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ON EXTREMAL MAXIMAL SELF-ORTHOGONAL CODES OF TYPE I-IV

Annika Meyer

Lehrstuhl D für Mathematik
Templergraben 64, 52062 Aachen, Germany
RWTH Aachen University

Abstract. For a Type $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ and length N where there exists no self-dual Type T code of length N, upper bounds on the minimum weight of the dual code of a self-orthogonal Type T code of length N are given, allowing the notion of *dual extremal* codes. It is proven that the Hamming weight enumerator of a dual extremal maximal self-orthogonal code of a given length is unique.

1. **Introduction.** Let $\mathbb{F}$ be a finite field. A linear code is a subspace $C \leq \mathbb{F}^N$. The *dual code* of C is

$$C^{\perp} = \{v \in \mathbb{F}^N \mid \sum_{i=1}^{N} v_i c_i^J = 0 \text{ for all } c \in C\},$$

where J is the identity or a field automorphism of order 2. If $C \subseteq C^{\perp}$ then C is called *self-orthogonal* and if $C = C^{\perp}$ then C is called *self-dual*.

A famous result by Gleason and Pierce states that if a certain divisibility condition on the *Hamming weights* $\mathrm{wt}(c) := |\{i \in \{1, \ldots, N\} \mid c_i \neq 0\}|$ is imposed on the codewords $c \in C$, there are basically four Types of codes:

Theorem 1.1. *[Gleason-Pierce Theorem](cf. [12]) Let $C = C^{\perp} \leq \mathbb{F}_q^N$ such that $\mathrm{wt}(c) \in m\mathbb{Z}$ for all $c \in C$ and some $m > 1$. Then one of the following holds.*

- (I) $q = 2$ *and* $m = 2$ *(self-dual binary codes),*
- (II) $q = 2$ *and* $m = 4$ *(doubly-even self-dual binary codes),*
- (III) $q = 3$ *and* $m = 3$ *(self-dual ternary codes),*
- (IV) $q = 4$*,* $m = 2$ *and* $J \neq \mathrm{id}$ *(Hermitian self-dual quaternary codes),*
- *(o)* $q = 4$*,* $m = 2$ *(certain Euclidean self-dual codes),*
- *(d)* q *is arbitrary,* $m = 2$ *and* C *is permutation equivalent to an orthogonal sum* $\perp^{N/2} (1, a)$ *of self-dual codes of length* 2 *where either* q *is even and* $a = 1$ *or* $q \equiv 1 \pmod 4$ *and* $a^2 = -1$ *or* J *has order 2 and and* $aa^J = -1$*.*

The first four of the above are named Type $\mathrm{I}, \mathrm{II}, \mathrm{III}$ and IV, respectively. For $T \in \{\mathrm{I}, \ldots, \mathrm{IV}\}$, the respective integer m from Theorem 1.1 will be denoted by m_T throughout this paper. The *Hamming weight enumerator*

$$\mathrm{we}(C)(x, y) := \sum_{c \in C} x^{N - \mathrm{wt}(c)} y^{\mathrm{wt}(c)} \in \mathbb{C}[x, y],$$

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a homogeneous polynomial of degree N, counts the number of codewords of each weight. Gleason showed that for a Type $T \in \{\mathrm{I}, \dots, \mathrm{IV}\}$, the weight enumerators of self-dual Type T codes lie in a polynomial ring $\mathbb{C}[f_T, g_T]$, where f_T and g_T themselves are linear combinations of products of weight enumerators of self-dual Type T codes ([4], see Theorem 4.1 in this paper).

This very powerful result provides an overview of the possible weight distributions of such codes, and in particular allows to derive the upper bounds on the *minimum weight*, $d(C) := \min_{0 \neq c \in C} \mathrm{wt}(c)$, cited in Theorem 3.1. The closer the minimum weight comes to this bound, the better the error-correcting capability of the code. A self-dual code is called *extremal* if its minimum weight reaches the respective upper bound.

Moreover, it follows immediately from Gleason's Theorem that the length of a self-dual Type T code is always a multiple of $o_T := \min(\deg(f_T), \deg(g_T)) = \gcd(\deg(f_T), \deg(g_T))$.

In this paper, we consider the case where N is no multiple of o_T. By the above, there exists no self-dual Type T code of length N, but one may still consider *maximal self-orthogonal* codes - recall that a code C is called maximal self-orthogonal if C is self-orthogonal and there exists no self-orthogonal code D which properly contains C. The main theorem below gives upper bounds on the *dual minimum weight* $d(C^\perp)$ of a maximal self-orthogonal Type T code C (and thus on the dual minimum weight of any self-orthogonal Type T code), which gives rise to the notion of *dual extremal* maximal self-orthogonal codes.

Theorem 1.2. *Let $T \in \{\mathrm{I}, \dots, \mathrm{IV}\}$ and let C be a maximal self-orthogonal Type T code. Then $d(C^\perp) \leq d_{\max}(T, N)$, where $d_{\max}(T, N)$ is given in Table 1 below.*

The bounds are, basically, developed in two ways, depending on the parameters of C. If the code length N, writing $n \cdot o_T \leq N \leq (n+1) \cdot o_T$ with some integer n, is closer to $(n+1) \cdot o_T$, then one may *extend* $C^\perp$ to a self-dual Type T code of length $(n+1) \cdot o_T$ (cf. Section 2), and then use the well-known bounds on the minimum weight of self-dual Type T codes and *design theory* to upper bound $d(C^\perp)$ (cf. Section 3.2). If N is closer to $n \cdot o_T$ then it is more appropriate to use the structure of the complex vector space $I_T^{(k)}$ spanned by the dual Hamming weight enumerators of maximal self-orthogonal codes of length equivalent to $k \pmod{o_T}$, in Section 4. The case $T = \mathrm{II}$ and $N \equiv 4 \pmod 8$ is exceptional here, since the extension and shortening procedure introduced in this paper fail to construct a self-dual code from a maximal self-orthogonal Type II code of length $N \equiv 4 \pmod 8$. However, one obtains upper bounds on $d(C^\perp)$ using the *shadow* of a self-dual Type I code of length N and a result by Bachoc and Gaborit in [1] (cf. Section 3.4).

The structure of $I_T^{(k)}$ is investigated in Section 4. Clearly $I_T^{(k)}$ is a module for $\mathbb{C}[f_T, g_T]$, since the orthogonal sum of a self-dual and a maximal self-orthogonal code is again a maximal self-orthogonal code. As a $\mathbb{C}[f_T, g_T]$-module, $I_T^{(k)}$ is finitely generated and free ([6, Ch. 10]).

Based on the latter observation, one obtains results on the weight distribution similar to those in the case of self-dual Type T codes. In particular, it is shown in Section 4 that the Hamming weight enumerator of a dual extremal maximal self-orthogonal Type T code is uniquely determined.

2. **Constructing self-dual codes from maximal self-orthogonal codes.**

Let C be a maximal self-orthogonal code of Type $T \in \{\mathrm{I}, \dots, \mathrm{IV}\}$ and length $N = k +$

$n \cdot o_T$, with $1 \le k \le o_T - 1$. In what follows, two methods are presented to construct a self-dual code from C. The first method, an *extension* of $C^\perp$, applies whenever $k \ge \frac{o_T}{2}$. The thus extended self-dual code $\mathrm{ext}(C)$ will have length $(n+1) \cdot o_T$. With binary codes (Type I), this is nothing but the well-known procedure of adding an overall parity check (cf. [8, Ch. 1]). The second method, a *shortening* of C, applies when $t \le \frac{o_T}{2}$, and results in a self-dual code of length $n \cdot o_T$. This method is a generalization of the puncturing process for Type I codes (see, again, [8, Ch. 1]).

