

# Self-Dual Codes over Chain Rings

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Projects:

$\mathbb{F}_p$ -linear self-dual codes with an automorphism of order  $p$   
(this talk)

Extremal  $p$ -modular lattices with an automorphism of order  $p$   
(current)

(Relative) projective group ring codes over chain rings

A code  $C$  of length  $n$  over some finite field  $\mathbb{F}$  is a subspace of  $\mathbb{F}^n$ .

Let  $\bar{\phantom{x}}$  be a field automorphism of order 1 or 2 and define

$$(v, w) := \sum_{i=1}^n v_i \bar{w}_i.$$

The dual code  $C^\perp$  of  $C$  is the orthogonal space w.r.t. this inner product.

If  $C = C^\perp$ , then  $C$  is called (hermitian) self-dual.

# Automorphism group of a code

Full Monomial group of a field:  $\mathbb{F}^* \wr S_n$ , acting on coordinates of  $\mathbb{F}^n$   
(permutation matrices with non-zero entries in  $\mathbb{F}^*$  instead of 1)

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Define

$$U := \{\alpha \in \mathbb{F} \mid \alpha \bar{\alpha} = 1\} \leq \mathbb{F}^* .$$

Then

$$|U| = \left\{ \begin{array}{ll} 1 & \text{if } \text{char}(\mathbb{F}) = 2 \text{ and } \bar{\phantom{x}} = \text{id} \\ 2 & \text{if } \text{char}(\mathbb{F}) \neq 2 \text{ and } \bar{\phantom{x}} = \text{id} \\ \sqrt{|\mathbb{F}|} + 1 & \text{if } \bar{\phantom{x}} \neq \text{id} \end{array} \right\} .$$

Let

$$\text{Mon}_n(\mathbb{F}) := U \wr S_n,$$

then the automorphism group of a code  $C$  is

$$\text{Aut}(C) := \{g \in \text{Mon}_n(\mathbb{F}) \mid C \cdot g = C\}.$$

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Every  $g \in \text{Mon}_n(\mathbb{F})$  has unique decomposition

$$g = \text{diag}(\alpha_1, \dots, \alpha_n)\pi(g), \quad \pi(g) \in S_n, \quad \alpha_i \in U$$

and the fixcode of  $g$  is

$$C(g) := \{c \in C \mid c \cdot g = c\}.$$

### Remark

Let  $g \in \text{Mon}_n(\mathbb{F})$  be an element of order  $r$  such that  $\gcd(r, |U|) = 1$ . Then  $g$  is conjugate in  $\text{Mon}_n(\mathbb{F})$  to some element of  $S_n$ .



## Theorem (MacWilliams, Mallos, Sloane, Ward, Rains)

Let  $C$  be a self-dual code in  $\mathbb{F}_q^n$  with minimum distance  $d$ .

If  $q = 2$  and  $C$  is even, then

$$d \leq \begin{cases} 4 \cdot \lfloor n/24 \rfloor + 4 & \text{if } n \not\equiv 22 \pmod{24} \\ 4 \cdot \lfloor n/24 \rfloor + 6 & \text{if } n \equiv 22 \pmod{24} \end{cases}.$$

If  $q = 2$  and  $C$  is doubly-even, then  $d \leq 4 \cdot \lfloor n/24 \rfloor + 4$ .

If  $q = 3$ , then  $d \leq 3 \cdot \lfloor n/12 \rfloor + 3$ .

If  $q = 4$  and  $C$  is Hermitian self-dual, then  $d \leq 2 \cdot \lfloor n/6 \rfloor + 2$ .

## Definition

Self-dual codes, which achieve those bounds are called extremal.

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### Theorem (Huffmann, 1992)

Let  $C \leq \mathbb{F}_3^{36}$  be a ternary, extremal ( $d = 12$ ) code of length 36 such that  $|\text{Aut}(C)|$  is divided by some prime  $p \geq 5$ . Then  $C$  is isomorphic to the Pless-Code  $P_{36}$ .

### Theorem (Nebe, 2011)

Let  $C \leq \mathbb{F}_3^{48}$  be a ternary, extremal ( $d = 15$ ) code of length 48 such that  $|\text{Aut}(C)|$  is divided by some prime  $p \geq 5$ . Then  $C$  is isomorphic to the Pless-Code  $P_{48}$  or to the extended quadratic residue code  $Q_{48}$ .

# Decomposition theory of Huffman (1988)

Basic idea of the proofs:

Let  $C$  be an  $\mathbb{F}$ -linear code with an automorphism  $g$  of prime order  $p \neq \text{char}(\mathbb{F})$ .

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Let  $C$  be an  $\mathbb{F}$ -linear code with an automorphism  $g$  of prime order  $p \neq \text{char}(\mathbb{F})$ .

