minimal classes for $\mathbb{Q}(\sqrt{-6})$

| $L = L_0$ | | | $L = L_1$ | | |
|---|---|---|---|---|---|
| $C$ | $G = \mathrm{Aut}_L(C)$ | max. | $C$ | $G = \mathrm{Aut}_L(C)$ | max. |
| $P_1$ | $\mathrm{SL}(2,3)$ | $\checkmark$ | $P_1$ | $Q_8$ | $\checkmark$ |
| $C_1$ | $D_{12}$ | $\checkmark$ | $P_2$ | $C_3 \rtimes C_4$ | $\checkmark$ |
| $C_2$ | $D_{12}$ | $\checkmark$ | $C_1$ | $D_8$ | $\checkmark$ |
| $C_3$ | $C_4$ | $\times$ | $C_2$ | $C_4$ | $\times$ |
| $C_4$ | $D_8$ | $\checkmark$ | $C_3$ | $C_4$ | $\times$ |
| $D_1$ | $D_8$ | $\checkmark$ | $C_4$ | $D_{12}$ | $\checkmark$ |
| $D_2$ | $D_8$ | $\checkmark$ | $D_1$ | $C_2 \times C_2$ | $\checkmark$ |
| $D_3$ | $C_2 \times C_2$ | $\checkmark$ | $D_2$ | $C_2 \times C_2$ | $\checkmark$ |

$$\Longrightarrow \mathrm{GL}(L_0) \ncong \mathrm{GL}(L_1)$$

# Overview

Numer of conjugacy classes of maximal finite subgroups

| | $D_8$ | $D_{12}$ | $C_2 \times C_2$ | $\mathrm{SL}(2,3)$ | $Q_8$ | $C_3 \rtimes C_4$ |
|---|---|---|---|---|---|---|
| $K = \mathbb{Q}(\sqrt{-15})$ | | | | | | |
| $\mathrm{St}(L) = [\mathcal{O}_K]$ | 2 | 2 | 2 | - | - | - |
| $\mathrm{St}(L) = [\mathfrak{p}_2]$ | 2 | 1 | 1 | - | - | 1 |
| $K = \mathbb{Q}(\sqrt{-5})$ | | | | | | |
| $\mathrm{St}(L) = [\mathcal{O}_K]$ | 3 | 2 | 1 | - | 1 | - |
| $\mathrm{St}(L) = [\mathfrak{p}_2]$ | 1 | 2 | 1 | 1 | - | - |
| $K = \mathbb{Q}(\sqrt{-6})$ | | | | | | |
| $\mathrm{St}(L) = [\mathcal{O}_K]$ | 3 | 2 | 1 | 1 | - | - |
| $\mathrm{St}(L) = [\mathfrak{p}_2]$ | 1 | 1 | 2 | - | 1 | 1 |

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