An important preparation is the following basic result on the dimension of maximal self-orthogonal codes. By the theory of Witt groups (see [11, Ch.1,2]), the isomorphism type of the quadratic module $C^\perp/C$ is independent from the choice of C (see e.g. [6, Ch.]). In particular we have the following.

Lemma 2.1. *Let $D_T(N)$ be the dimension of a maximal self-orthogonal Type T code C of length $N = k + n \cdot o_T$, with $1 \le k \le o_T - 1$, and let $D'_T(N) = N - D_T(N)$ be the dimension of $C^\perp$. Then $D_T(N)$ (and hence $D'_T(N)$) is well-defined, i.e. independent from the choice of C.*

For the extension and shortening procedures, the values of $D_T(N)$ and $D'_T(N)$ are particularly important.

Lemma 2.2. *Let $N = k + n \cdot o_T$, where n and k are integers and $1 \le k \le o_T - 1$.*

(i) If $k \ge \frac{o_T}{2}$ then $D'_T(N) - D_T(N) = o_T - k$ and $D'_T(N) = \frac{(n+1)\cdot o_T}{2}$, except in the case when $T = \mathrm{II}$ and $k = 4$.
(ii) If $k \le \frac{o_T}{2}$ then $D'_T(N) - D_T(N) = k$ and $D_T(N) = \frac{n \cdot o_T}{2}$, except in the case when $T = \mathrm{II}$ and $k = 4$.
(iii) If $k = 4$ then $D_{\mathrm{II}}(N) = \frac{n \cdot o_T}{2} + 1$.

Proof. For $n = 0$, the claim of the lemma is easily verified. Now if C is a maximal self-orthogonal Type T code of length N and C' is a self-dual code of length o_T, then $C \oplus C'$ is a maximal self-orthogonal Type T code of length $N + o_T$ and dimension $D_T(N) + \frac{o_T}{2}$. Hence

$$D'_T((n+1) \cdot o_T) = D'_T(n \cdot o_T) + \frac{o_T}{2}.$$

The rest is induction on N, using the relation $D_T(N) = N - D'_T(N)$. $\square$

2.1. **Extension.** This is a special way of *gluing* codes together (cf. [8, Ch.3, Sect. 11.11.1]). Assume that $k \ge \frac{o_T}{2}$, but not $T = \mathrm{II}$ and $k = 4$. We construct a linear map $f : C^\perp \to \mathbb{F}^{o_T - k}$ with kernel C such that $(f(c'), f(c'')) = -(c', c'')$ for all $c', c'' \in C^\perp$. Then $\mathrm{ext}(C) := \{(c', f(c')) \mid c' \in C^\perp\}$ is a self-orthogonal Type T code of length $(n+1) \cdot o_T$, which is even self-dual since according to Lemma 2.2,

$$\dim(\mathrm{ext}(C)) = \dim(C^\perp) = \frac{(n+1) \cdot o_T}{2}.$$

The map f will be explicitly given in Table 2.1, since the weight distribution of the code $\mathrm{ext}(C)$ is the primary interest here. However, we shall mention that the existence of f has the following theoretical background (see [11, Ch.1]): Let $\beta^{(N)}$ denote the (Euclidian or Hermitian) scalar product on $\mathbb{F}^N$ which defines orthogonality in the context of the respective Type T. If $C \le \mathbb{F}^N$ is a self-orthogonal Type T code then one obtains another well-defined scalar product

$$\beta^{(N)}/C : \; C^\perp/C \times C^\perp/C \to \mathbb{F}, \;\; (c' + C, c'' + C) \mapsto \beta^{(N)}(c', c'').$$

If C is maximal self-orthogonal then the space $(C^\perp/C, -\beta^{(N)}/C)$ is isometric to $(\mathbb{F}^{o_T-k}, \beta^{(o_T-k)})$ (that the two spaces have the same dimension is already in Lemma 2.1, the rest is, again, a result of the theory of Witt groups). Hence the orthogonal sum

$$(C^\perp/C, \beta^{(N)}/C) \oplus (\mathbb{F}^{o_T-k}, \beta^{(o_T-k)})$$

contains a self-dual code $\widetilde{C}$. Now for $c' \in C^\perp$, $f(c')$ is the unique element of $\mathbb{F}^{o_T-k}$ such that $(c'+C, f(c')) \in \widetilde{C}$, i.e. $\mathrm{ext}(C) = \{(c', f(c')) \mid (c'+C, f(c')) \in \widetilde{C}\}$.

Table 2.1 shows explicitly how to extend a maximal self-orthogonal Type T code. There $\mathcal{B}$ denotes an ordered set of vectors in $\mathbb{F}^N$ such that $(b + C \mid b \in \mathcal{B})$ is a basis for $C^\perp/C$, and $\mathcal{G}$ is the Gram matrix of $\beta^{(N)}/C$ with respect to $\mathcal{B}$. If $T = \mathrm{II}$ then, additionally, the table gives the values of the quadratic form

$$Q: \ C^\perp/C \to \mathbb{Q}/\mathbb{Z}, \ c' + C \mapsto \frac{1}{4}\,\mathrm{wt}(c') + \mathbb{Z}.$$

Note that, writing $N = k + n \cdot o_T$ as above, up to isometry $\mathcal{G}$ and Q do not depend on n - for $\mathcal{G}$, this has already been mentioned, and there is an analogous result for quadratic forms (cf. [11, Ch.2]).

One observes that a word in $C^\perp$ is extended to a word with the least possible Hamming weight which is a multiple of m_T. Technically speaking, the extension procedure has the following effect on the Hamming weight enumerator.

Remark 1. Let C be a maximal self-orthogonal Type T code of length $N = n \cdot o_T + k$, where $\frac{o_T}{2} \leq k \leq n-1$, and let $\mathrm{ext}(C)$ be the self-dual Type T code of length $(n+1)\cdot o_T$ obtained by extension of C as described above. If $\mathrm{we}(C^\perp) = x^N + \sum_{i=d(C^\perp)}^{N} a_i x^{N-i} y^i$ then

$$\mathrm{we}(\mathrm{ext}(C)) = x^{N+o_T-k} + \sum_{i=d(C^\perp)}^{N+o_T-k} b_i x^{N+o_T-k-i} y^i,$$

where

$$b_i = \begin{cases} a_i + a_{i-1} + \cdots + a_{i-m(T)+1}, & i \equiv 0 \pmod{m_T} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, there exists an integer t, depending on the length of C, such that $d(C^\perp) \leq t \cdot m_T \leq d(\mathrm{ext}(C))$. This will be used in Section 3 to derive upper bounds on $d(C^\perp)$.

2.2. **Shortening.** Assume that $k \leq \frac{o_T}{2}$, but not $T = \mathrm{II}$ and $k = 4$. Moreover, assume that $D_T(N) \geq k$. This only excludes the case $N = k = 1$, for $T \in \{\mathrm{I}, \dots, \mathrm{IV}\}$, or $T \in \{\mathrm{II}, \mathrm{III}\}$ and $k = N = 2$. In these cases, the zero code of length 0 is appropriate as the shortened code.

Otherwise, since $D_T(N) \geq k$, after some suitable permutation of the coordinates the map

$$\pi: \ C \to \mathbb{F}^k, \ (c_1, \dots, c_N) \mapsto (c_{N-k+1}, \dots, c_N)$$

which maps a codeword to its last k components is surjective. There are possibly lots of suitable coordinate permutations, which may result in different shortened codes (cf. Example 1). However, in the context of this paper it suffices to consider just any of these, keeping in mind that the obtained shortened code depends on this choice.