$C$  is an  $\mathbb{F}[g]$ -module, this ring is isomorphic to  $\mathbb{F}[x]/(x^p - 1)$  and is the direct sum of field extensions of  $\mathbb{F}_q$

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What happens if  $p = \text{char}(\mathbb{F})$ ? (motivation: prove the Theorems for  $p = 3$ )

Let  $C \leq \mathbb{F}^n$  be a self-dual code with an automorphism  $g$  of order  $p = \text{char}(\mathbb{F})$ .



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$$\mathbb{F}[g] \supset \langle (1 - g) \rangle \supset \langle (1 - g)^2 \rangle \supset \cdots \supset \langle (1 - g)^q \rangle = \{0\}.$$

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This chain gives rise to a chain of subcodes:

$$C \supset C \cdot (1 - g) \supset C \cdot (1 - g)^2 \supset \cdots \supset C \cdot (1 - g)^q = \{0\}.$$

### Theorem (E.)

Let  $C \leq \mathbb{F}_3^n$  be a ternary, extremal code of length 36 resp. 48 and let  $g \in \text{Aut}(C)$  be of order 3. We can assume that

$$g = (1, 2, 3) \dots (3t - 2, 3t - 1, 3t)(3t + 1) \dots (n).$$

Then  $g$  has no fixpoints (i.e.  $3t = n$ ) and  $C$  is a free  $\mathbb{F}_3[g]$ -module, i.e. isomorphic to  $\mathbb{F}_3[g]^6$  resp. to  $\mathbb{F}_3[g]^8$ .

If  $g$  acts without fixpoints on the coordinates, then the map

$$\mathbb{F}^n \rightarrow \mathbb{F}[g]^t, (c_1, \dots, c_{pt}) \mapsto \left( \sum_{i=1}^p c_i g^{i-1}, \dots, \sum_{i=1}^p c_{(t-1)p+i} g^{i-1} \right)$$

is a bijection between the self-dual codes in  $\mathbb{F}^n$  and the self-dual codes in  $\mathbb{F}[g]^t$  with respect to an inner product defined later.

(Example:  $(0, 1, 2, 1, 1, 0) \mapsto (g + 2 \cdot g^2, 1 + g) \in \mathbb{F}_3[g]$ ).

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What is the structure of self-dual codes over chain rings?

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Let  $R$  be a commutative Artinian chain ring with 1 and let  $\bar{\phantom{x}} : R \rightarrow R$  be an involution (i.e. automorphism of order 1 or 2).

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Fix generator  $x$  of  $\mathfrak{m}$  such that

$$\bar{x} \equiv \epsilon x \pmod{Rx^2}, \quad \epsilon \in \{1, -1\}.$$

Let  $a \in \mathbb{N}_0$ , such that

$$R \supset Rx \supset Rx^2 \supset \dots \supset Rx^{a+1} = \{0\}$$

is the complete chain of ideals in  $R$ .



# Example

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We have

$$\begin{aligned} \overline{(1-g)} &= -(1-g) + (1-g) \cdot (1-g^{-1}) \\ &= -(1-g) - (1-g)^2 \\ &\equiv -(1-g) \pmod{\langle (1-g)^2 \rangle} \end{aligned}$$

→ choose  $x := (1-g)$  as generator with  $\epsilon = -1$ .

All indecomposable  $R$ -modules:

$$S_b := Rx^b \text{ for some } 0 \leq b \leq a,$$

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$V := R^t = \{(v_1, \dots, v_t) \mid v_i \in R\}$  denotes the free  $R$ -module of rank  $t$ .

An  $R$ -submodule  $C$  of  $V$  is called code of length  $t$ .

Theorem of Krull, Remak, Schmidt:

$$C = S_0^{t_0} \oplus S_1^{t_1} \oplus \dots \oplus S_a^{t_a}.$$

Define (Hermitian) inner product

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow R, \langle v, w \rangle := \sum_{j=1}^t v_j \overline{w_j}.$$

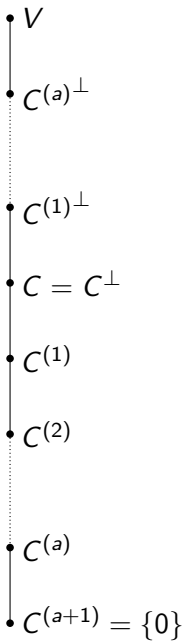
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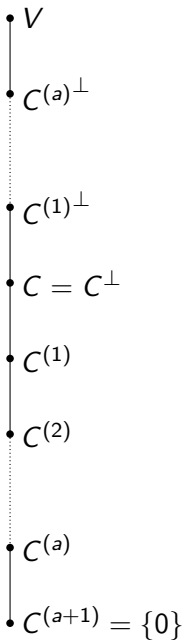
$$\langle \cdot, \cdot \rangle : V \times V \rightarrow R, \langle v, w \rangle := \sum_{j=1}^t v_j \overline{w_j}.$$

Now let  $C = C^\perp$  be a self-dual code which is a free  $R$ -module, i.e.  $C \cong R^{t/2}$ . Then the subcodes

$$C^{(i)} := Cx^i \cong S_i^{t/2}$$

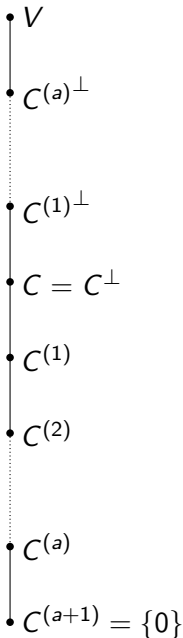
form the following chain:





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Assume we know  $C^{(i+1)}$ , how can we construct all possible  $C^{(i)}$ ?

