Homology Groups of Unit Groups of Orders

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Overview

1. Some Definitions

2. Resolutions for Unit Groups of Orders
   - Quadratic Forms
   - Perturbations
   - The Well-Rounded Retract

3. Computational Example
   - $\mathbb{Q}(\sqrt{-5})$
1 Some Definitions

2 Resolutions for Unit Groups of Orders
   - Quadratic Forms
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3 Computational Example
   - $\mathbb{Q}(\sqrt{-5})$
Definition: Chain Complexes

A positive chain complex $C = \{ C_n, \partial_n \mid n \geq 0 \}$ over the ring $R$ is a family of $R$-modules $C_n, n \geq 0$, together with a family of morphisms $\partial_n : C_n \to C_{n-1}, n \geq 1$, with the property $\partial_n \partial_{n+1} = 0$ for all $n$.

$$\cdots \partial_{n+1} \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \partial_2 \to C_1 \xrightarrow{\partial_1} C_0 \to 0$$
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Definition

- $H_n(C) := \ker(\partial_n) / \img(\partial_{n+1})$ is called the $n$th homology module of $C$. 
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Definition

- $H_n(C) := \ker(\partial_n)/\text{img}(\partial_{n+1})$ is called the $n$th homology module of $C$.
- $C$ is called acyclic, if $H_n(C) = 0$ for all $n \geq 1$.
- $C$ is called projective (free), if $C_n$ is projective (free) for all $n \in \mathbb{Z}$. 
Definition: Resolution

$A$ an $R$-module. An acyclic und projective (free) chain complex of the form

$$P : \ldots \rightarrow P_n \rightarrow \ldots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

with $H_0(P) \cong A$ is called a projective (free) resolution of $A$. 
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Remark

Every $R$-module has a free resolution.
**Definition: The Functor Tor**

$B$ an $R$-right module. $\text{Tor}_n^R(B, -) : R - \text{mod} \to \text{Ab}$ is the functor, whose value $\text{Tor}_n^R(B, A)$ may be computed in the following way:
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Group Homology

$G$ group, $R = \mathbb{Z}$ $G$-module. $\text{H}_n(G, B)$ is called the $n$th homology group of $G$ with coefficients in $B$. 

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**Situation:** $D$ a finite-dimensional $\mathbb{Q}$-division algebra, $K = Z(D)$, $R$ the $\mathbb{Z}$-maximal order in $K$, $A = D^{n \times n}$ a simple $\mathbb{Q}$-algebra, $\mathcal{O}$ a $R$-maximal order in $D$. $L$ an $\mathcal{O}$-lattice in $V = D^n$. 
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**Notation:**

1. $D_R := D \otimes_\mathbb{Q} R$ is a direct sum of matrix rings over $\mathbb{R}, \mathbb{C}, \mathbb{H}$. 

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3. $\Sigma := \{ F \in A_R \mid F^\dagger = F \}$ admits a positive definite inner product $\langle F, F' \rangle := \text{Tr}(FF')$ (reduced trace).
Idea: Find a cell complex, which admits a cellular $G$-action and use its cellular chain complex.
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**Definition: Shortest Vectors**

Any $F \in \Sigma$ defines a quadratic form on $V_{\mathbb{R}}$ via $F[x] := \langle F, xx^\dagger \rangle$. Let $P \subset \Sigma$ be the set of elements whose corresponding forms are positive definite.
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2. $\min_L(F) := \min_{0 \neq x \in L} F[x]$, $S_L(F) := \{x \in L \mid F[x] = \min_L(F)\}$. 
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**The Cell Structure**

$F \in P$. Define $\text{Cl}_L(F) := \{ F' \in P \mid S_L(F) = S_L(F') \}$ the *minimal class* corresponding to $F$. 
Properties of this decomposition

1. $G$ acts on $P$ via $gF := g^\dagger Fg$. 