Since $D'_T(N) - D_T(N) = k$ due to Lemma 2.2, there exists a subset $\mathcal{B} = \{v_1, \ldots, v_k\} \subset \mathbb{F}^N$ with $\langle C, v_1, \ldots, v_k \rangle = C^\perp$. The v_i may be chosen to satisfy

$$(v_i, v_j) = (\pi(v_i), \pi(v_j))$$

for all $i, j \in \{1, \ldots, k\}$, possibly after adding suitable elements of C, due to the surjectivity of π. Now we define a map $f : C^\perp \to \mathbb{F}^{o_T - k}$ (given explicitly in Table 2.2) as in the case where $k \geq \frac{o_T}{2}$, to obtain a self-orthogonal code $E := \{(c', f(c'))\}$ of length $(n+1) \cdot o_T$ and dimension $D'_T(N)$. In general, the code E is not self-dual. However, the code D formed by the last o_T coordinates of the vectors $(v_i, f(v_i)),\ i \in \{1, \ldots, k\}$ is self-orthogonal. Define

$$C_{(k)} := \{(c_1, \ldots, c_{n \cdot o_T}) \mid (c_1, \ldots, c_{(n+1)\cdot o_T}) \in E,\ (c_{n \cdot o_T + 1}, \ldots, c_{(n+1)\cdot o_T}) \in D\}.$$

This procedure is called *subtraction* of D from E (cf. [8]). The code $C_{(k)}$ is clearly self-orthogonal. Its length is $n \cdot o_T$ and, by Lemma 2.2, its dimension is

$$\dim(\ker(\pi)) + \dim(C^\perp / C) = \dim(C) - k + k = \dim(C) = \frac{n \cdot o_T}{2}.$$

Hence $C_{(k)}$ is a self-dual Type T code. From Table 2.2 one easily tells that there exists an integer t, depending on the length of C, such that $d(C^\perp) - k \leq t \cdot m_T \leq d(C_{(k)})$. It is not possible, though, to foretell the effect of the shortening procedure on the weight distribution of C. Example 1 presents two ways of shortening a code, such that the shortened codes have different weight enumerators.

Example 1. Let C be the maximal self-orthogonal ternary $[13, 6, 3]$ code such that $C^\perp$ has generator matrix

$$B := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 1 \end{pmatrix}$$

The first six rows of B form a generator matrix of C. One verifies that C is a direct sum of the self-dual tetracode t_4 of length 4 and the maximal self-orthogonal $[9, 4, 3]$ code e_3^{3+} (see [6] for more details). A generator matrix of a shortening $C_{(1)}$ of C is obtained by deleting the penultimate row and last column of B. Using Magma, one verifies that

$$\mathrm{we}(C_{(1)}) = x^{12} + 8x^9y^3 + 240x^6y^6 + 464x^3y^9 + 16y^{12}.$$

Let C' be the code obtained from C by a cyclic left coordinate shift. A generator matrix for a shortening $C'_{(1)}$ of C' is obtained from B by deleting the first row and column. One observes that $C'_{(1)}$ has a direct summand t_4, while $C_{(1)}$ has not; hence their weight enumerators must be distinct. In fact,

$$\mathrm{we}(C'_{(1)}) = x^{12} + 24x^9y^3 + 192x^6y^6 + 512x^3y^9.$$

In conclusion, the weight distribution of the extended code only depends on the weight distribution of the original code, while shortening does not even respect permutation equivalence. In Section 5, it will be shown that no way of shortening can be found such that

(1) shortening is well-defined on the weight enumerator level and, in this sense,

(2) the "shortening" of weight enumerators extends to a homomorphism $U_T^{(k)} \to \mathbb{C}[f_T, g_T]$ of $\mathbb{C}[f_T, g_T]$-modules (cf. Section 1), such that
(3) a polynomial $x^N + a_d y^d x^{N-d} + \dots a_N y^N$ is "shortened" to a polynomial of the form $x^{N-s} + b_{d-s} y^{d-s} + \cdots + b_{N-s} y^{N-s}$, where s is the number of positions shortened, i.e. shortening allows to derive upper bounds on the minimum distance even on the polynomial level.

Table 1. Value of $d_{\max}(T, N)$

T	m_T	o_T	$\deg(g_T)$	$N \pmod{\deg(g_T)}$	$d_{\max}(T, N)$
I	2	2	8	1, 3 or 5	$1 + 2\lfloor \frac{N}{8} \rfloor$
				7	$3 + 2\lfloor \frac{N}{8} \rfloor$
II	4	8	24	1, 9 or 17	$1 + \lfloor \frac{N}{24} \rfloor + 3\lfloor \frac{N+7}{24} \rfloor$
				2	$\lfloor \frac{N+8}{6} \rfloor$
				3,11 or 19	$1 + 2\lfloor \frac{N}{24} \rfloor + \lfloor \frac{N+5}{24} \rfloor + \lfloor \frac{N+13}{24} \rfloor$
				4	$\frac{N+8}{6}$
				5	$1 + 4\lfloor \frac{N}{24} \rfloor$
				6	$2 + 4\lfloor \frac{N}{24} \rfloor$
				7, 13, 14 or 15	$3 + 4\lfloor \frac{N}{24} \rfloor$
				10 or 18	$1 + \lfloor \frac{N}{8} \rfloor + \lfloor \frac{N+8}{24} \rfloor$
				12	$\frac{N}{6}$
				20	$\frac{N+4}{6}$
				21	$5 + 4\lfloor \frac{N}{24} \rfloor$
				22	$6 + 4\lfloor \frac{N}{24} \rfloor$
				23	$7 + 4\lfloor \frac{N}{24} \rfloor$
III	3	4	12	1, 5 or 9	$3 + 3\lfloor \frac{N}{12} \rfloor$
				2	$1 + 3\lfloor \frac{N}{12} \rfloor$
				3, 6 or 7	$2 + 3\lfloor \frac{N}{12} \rfloor$
				10	$4 + 3\lfloor \frac{N}{12} \rfloor$
				11	$5 + 3\lfloor \frac{N}{12} \rfloor$
IV	2	2	6	1 or 3	$1 + 2\lfloor \frac{N}{6} \rfloor$
				5	$3 + 2\lfloor \frac{N}{6} \rfloor$

Table 2. Extension

T	o_T	k	$\mathcal{B}$	$f(\mathcal{B})$
I	2	1	(v), $\mathcal{G}(v) = (1)$	$f(v) = (1)$
II	8	5	(u,v,w), $\mathcal{G}(u,v,w) = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$, $Q(u,v,w) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$	$f(u) = (1\ \ 1\ \ 0)$, $f(v) = (0\ \ 1\ \ 1)$, $f(w) = (1\ \ 1\ \ 1)$
II	8	6	(u,v), $\mathcal{G}(u,v) = \begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$, $Q(u,v) = (\frac{3}{4}, \frac{3}{4})$	$f(u) = (1\ \ 0)$, $f(v) = (0\ \ 1)$
II	8	7	(v), $\mathcal{G}(v) = (1)$, $Q(v) = (\frac{3}{4})$	$f(v) = (1)$
III	4	2	(u,v), $\mathcal{G}(u,v) = \begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$	$f(u) = (1\ \ 1)$, $f(v) = (1\ \ 2)$
III	4	3	(v), $\mathcal{G}(v) = (2)$	$f(v) = (1)$
IV	2	1	(v), $\mathcal{G}(v) = (1)$	$f(v) = (1)$