Multiplication by  $x^a$  defines an isomorphism between the residue field  $\mathbb{F}$  and the socle of  $R$ :

$$\varphi : \mathbb{F} = R/Rx \xrightarrow{\sim} Rx^a = S_a, r + Rx \mapsto rx^a.$$

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This defines an  $\mathbb{F}$ -linear isomorphism:

$$\pi : \text{soc}(V) = Vx^a \rightarrow \mathbb{F}^t, (v_1, \dots, v_t) \mapsto (\varphi^{-1}(v_1), \dots, \varphi^{-1}(v_t))$$

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What does that mean?

The socle  $C^{(a)}$  is generated by some matrix  $Mx^a$ , where  $M \in \mathbb{F}^{\frac{t}{2} \times t}$  generates a self-dual code in  $\mathbb{F}^t$  (only true if  $C$  is a free module, in general it's the dual of a self-orthogonal code)

→ those are classified (for moderate length  $t$ )

# Example

Let  $C$  be an extremal, ternary code of length 36 (resp. 48) with an automorphism  $g$  of order 3 (which is then fixpoint-free).

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The socle  $C \cdot (1 - g)^2$  is the fixcode  $C(g)$  and is generated by some matrix

$$M \otimes \begin{pmatrix} 1 & 1 & 1 \end{pmatrix},$$

where  $M$  generates a self-dual code in  $\mathbb{F}_3^{12}$  (resp.  $\mathbb{F}_3^{16}$ ).

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We must have  $d(\langle M \rangle) \geq \frac{12}{3} = 4$  (resp.  $\geq \frac{15}{3} = 5$ )

$\rightarrow \langle M \rangle$  is an extremal, ternary code, hence unique.



# The lifting process

Let  $0 \leq i \leq a$  and fix some  $C^{(i+1)}$ .

We want to find all lifts, i.e. all codes  $D$  which are self-orthogonal and  $D\mathfrak{x} = C^{(i+1)}$ .

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Define

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with the inner product

$$(\cdot, \cdot)_i : W_i \times W_i \rightarrow \mathbb{F}, (Ax^i + C^{(i+1)}, Bx^i + C^{(i+1)})_i := \varphi^{-1}(\langle A, B \rangle x^i).$$

which is well-defined, non-degenerate and Hermitian in the Hermitian case and  $\epsilon^{(i+a)}$ -symmetric otherwise.

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How does one construct those complements?

Choose some isotropic complement  $Y_i$  of  $X_i$  in  $W_i$ .

There are bases  $B_1$  and  $B_2$  of  $X_i$  resp.  $Y_i$  such that the Gram matrix of  $(\cdot, \cdot)_i$  w.r.t  $(B_1, B_2)$  is

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→ all self-dual complements of  $X_i$  are given by  $\langle B_2 + B_1 \cdot A \rangle$ , where

$$A \in \mathbb{F}^{t/2 \times t/2} \text{ such that } \overline{A}^{\text{tr}} + \epsilon^{a+i} A = 0.$$

(the set of all thoses matrices form a vector space over  $\mathbb{F}$ )

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Vectorspace of all  $A \in \mathbb{F}_3^{6 \times 6}$  such that

$$\begin{aligned} \overline{A}^{\text{tr}} + \epsilon^{a+i} A &= 0 \\ \Leftrightarrow A^{\text{tr}} - A &= 0 \end{aligned}$$

has dimension 21 over  $\mathbb{F}_3$ , the centralizer of  $g$  in  $\text{Aut}(C^{(2)})$  has 16 orbits on this set, so we have 16 possibilities for  $C^{(1)}$ .

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For these 16 codes we constructed all  $3^{15}$  codes  $C^{(0)}$  and have shown:

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### Remark

The search spaces for  $n = 48$  has size  $3^{36}$  resp.  $3^{28} - 1$  can wait a few days for a result, but not 10000 years.

If  $C$  is a  $[72, 36, 16]$ -code with an automorphism  $g$  of order 2, then  $g$  has no fixpoints and  $C$  is a free  $\mathbb{F}_2[g]$ -module  $\rightarrow$  there are 41 possibilities for  $C^{(1)}$ , the search space for  $C^{(0)}$  has dimension 171 over  $\mathbb{F}_2$ .