2. Partial ordering on the minimal classes: $C \preceq C' \iff SL(C) \subseteq SL(C')$.

3. The decomposition as well as the partial ordering are compatible with the $G$-action.

4. We have $C = \bigcup_{C \preceq C'} C'$.

The cellular chain complex

The decomposition yields an acyclic chain complex $C$, where $C_n$ is the free Abelian group on the minimal classes in dimension $n$. $C_n$ becomes a $G$-module by means of the $G$-action.
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The cellular chain complex

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**Problem:** The modules $C_n$ are not necessarily free.
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**Perturbations (C. T. C. Wall 1960)**

Assume we are given a free $\mathbb{Z}G$-resolution $A_{p,*}$ (boundary $d_0$) of $C_p$ for all $p$. Then there are homomorphisms $d_k : A_{p,q} \to A_{p-k,q+k-1}$, such that

$$d := d_0 + d_1 + d_2 + \ldots : R_n := \bigoplus_{p+q=n} A_{p,q} \to R_{n-1} := \bigoplus_{p+q=n-1} A_{p+q}$$

is the boundary of an acyclic chain complex of free $\mathbb{Z}G$-modules, where $H_0(R) \cong \mathbb{Z}$. 
A Diagramm:

\[ \cdots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{} H_0(C) \]
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\begin{array}{ccc}
\cdots & A_{2,1} & A_{1,1} & A_{0,1} \\
\downarrow & d_0 & d_0 & d_0 \\
\cdots & A_{2,0} & A_{1,0} & A_{0,0} \\
\downarrow & \epsilon & \epsilon & \epsilon \\
\cdots & \partial & \partial & \partial & \partial & \partial \\
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\]
The situation in the space of positive definite forms:
Let \((C_k,i)_i\) be a system of representatives of the \(G\)-orbits of minimal classes and \(S_{k,i} := \text{Stab}_G(C_k,i)\).
The situation in the space of positive definite forms:
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Then:
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C_k \cong \bigoplus_i \mathbb{Z} G \otimes \mathbb{Z} S_{k,i} \mathbb{Z} \chi_{k,i}
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\((\chi_{k,i}(s) \in \{\pm 1\} \text{ describes how } s \text{ acts on the orientation of the cell.})\)
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Hence we need $\mathbb{Z}S_{k,i}$-resolutions of $\mathbb{Z}$, to get the algorithm running.
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Hence we need $\mathbb{Z} S_{k,i}$-resolutions of $\mathbb{Z}$, to get the algorithm running.

**Problem:** $S_{k,i}$ is an infinite group for some classes.
Solution: Consider a certain retract of $P$
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**Definition: well-rounded**

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- $F \in P$ is called *well-rounded*, if $S_L(F)$ contains a $D$-Basis of $D^n$.
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**Properties of the well-rounded retract**

In $P_{\geq 1}^{wr}$ we have:
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In $\mathbb{P}_{\geq 1}^{wr}$ we have:
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**Properties of the well-rounded retract**

In $P_{\geq 1}^{wr}$ we have:

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**Properties of the well-rounded retract**

In $\mathbb{P}^{wr}_{\geq 1}$ we have:

- There are only finitely many $G$-orbits in any dimension and every stabilizer is finite.
- The topological closure of each cell is a polytope.
- $\mathbb{P}^{wr}_{\geq 1}$ is a retract of $\mathbb{P}$, especially we have that the cellular chain complex is again acyclic and $H_0 \cong \mathbb{Z}$ (A. Ash 1984).
The Well-Rounded-Retract for $SL_2(\mathbb{Z})$

Quelle: http://www.uncg.edu/mat/numbertheory/quadratic_form.html
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3 Computational Example
   - $\mathbb{Q}(\sqrt{-5})$
$D := \mathbb{Q}(\sqrt{-5}), \ A := D^2 \times 2, \ R := \mathbb{Z} [\sqrt{-5}]$.

$\mathcal{O}_i := \text{End}_R(L_i)$ mit $L_1 := R^2, \ L_2 := R \oplus \wp$ mit $\wp^2 = (2)$. 

Especially:

$G_1 \nmid G_2$. 

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1. $G_1 := \text{GL}(L_1)$:

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H_n(G_1, \mathbb{Z}) = \begin{cases} 
C_2^5 & n = 1 \\
C_4^2 \times C_{12} \times \mathbb{Z} & n = 2 \\
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2. $G_2 := \text{GL}(L_2):$

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Especially: \( G_1 \not\cong G_2 \).