TABLE 3. Shortening

T	o_T	k	$\mathcal{B}$	$f(\mathcal{B})$	D
I	2	1	(v), $\mathcal{G}(v) = (1)$	$f(v) = (1)$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$
II	8	1	(v), $\mathcal{G}(v) = (1)$, $Q(v) = \frac{1}{4}$	$f(v) = (1\ 1\ 1\ 0\ 0\ 0\ 0)$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$
II	8	2	(v,w), $\mathcal{G}(v,w) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $Q(v,w) = (\frac{1}{4}, \frac{1}{4})$	$f(v) = (1\ 1\ 1\ 0\ 0\ 0)$, $f(w) = (0\ 0\ 0\ 1\ 1\ 1)$	$\begin{pmatrix} v_{N-1} & v_N & 1 & 1 & 1 & 0 & 0 & 0 \\ w_{N-1} & w_N & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$
II	8	3	(u,v,w), $\mathcal{G}(u,v,w) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $Q(u,v,w) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$f(u) = (1\ 1\ 1\ 0\ 0)$, $f(v) = (0\ 1\ 1\ 1\ 0)$, $f(w) = (0\ 1\ 1\ 0\ 1)$	$\begin{pmatrix} u_{N_2} & u_{N-1} & u_N & 1 & 1 & 1 & 0 & 0 \\ v_{N-2} & v_{N-1} & v_N & 0 & 1 & 1 & 1 & 0 \\ w_{N-2} & w_{N-1} & w_N & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$
III	4	1	(v), $\mathcal{G}(v) = (1)$	$f(v) = (1\ 1\ 0)$	$\begin{pmatrix} v_N & 1 & 1 & 0 \end{pmatrix}$
III	4	2	(v,w), $\mathcal{G}(v,w) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$f(v) = (1\ 1)$, $f(w) = (1\ 2)$	$\begin{pmatrix} v_{N-1} & v_N & 1 & 1 \\ w_{N-1} & w_N & 1 & 2 \end{pmatrix}$
IV	2	1	(v), $\mathcal{G}(v) = (1)$	$f(v) = (1)$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$

3. **Bounds on the dual distance of self-orthogonal codes of Type** $\mathrm{I}-\mathrm{IV}$**.** In the previous section, two procedures have been described to obtain a self-dual Type T code D from a maximal self-orthogonal Type T code C. Now the connection between $d(C)$ and $d(D)$ is studied, to obtain an upper bound for $d(C^{\perp})$ from upper bounds that are known for $d(D)$.

For self-orthogonal codes of length $N = k + n \cdot o_T$, where n and k are integers with $k \in \{\frac{o_T}{2}, \ldots, o_T - 1\}$ (cf. Section 3.2), the extension procedure is applied, and the obtained bounds (cf. Theorem 3.5) are sharp for small N. An important tool in developing these bounds is a result by Assmus and Mattson (cf. Theorem 3.4), which says that the words of minimum weight in an extremal self-dual code (cf. Definition 3.3) hold a t-design, where t depends on the Type and length of the code.

When $k \in \{1, \ldots, \frac{o_T}{2} - 1\}$ (cf. Section 3.3), the shortening procedure applies, but the thus obtained bounds on $d(C^{\perp})$ are not satisfactory, not even for small N. In Section 5, a different approach is pursued, using the algebraic structure of the space spanned by the Hamming weight enumerators of maximal self-orthogonal codes, to obtain the sharp upper bounds from Theorem 1.2.

In the exceptional case $T = \mathrm{II}$ and $N \equiv 4 \pmod 8$, sharp upper bounds on $d(C^{\perp})$ are derived in Section 3.4.

3.1. **Some known results on extremal self-dual codes.** Recall that the Hamming weight enumerators of self-dual Type T codes, for $T \in \{\mathrm{I}, \ldots, \mathrm{IV}\}$, lie in a polynomial ring $\mathbb{C}[f_T, g_T]$, where f_T, g_T are themselves are linear combinations of products of weight enumerators of self-dual Type T codes (cf. Section 1). This has been used by several authors to derive upper bounds on the minimum distance of these codes, as follows.

Theorem 3.1. *(cf. [8, Ch.3, Th.28]) Let C be a self-dual Type T code of length N, where $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$. Then $d(C) \leq m_T + m_T \lfloor \frac{N}{\deg(g_T)} \rfloor$.*

For Type I codes, this bound can be improved using the concept of the shadow of a code (cf. Section 3.4). The following is due to Rains ([10]).

Theorem 3.2. *Let C be a self-dual Type I code of length N. Then $d(C) \leq 4+4\lfloor \frac{N}{24} \rfloor$, except if $N \equiv 22 \pmod{24}$, in which case $d(C) \leq 6 + 4\lfloor \frac{N}{24} \rfloor$.*

These very powerful results allow a notion of extremality for self-dual codes of Type I-IV.

Definition 3.3. For $T \in \{\mathrm{II}, \mathrm{III}, \mathrm{IV}\}$, a self-dual Type T code is called *extremal* if its minimum weight equals the bound given in Theorem 3.1. A self-dual Type I code is called *extremal* if its minimum weight equals the bound given in Theorem 3.2.

In either case, we denote the minimum weight of an extremal self-dual code of Type T and length N by $d_{\max}(T, N)$. The set of all words of minimum weight in an extremal code of Type $\mathrm{II} - \mathrm{IV}$ has a particularly nice structure.

Theorem 3.4. *[Assmus-Mattson, see [5, Th. 9.3.10]] Let $T \in \{\mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ and let C be an extremal self-dual Type T code of length $N > 0$. Then the words of minimum weight in C hold a $t(T, N)$-design according to the following table.*

T	II			III			IV		
N *(mod* $\deg(g_T)$*)*	0	8	16	0	4	8	0	2	4
$t(T,N)$	5	3	1	5	3	1	5	3	1

In an extremal self-dual Type I code whose length is no multiple of 24, the words of minimum weight in general do not hold a design. There are even extremal self-dual Type I codes where the supports of the words of minimum weight are contained in a proper subset of $\{1,\ldots,N\}$. Hence we define $t(\mathrm{I},N)=0$ for $N\not\equiv 0 \pmod{24}$. If N is a multiple of 24 then an extremal Type I code is also Type II, and of course extremal in the sense of Type II. In this case the words of minimum weight form a 5-design. Correspondingly, we define $t(\mathrm{I},N)=5$ if $N\equiv 0 \pmod{24}$.

3.2. **Bounds in the extension case.**

Theorem 3.5. *Let $T\in\{\mathrm{I},\ldots,\mathrm{IV}\}$ and let C be a self-orthogonal Type T code of length $N\equiv k$ (mod o_T), where $k\geq \frac{o_T}{2}$. Let $d_{\max}(T,N+o_T-k)$, $t(T,N+o_T-k)$ be as in the previous section. Then*

$$d(C^\perp)\leq d_{\max}(T,N):=d_{\max}(T,N+o_T-k)-\min(t(T,N+o_T-k),o_T-k).$$

Proof. Assume that $d(C^\perp)>d_{\max}(T,N)$, and let $\delta:=\min(t(T,N+o_T-k),o_T-k)$. Since always $\delta\leq m_T$, the self-dual code $\mathrm{ext}(C)$ is extremal, and the words of minimum weight in $\mathrm{ext}(C)$ have less than δ nonzero entries in their last o_T-k coordinates, since due to our assumption $d(C)>d(\mathrm{ext}(C))-\delta$. But for an arbitrary δ-subset M of the last o_T-k coordinates, there exists a word of minimum weight in $\mathrm{ext}(C)$ whose support contains M, since $\delta\leq t(T,N+o_T-k)$ and the words of minimum weight in $\mathrm{ext}(C)$ form a $t(T,N+o_T-k)$-design. This is a contradiction, and hence always $d(C^\perp)\leq d_{\max}(T,N)$. □

Lemma 3.6. *With the notation from Theorem 3.5, assume that $o_T-k\leq t(T,N+o_T-k)$. Then* ext *establishes a correspondence between the set of all maximal self-orthogonal Type T codes of length N and dual minimum weight $d_{\max}(T,N)$ and the set of all extremal self-dual Type T codes of length $N+o_T-k$. In this case,* ext *inverts the* puncturing *process on these two sets, i.e. one obtains the dual of a maximal self-orthogonal code C, with $d(C^\perp)=d_{\max}(T,N)$, by deleting the last o_T-k coordinates in an extremal self-dual code of length $N+o_T-k$.*

Proof. If D is an extremal self-dual code of length $N+o_T-k$, then D contains a word of minimum weight whose support contains the last o_T-k coordinate positions. Hence deleting the last o_T-k coordinates in D yields the dual of a maximal self-orthogonal code C, with $d(C^\perp)=d(D)-\delta=d_{\max(T,N)}$. □

Example 2. (i) There is in general no correspondence between the dual extremal maximal self-orthogonal Type III codes of length $N\equiv 6 \pmod{12}$ and the extremal self-dual Type III codes of length $N+2$ (in this case, with the notation from Theorem 3.5, $o_T-k=2$ and $t(T,N+2)=1$). For instance, the extremal ternary $[20, 10, 6]$ code D with generator matrix $(I_{10} \mid H)$, where

$$H = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 0 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 0 & 2 & 0 & 2 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 & 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 2 \end{pmatrix},$$

(cf. [7, 9]) cannot be obtained by extending an extremal maximal self-orthogonal Type III code of length 18. This is due to the fact that D contains codewords of weight 6 whose penultimate and ultimate entries are both nonzero, so any word in a ternary code of length 18 that extends to these must have minimum weight $4 < 5 = d_{\max(\mathrm{III},18)}$. Although the words of weight 6 in C do not form a 2-design, every pair of coordinate positions is contained in the support of at least 4 words of weight 6 in D. Hence extension of an extremal self-orthogonal Type III code never leads to a permutation equivalent of D, i.e. here ext does not even establish a correspondence between the permutation equivalence classes of the respective codes.

(ii) Similarly, there is in general no correspondence between the extremal maximal self-orthogonal Type II codes of length $N \equiv 13$ or $14 \pmod{24}$ and the extremal self-dual Type II codes of length $N + 3$ or $N + 2$, respectively.

(iii) In all the other cases (i.e. except when $T = \mathrm{III}$ and $N \equiv 6 \pmod{12}$ or $T = \mathrm{II}$ and $N \equiv 13$ or $14 \pmod{24}$), ext establishes a bijection between the set of all dual extremal self-orthogonal codes of length N and the set of all extremal self-dual codes of length $N + o_T - k$, due to Theorem 3.5. The inverse map consists in *puncturing* (and then changing to the dual code), i.e. in an extremal self-dual code, the last $o_T - k$ coordinates can be omitted to obtain the dual of a dual extremal maximal self-orthogonal code. Puncturing 1,2 or 3 coordinates in the extremal extended Golay code $g_{24} = g_{\mathrm{II}}$, for instance, leads to an extremal $[23, 12, 7]$ or $[22, 12, 6]$ or $[21, 12, 5]$ code, respectively.

3.3. **Bounds in the shortening case.** As announced above, in this case one obtains only very weak bounds, using the fact that the minimum distance of the shortened code is always less than or equal to the minimum distance of the original code. The reason is that, contrarily to the extension case, shortening means a lot of information loss. The following preliminary result is improved in Section 4.

Remark 2. Let C be a self-orthogonal Type T code of length N, where $T \in \{\mathrm{I}, \dots, \mathrm{IV}\}$ and assume that $N \equiv k \pmod{o_T}$, where $k \in \{1, \dots, \frac{o_T}{2} - 1\}$. Then $d(C^\perp) \leq d_{\max}(T, N - k) + k$.

3.4. **Bounds from shadows.** In this section, an upper bound on the dual distance of a maximal self-orthogonal Type II code of length $N \equiv 4 \pmod 8$ is given, using the concept of the *shadow* (cf. [2]). For a self-dual Type I code D of length N which is not Type II, the shadow $S(D)$ is the set of all vectors $v \in \mathbb{F}_2^N$ such that

$$2\sum_{i=1}^{N} v_i d_i \equiv \mathrm{wt}(d) \pmod 4$$

for all $d \in D$. Equivalently, if $C := \{d \in D \mid \mathrm{wt}(d) \equiv 0 \pmod 4\}$ is the doubly-even subcode of D, then $S(D) = C^\perp - C$. Note that $\dim(C) = \dim(D) - 1$, i.e. C is a maximal self-orthogonal Type II code. The following has been shown in [1] (see also [3]).

Theorem 3.7. *Let C be a self-dual Type* I *code which is not Type* II*. Let $d(S(C)) := \min\{\mathrm{wt}(s) \mid s \in S(C)\}$. Then $2d(C) + d(S(C)) \leq 4 + \frac{N}{2}$, unless $N \equiv 22 \pmod{24}$, when $2d(C) + d(S(C)) \leq 6 + \frac{N}{2}$.*

Assume that N is even, but no multiple of 8, i.e. there exists a self-dual Type I code, but no self-dual Type II code of length N. Let C be a maximal self-orthogonal Type II code of length N, and let D be a self-dual Type I code which contains C, i.e. C is the doubly-even subcode of D, and $S(D) = C^\perp - C$. Then, due to Theorem 3.7,

$$d(C^\perp) = \min\{d(S(D)), d(C)\} \leq \frac{1}{3}(2S(D) + d(C)) \leq \frac{N+8}{6}.$$

Based on this observation, one easily concludes the proof of Theorem 1.2 in the case $T = \mathrm{II}$ and $N \equiv 4 \pmod 8$, and in the case $T = \mathrm{II}$ and $N \equiv 2 \pmod{24}$.

4. **The weight distribution of a dual extremal code of Type** I-IV **is unique.**

Using the structure of the ring $\mathbb{C}[f_T, g_T]$, it is not hard to see that the weight enumerator of an extremal self-dual code of Type $T \in \{\mathrm{I}, \ldots, \mathrm{IV}\}$ is uniquely determined (cf. [8, Ch. 3]). For lengths where there exists no self-dual Type T code, this section uses the structure of the complex vector space $I_T^{(k)}$ spanned by the Hamming weight enumerator of maximal self-orthogonal Type T codes of length congruent to $k \pmod{o_T}$ as a $\mathbb{C}[f_T, g_T]$-module (free and finitely generated, cf. Section 4.1) to prove an analogous result for maximal self-orthogonal Type T codes. It follows from the fact that $I_T^{(k)}$ has the *Weierstrass property* that for every integer $N \equiv k \pmod{o_T}$ there exists a unique homogeneous polynomial $p \in I_T^{(k)}$ of the form

$$x^N + a_{\delta((I_T^{(k)})_N)} y^{\delta((I_T^{(k)})_N)} x^{N-\delta((I_T^{(k)})_N)} + \cdots + a_N y^N,$$

where δ depends basically on the dimension of $(I_T^{(k)})_N$. On the one hand, in the case $k \geq \frac{o_T}{2}$, where Theorem 1.2 has been proven in the previous section, it turns out that the dual weight enumerator of a dual extremal maximal self-orthogonal Type T code of lenght N is the unique element of $(I_T^{(k)})_N$ of the above form.

On the other hand, the least non-vanishing term in p provides an upper bound for the minimum weight of a maximal self-orthogonal Type T code C of length N, and the weight enumerator of a code meeting that bound is of course unique. This bound is easy to calculate for small values of N. For $k < \frac{o_T}{2}$ and lengths N exceeding a certain range, one obtains upper bounds on $d(C^\perp)$ exploiting the fact that as $d(C^\perp)$ grows sufficiently large, one may shorten C to an extremal self-dual code of almost the same length. But extremal self-dual codes do not exist for sufficiently large N. The exact spectrum, i.e. the meaning of "sufficiently large", is in [8, Ch. 9.3]. In conclusion, one thus obtains upper bounds on $d(C^\perp)$ also when $k < \frac{o_T}{2}$, which completes the proof of Theorem 1.2.

4.1. **Gleason's Theorem and maximal self-orthogonal codes.**

Theorem 4.1. *[Gleason's Theorem] If C is a self-dual Type T code, where $T \in \{\mathrm{I}, \dots \mathrm{IV}\}$, then $\mathrm{we}(C) \in \mathbb{C}[f_T, g_T]$, where f_T and g_T are themselves linear combinations of products of weight enumerators of self-dual Type T codes, according to Table 4.1.*

Table 4. Gleason Polynomials

T	f_T	g_T
I	$x^2 + y^2$	$x^2y^2(x^2 - y^2)^2$
II	$x^8 + 14x^4y^4 + y^8$	$x^4y^4(x^4 - y^4)^4$
III	$4x^4 + 8xy^3$	$y^3(x^3 - y^3)^3$
IV	$x^2 + 3y^2$	$y^2(x^2 - y^2)^2$

The direct sum of a self-dual Type T code and a maximal self-orthogonal Type T code is again a maximal self-orthogonal Type T code, and the weight enumerator of a direct sum is the product of the weight enumerators of the summands. Hence, if $I_T^{(k)}$ is as above, then

Remark 3. $I_T^{(k)}$ is a module for $\mathbb{C}[f_T, g_T]$.

It has been shown in [6, Ch. 10] that for $T \in \{\mathrm{I}, \dots, \mathrm{IV}\}$, the module $I_T^{(k)}$ is free and generated by finitely many weight enumerators of maximal self-orthogonal Type T codes (see Table 4.3 in this paper).

4.2. **The Weierstrass property.** For a subspace $W \subseteq \mathbb{C}[x, y]$ and an integer N, let W_N be the subspace of W formed by the homogeneous elements of degree N.

Definition 4.2. Let $W \subseteq \mathbb{C}[x, y]$ be a subspace and let

$$J := \{j \in \mathbb{N} \mid \operatorname{coef}(p, y^j x^i) = 0 \text{ for all } p \in W \text{ and all } i \in \mathbb{N}\}.$$

The space W is said to have the *Weierstrass property* if, for every $N \in \mathbb{N}$, every element of W_N is uniquely determined by its first $\dim(W_N)$ coefficients which do not belong to J, i.e. by the coefficients in $x^N, x^{N-1}y, \dots, x^{N-(\delta(W_N)-1)}y^{\delta(W_N)-1}$, where

$$\delta(W_N) = \min\{n \in \mathbb{N} \mid |\{0, \dots, n-1\} - J| = \dim(W_N)\}.$$

It is well-known that the spaces $\mathbb{C}[f_T, g_T]$, for $T \in \{\mathrm{I}, \dots, \mathrm{IV}\}$, have the Weierstrass property (cf. [8, Ch. 3]). This allows to define extremality of self-dual Type T codes. In what follows this concept is described for a general space W, which is assumed to have the Weierstrass property: For every positive integer N there is an injective linear map

$$W_N \to \mathbb{C}^{\delta(W_N)}, \quad \sum_{i=0}^{N} a_i y^i x^{N-i} \mapsto (a_0, \dots, a_{\delta(W_N)}).$$

This gives rise to a notion of extremality, as follows.

Definition 4.3. Assume that the space $W \subseteq \mathbb{C}[x,y]$ has the Weierstrass property. Then for every positive integer N the space W_N contains a unique element of the form

$$x^N + a_{\delta(W_N)} y^{\delta(W_N)} x^{N-\delta(W_N)} + a_{\delta(W_N)+1} y^{\delta(W_N)+1} x^{N-(\delta(W_N)+1)} + \cdots + a_N y^N,$$

i.e. where the sequence formed by the first $\delta(W_N)$ coefficients is $(1,0,\ldots,0)$. This element is called the *extremal* element of W_N.

Recall that for Hamming weight enumerators of maximal self-orthogonal Type T codes, for $T \in \{\mathrm{I},\ldots,\mathrm{IV}\}$ and $k \geq \frac{o_T}{2}$, a notion of extremality has been introduced in Section 3.2, using the fact that a self-orthogonal Type T code of length N satisfies $d(C^\perp) \leq d_{\max}(T,N)$. In the subsequent section it is shown that the latter notion of extremality coincides with the one defined via the Weierstrass property.

4.3. **Proof of the uniqueness of the extremal weight enumerator.** In this section it is proven that the spaces $I_T^{(k)}$ spanned by the dual Hamming weight enumerators of maximal self-orthogonal Type T codes of length $N \equiv k \pmod{o_T}$ have the Weierstrass property, for $T \in \{\mathrm{I},\ldots,\mathrm{IV}\}$ and $k \in \{1,\ldots,o_T-1\}$. In particular, the space $(I_T^{(k)})_N$ contains a unique *extremal* polynomial of the form

$$x^N + a_{\delta((I_T^{(k)})_N)} y^{\delta((I_T^{(k)})_N)} x^{N-\delta((I_T^{(k)})_N)} + \cdots + a_N y^N,$$

i.e. where the sequence formed by the first $\delta((I_T^{(k)})_N)$ coefficients is $(1,0,\ldots,0)$. Note that the coefficient $a_{\delta((I_T^{(k)})_N)}$ may be zero. Now if $k \geq \frac{o_T}{2}$ then the weight enumerator of an extremal maximal self-orthogonal Type T code is the extremal element in $(I_T^{(k)})_N$, since always $d_{\max}(T,N) \leq \delta((I_T^{(k)})_N)$ (cf. Remark 4). In particular, this completes the proof of the uniqueness of the extremal weight enumerator for $k \geq \frac{o_T}{2}$.

It remains to show that the spaces $I_T^{(k)}$ have the Weierstrass property. To this aim a triangular basis of $(I_T^{(k)})_N$ is constructed below. The construction starts with an appropriate basis for $\mathbb{C}[f_T,g_T]$. It is equivalent with the fact that the space $\mathbb{C}[f_T,g_T]$ has the Weierstrass property (cf. Section 4.2) that

Corollary 1. *For every integer n which is a multiple of o_T, the complex vector space $(\mathbb{C}[f_T,g_T])_n$ has a basis $(p_0,\ldots,p_{s_n})$ which is of triangular shape, i.e. p_i is a multiple of $y^{m_T i}$, for $i \in \{0,\ldots,s\}$.*

Table 4.3 shows that for every $k \in \{1,\ldots,o_T-1\}$, the $\mathbb{C}[f_T,g_T]$-module $I_T^{(k)}$ has a basis $(q_1,\ldots,q_{t_{T,k}})$ which is triangular as well: If i is the largest integer such that q_j is a multiple of y^i, then q_{j+1} is a multiple of y^{i+1}, for $j \in \{1,\ldots,t_{T,k}-1\}$.

Moreover, one observes that in most cases there are some regular "gaps" in the weight distribution of the q_j, i.e. the set

$$J_T^{(k)} := \{i \in \mathbb{Z} \mid \operatorname{coef}(q_j(1,y), y^{i+m_T z}) = 0 \text{ for all } j \in \{1,\ldots,t_{T,k}\} \text{ and all } z \in \mathbb{Z}\}$$

is non-empty. Since all the weights of an element of $\mathbb{C}[f_T,g_T]$ are multiples of m_T, it even holds that $\operatorname{coef}(p(1,y),y^i) = 0$ for all $i \in J_T^{(k)}$ and all $p \in I_T^{(k)}$.

One observes from Table 4.3 that, metaphorically speaking, if one ignores the columns belonging to the coefficients of $q_j(1,y)$ at y^i, for $i \in J_T^{(k)}$, then the triangle formed by the basis vectors q_j is even isosceles. In particular,

$$t_{T,k} = |\{i \in \{0,\ldots,o_T-1 \mid i \notin J_T^{(k)}\}\}|.$$

Now one forms a triangular basis of $(I_T^{(k)})_N$, where $N \equiv k \pmod{o_T}$, as follows. For $j \in \{1, \ldots, t\}$, let $\eta_j := \deg(q_j)$. Then $N - \eta_j$ is a multiple of o_T. Choose a basis $\mathcal{B}_j = \{p_{0,j}, \ldots, p_{s,j}\}$ of $(\mathbb{C}[f_T, g_T])_{N-\eta_j}$ as in Corollary 1. Then

$$\mathcal{C} := \dot\bigcup_{j=1}^{t_{T,k}} \{q_j b \mid b \in \mathcal{B}_j\}$$

is a basis of $(I_T^{(k)})_N$ which, if $\mathcal{C} = \{c_1, \ldots, c_u\}$ is ordered appropriately, has the property that if i is the largest integer such that c_j is a multiple of y^i, then c_{j+1} is a multiple of y^{i+1}. As an immediate consequence,

Corollary 2. *The spaces $I_T^{(k)}$, for $T \in \{\mathrm{I}, \ldots, \mathrm{IV}\}$ and $k \in \{1, \ldots, o_T - 1\}$, have the Weierstrass property.*

As announced above, the last necessary result, which is easy to verify by induction on N, e.g., to prove the uniqueness of the extremal weight enumerator of a maximal self-orthogonal Type T code is

Remark 4. $d_{\max}(T, N) \leq \delta((I_T^{(k)})_N)$.

In Table 4.3, all polynomials are given evaluated at $x = 1$ to shorten notation. A small index indicates the total degree. If a polynomial p is symmetric, i.e. $p(x, y) = p(y, x)$, then its redundant coefficients are omitted, which is indicated by an index $_{sym}$. For instance, $[y + 3y^2 - 9y^3 + 5y^4 - 6y^5 + 6y^6]_{13,sym}$ denotes the polynomial

$$\begin{array}{l} x^{12}y + 3x^{11}y^2 - 9x^{10}y^3 + 5x^9y^4 - 6x^8y^5 + 6x^7y^6 \\ +6x^6y^7 - 6x^5y^8 + 5x^4y^9 - 9x^3y^{10} + 3x^2y^{11} + xy^{12}. \end{array}$$

In addition, Table 4.3 describes how these polynomials can be obtained from weight enumerators of maximal self-orthogonal codes. For the notation of these codes, the reader is referred to [6].

TABLE 5. Triangular bases for the spaces $I_T^{(k)}$

T	k	Basis for $I_T^{(k)}$	Coefficients	$J_T^{(k)}$
I	1	$\mathrm{we}(i_1)$ $\mathrm{we}(i_1)\,\mathrm{we}(i_2)^3 - \mathrm{we}(e_7)$	$[1+y]_1$ $[\quad y+3y^2-4y^3-4y^4+3y^5+y^6]_7$	$\{\,\}$
II	1	$\mathrm{we}(i_1)$ $\mathrm{we}(i_1)\,\mathrm{we}(e_8)^2 - \mathrm{we}(d_{10}e_7^+)$	$[1+y]_1$ $[\quad y+11y^4-23y^5-176y^8+198y^9-23y^{12}$ $+11y^{13}+y^{16}]_{17}$	$2+4\mathbb{Z}\cup 3+4\mathbb{Z}$
	2	$\mathrm{we}(d_2)$ $\frac{1}{2}(\mathrm{we}(d_2)\,\mathrm{we}(e_8) - \mathrm{we}(d_{10}))$ $-\frac{1}{4}(\mathrm{we}(d_{10})\,\mathrm{we}(e_8) - \mathrm{we}(d_{18}))$	$[1+2y+y^2]_2$ $[\quad y-2y^2+2y^4-2y^5+2y^6-2y^8+y^9]_{10}$ $[\quad y^2+3y^4-8y^5+y^6-5y^8+16y^9-5y^{10}+y^{12}-8y^{13}$ $+3y^{14}+y^{16}]_{18}$	$3+4\mathbb{Z}$
	3	$\mathrm{we}(d_3)$ $\frac{1}{2}(\mathrm{we}(d_3)\,\mathrm{we}(e_8) - \mathrm{we}(d_{11}))$ $-\frac{1}{4}(\mathrm{we}(d_{11})\,\mathrm{we}(e_8) - \mathrm{we}(g_{19}))$ $\frac{1}{7}(-\frac{1}{2}\,\mathrm{we}(d_3)\,\mathrm{we}(e_8)^3 + \frac{3}{2}\,\mathrm{we}(d_{11})\,\mathrm{we}(e_8)^2)$	$[1+3y+3y^2+y^3]_3$ $[\quad y-y^2-2y^3+2y^4+2y^7-2y^8-y^9+y^{10}]_{11}$ $[\quad y^2+y^3+3y^4-5y^5-7y^6+y^7-5y^8+11y^9]_{19,sym}$ $[\quad y^3+5y^4+6y^5+30y^6-7y^7-11y^8-48y^9-96y^{10}$ $+10y^{11}+2y^{12}+108y^{13}]_{27,sym}$	$\{\,\}$
	4	$\mathrm{we}(d_4)$ $\mathrm{we}(d_4)\,\mathrm{we}(e_8)^2 - \mathrm{we}(g_{20})$	$[1+6y^2+y^4]_4$ $[\quad y^2+4y^4-12y^6-4y^8+22y^{10}]_{20,sym}$	$1+2\mathbb{Z}$
	5	$\mathrm{we}(d_5)$ $\mathrm{we}(d_5)\,\mathrm{we}(e_8) - \mathrm{we}(e_7d_6)$ $\frac{1}{3}(\mathrm{we}(e_7d_6)\,\mathrm{we}(e_8) - \mathrm{we}(e_7^3))$ $\frac{1}{21}(\mathrm{we}(e_7)^3 - \mathrm{we}(g_{21}))$	$[1+y+6y^2+6y^3+y^4+y^5]_5$ $[\quad y+3y^2-9y^3+5y^4-6y^5+6y^6]_{13,sym}$ $[\quad y^2-2y^3+y^4+7y^5-9y^6-3y^7+5y^8-13y^9+13y^{10}]_{21,sym}$ $[\quad y^3+y^4-y^5-y^6-3y^7-3y^8+3y^9+3y^{10}]_{21,sym}$	$\{\,\}$
	6	$\mathrm{we}(d_6)$ $\frac{1}{2}(\mathrm{we}(d_6)\,\mathrm{we}(e_8) - \mathrm{we}(d_{14}))$ $-\frac{1}{12}\,\mathrm{we}(d_{14})\,\mathrm{we}(e_8) + \frac{1}{4}\,\mathrm{we}(d_6)\,\mathrm{we}(e_8)^2$ $-\frac{1}{6}\,\mathrm{we}(g_{22})$	$[1+3y^2+8y^3+3y^4+y^6]_6$ $[\quad y^2-2y^3+y^4-2y^6+4y^7]_{14,sym}$ $[\quad y^3+2y^4-2y^6-4y^7-6y^8+6y^{10}+6y^{11}]_{22,sym}$	$1+4\mathbb{Z}$
	7	$\mathrm{we}(e_7)$ $\frac{1}{7}(\mathrm{we}(e_7)\,\mathrm{we}(e_8)^2 - \mathrm{we}(g_{23}))$	$[1+7y^3+7y^4+y^7]_7$ $[\quad y^3+5y^4-8y^7-16y^8+18y^{11}]_{23,sym}$	$1+4\mathbb{Z}\cup 2+4\mathbb{Z}$
III	1	$\mathrm{we}(i_1)$ $\mathrm{we}(i_1)\,\mathrm{we}(t_4)^2 - \mathrm{we}(e_3^{3+})$	$[1+y]_1$ $[\quad y+10y^3-38y^4-2y^6-44y^7-8y^9]_9$	$2+3\mathbb{Z}$
	2	$\mathrm{we}(i_0^\perp)$ $\frac{1}{4}(\mathrm{we}(i_0^\perp)\,\mathrm{we}(t_4) - \mathrm{we}(e_3^2))$ $\frac{1}{12}(\mathrm{we}(e_3^2)\,\mathrm{we}(t_4) - \mathrm{we}(g_{10}^\perp))$	$[1+4y+4y^2]_2$ $[\quad y-2y^2+y^3-y^4+2y^5-y^6]_6$ $[\quad y^2+y^3-2y^4-2y^5-2y^6+4y^7+y^8+y^9-2y^{10}]_{10}$	$\{\,\}$
	3	$\mathrm{we}(e_3^\perp)$ $\frac{1}{6}(\mathrm{we}(e_3^\perp)\,\mathrm{we}(t_4)^2 - \mathrm{we}(g_{11}^\perp))$	$[1+6y^2+2y^3]_3$ $[\quad y^2+3y^3-6y^5+9y^8+3y^9-4y^{11}]_{11}$	$1+3\mathbb{Z}$
IV	1	$\mathrm{we}(i_1)$ $\frac{1}{3}(\mathrm{we}(i_1)\,\mathrm{we}(d_4) - \mathrm{we}(h_5^\perp))$	$[1+3y]_1$ $[\quad y+2y^2-4y^3-2y^4+3y^5]_5$	$\{\,\}$

5. **Extremal polynomials and extremal codes.** Let $T \in \{\mathrm{I}, \dots, \mathrm{IV}\}$ and let k, N be integers with $1 \le k \le o_T - 1$ and $N \equiv k \pmod{o_T}$. Recall from Section 4 that there exists a unique *extremal* element $p \in (U_T^{(k)})_N$, of the form

$$p(x, y) = x^N + a_{\delta((I_T^{(k)})_N)} y^{\delta((I_T^{(k)})_N)} x^{N-\delta((I_T^{(k)})_N)} + \dots + a_N y^N.$$

Of course p is not necessarily the weight enumerator of a code. However, it is interesting to observe that for small lengths N, say up to 4000, the position of the least non-vanishing coefficient in p, i.e.

$$d(p) := \min\{i \in \{\delta((I_T^{(k)})_N), \dots, N\} \mid a_i \neq 0\}$$

satisfies the upper bound on the dual minimum distance of a putative maximal self-orthogonal Type T code of lenth N. This could probably be proven for arbitrary N using the Bürmann-Lagrange formula. In this section, though, it is shown that the extension procedure introduced in Section 2 can, to some extent, be useful to this aim, too. Regrettably, the shortening process fails to provide any upper bounds on $d(p)$. It is shown in Theorem 5.2 that this is no weakness of this particular shortening procedure, but that there exists no shortening procedure at all which is useful to this aim.

5.1. **A $\mathbb{C}[f_T, g_T]$-module homomorphism induced by code extension.** Assume that $N = k + n \cdot o_T$, where $k \ge \frac{o_T}{2}$ (here we exclude the exceptional case $T = \mathrm{II}$ and $k = 4$). Let C be a maximal self-orthogonal Type T code of length N. It is easy to observe that if D is a self-dual Type T code, then $\mathrm{ext}(D \oplus C) = D \oplus \mathrm{ext}(C)$. This suggests to define a $\mathbb{C}[f_T, g_T]$-module homomorphism

$$\alpha : I_T^{(k)} \to \mathbb{C}[f_T, g_T], \quad \mathrm{we}(C_i) \mapsto \mathrm{we}(\mathrm{ext}(C_i)),$$

where the $\mathrm{we}(C_i)$ form a $\mathbb{C}[f_T, g_T]$-basis for $I_T^{(k)}$. Recall that the weight distribution of the extended code $\mathrm{ext}(C)$, of length $N + o_T - k$, can easily be read off from the weight distribution of C (cf. Remark 1). Hence

Remark 5. If C is a maximal self-orthogonal Type T code then $\alpha(\mathrm{we}(C)) = \mathrm{we}(\mathrm{ext}(C))$, i.e. the map α extends the effect of the extension procedure on the weight enumerator to a homomorphism of $\mathbb{C}[f_T, g_T]$-modules.

This allows to upper bound the *generalized minimum distance* of polynomials in $I_T^{(k)}$.

Definition 5.1. Let $p \in \mathbb{C}[x, y]$ be a homogeneous polynomial of degree N, of the form

$$p(x, y) = x^N + a_d y^d x^{N-d} + \dots + a_N y^N.$$

Then the integer $d =: d(p)$ is called the *generalized minimum distance* of p.

Corollary 3. *If $k \ge \frac{o_T}{2}$ and $p \in I_T^{(k)}$ is homogeneous of degree N, such that $d(p)$ is defined, then $d(p) \le d(\alpha(p)) \le d_{\max}(T, N + o_T - k)$.*

As mentioned above, explicit calculations (the author used Magma) show that under the assumptions of Corollary 3, even $d(p) \le d_{\max}(T, N)$ for small lengths N. The author conjectures that this holds for arbitrary N, but at the present state of our knowledge, this question has to remain open for future research.

5.2. $\mathbb{C}[f_T, g_T]$**-module homomorphisms induced by shortening - a nonexistence result.** In this section, assume that $T \in \{\mathrm{II}, \mathrm{III}\}$ and $N = k + n\dot{o}_T$, where $k \in \{1, \ldots, \frac{o_T}{2} - 1\}$. We prove the nonexistence of a shortening procedure that is well-defined on the weight enumerator level and gives rise to a $\mathbb{C}[f_T, g_T]$-module homomorphism which allows to upper bound $d(p)$, for $p \in I_T^{(k)}$.

Theorem 5.2. *There exists no $\mathbb{C}[f_T, g_T]$-module homomorphism $\alpha : I_T^{(k)} \to \mathbb{C}[f_T, g_T]$ such that*

(1) if C is a maximal self-orthogonal Type T code of length N, then $\alpha(C)$ is a self-dual Type T code of length $N - k$ and

(2) $d(\alpha(p)) \geq d(p) - k$ for all $p \in I_T^{(k)}$ where $d(p)$ is defined.

Proof. We give the proof in the case $T = \mathrm{III}$ and $k = 1$ (the other cases, i.e. $T = \mathrm{II}$ and $k \in \{1, 2, 3\}$, are similar). The $\mathbb{C}[f_T, g_T]$-module $I_{\mathrm{III}}^{(1)}$ has a basis $(\mathrm{we}(i_1), \mathrm{we}(e_3^{3+}))$, according to Table 4.3 (see alse [6, Ch. 10]). Assume that α is a $\mathbb{C}[f_{\mathrm{III}}, g_{\mathrm{III}}]$-module homomorphism that satisfies the first condition of the theorem. Then $\alpha(\mathrm{we}(i_1)) = 1$ and $\alpha(\mathrm{we}(e_3^{3+})) = \mathrm{we}(t_4)^2$. Let p be the extremal element of $(I_{\mathrm{III}}^{(1)})_{25}$, then $d(p) = 7 = d_{\max}(\mathrm{III}, 25)$. But $d(\alpha(p)) = 3 < 7 - 1 = 6$, which contradicts the second condition. This shows the assertion. □

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E-mail address: annika.meyer@math.rwth-aachen.de