Induced Representations of split Iwahori-Hecke algebras are always reducible

Introduction

Let (W, S) be a finite Coxeter system and J a proper subset of S. Then $W_J := \langle J \rangle$ is a parabolic subgroup of W which is again a Coxeter group. It is very natural to study the representations of W induced by those of W_J. Such modules have been studied for several years and one key question is whether or not they are irreducible.

In 1974 Dragomir Djoković and Jerry Malzan considered the symmetric groups, i.e. Coxeter groups of type A_n, cf. [DM74]. In order to determine the imprimtive irreducible complex characters of the symmetric groups they showed that, if W_J is a proper parabolic subgroup of a Coxeter group W of type A_n, and M is a $\mathbb{C}[W_J]$-module, then its induced module $M \otimes_{\mathbb{C}[W_J]} \mathbb{C}[W]$ is always reducible.
In 2015 Gerhard Hiss, William J. Husen and Kay Magaard extended the result to all finite Weyl groups, cf. [HHM15, Lemma 8.2], to study the representations of finite groups of Lie type.

The purpose of this article is to prove an analogous result for the associated Iwahori-Hecke algebras:
Let K be a field containing invertible elements $u_s \in K^*$ for every $s \in S$ such that $u_s = u_t$ whenever s and t are conjugate in W and let $H := H_K(W, S, (u_s \mid s \in S))$ be the Iwahori-Hecke algebra of W over K with parameters u_s. For a proper subset $J \subseteq S$ of S denote by H_J the corresponding parabolic subalgebra. Again we consider induced modules and for an H_J-module M we define $\mathrm{Ind}_J^S(M) := M \otimes_{H_J} H$. The main result of this article is the following:

Theorem *Let M be an H_J-module and assume that K is a splitting field of H. Then*

$\mathrm{Ind}_J^S(M)$ *is reducible, i.e. it has a proper non-zero submodule.*

Note that the result is completely independent of the characteristic of K. In particular, it holds even if $\mathrm{Char}(K)$ is a so-called bad prime for W. Furthermore, the result depends neither on the type of W nor is it restricted to the equal parameter case. Finally, the results of [DM74] and [HHM15] are contained in ours as group algebras of Coxeter groups are special cases of Iwahori-Hecke algebras, i.e. $F[W] \cong H_F(W, S, (u_s = 1 \mid s \in S))$ for every field F.

Preliminaries

In this section we introduce the necessary notation and definitions as well as some well-known results on induced modules of Iwahori-Hecke algebras.

Throughout the remainder of this article we fix a finite Coxeter system (W, S) and a field K. Suppose that $u_s \in K^*$ for every $s \in S$ such that $u_s = u_t$ for all $s,\ t \in S$ whenever s and t are conjugate elements of W and define the Iwahori-Hecke algebra $H := H_K(W, S, (u_s \mid s \in S))$. This is an associative K-algebra with a basis $\{T_w \mid w \in W\}$ whose elements satisfy the braid relations and $T_sT_s = u_sT_s + (u_s - 1)T_s$ for every $s \in S$, cf. [GP00].
For a subset $L \subseteq S$ of S let H_L be the corresponding parabolic subalgebra, that is $H_L \cong H_K(W_L, L, (u_s \mid s \in L)) \subseteq H$. For an H-module N we denote the H_L-module obtained by restricting the action of H to that of H_L by $\mathrm{Res}_L^S(N)$.
Furthermore, we define X_L, the set of distinguished right coset representatives, by $X_L := \{w \in W \mid \ell(sw) > \ell(w)\ \forall\ s \in L\}$, cf. [GP00, Proposition 2.1.1]. Then X_J is a complete set of right W_L-coset representatives in W and we have an isomorphism of left H_L-modules, $H \cong \bigoplus_{x \in X_L} H_L \otimes_{H_L} T_x$.
If $L' \subseteq S$ is a second subset of S we define the set of distinguished double coset representatives $X_{LL'} := X_L \cap X_{L'}^{-1}$. This is a complete set of $(W_L, W_{L'})$-double coset representatives, cf. [GP00, Proposition 2.1.7].

Having established the necessary language we are now able to state a version of Mackey's formula that deals exclusively with the case of parabolic subgroups of Coxeter groups, cf. [GP00, Proposition 9.1.8]. It is one of the two key ingredients in our subsequent proof of the theorem.

Mackey Formula *Let $L,\ L' \subseteq S$ be subsets of S and M an H_L-module. Let $X_{LL'}$*

be the set of distinguished double coset representatives for the parabbolic subgroups W_L and $W_{L'}$. Then, for each $d \in X_{LL'}$, the subalgebra $H_{L^d \cap L'}$ acts on $M \otimes_{H_L} T_d \subseteq \mathrm{Ind}_L^S(M)$ by right multiplication: For $w \in W_{L^d \cap L'}$ and $m \in M$ we have

$$(m \otimes T_d)T_w = mT_{dwd^{-1}} \otimes T_d.$$

Moreover, there is an isomorphism of $H_{L'}$-modules

$$\mathrm{Res}_{L'}^S \circ \mathrm{Ind}_L^S(M) \cong \bigoplus_{d \in X_{LL'}} \mathrm{Ind}_{L^d \cap L'}^{L'} (M \otimes_{H_L} T_d).$$

Remark *Note that each direct summand in the above formula is a module induced from a parabolic subalgebra. This is essentially due to a result by Solomon and Kilmoyer, cf. [GP00, Theorem 2.1.12], which states that $W_L^d \cap W_{L'}$ is again a parabolic subgroup of W generated by $L^d \cap L' \subseteq S$.*

The second main tool to prove the theorem are Frobenius reciprocity and the Nakayama relations which hold in the context of Iwahori-Hecke algebras, cf. [GP00, Proposition 9.1.7]:

Frobenius Reciprocity & Nakayama Relations *Let M be an H_J module and M' an H-module. Then we have two isomorphisms of K-vector spaces:*

a) $\mathrm{Hom}_{H_J}\left(M, \mathrm{Res}_J^S(M')\right) \cong \mathrm{Hom}_H\left(\mathrm{Ind}_J^S(M), M'\right)$

b) $\mathrm{Hom}_{H_J}\left(\mathrm{Res}_J^S(M'), M\right) \cong \mathrm{Hom}_H\left(M', \mathrm{Ind}_J^S(M)\right)$

Proof of the Theorem

We are now fully prepared to prove the theorem. The technique is similar to that employed by Djoković and Malzan to prove the first case of the theorem in [DM74]. By using the above version of Mackey's formula for Coxeter groups instead of the general version for arbitrary finite groups we are able to prove the more general result.
We proceed by stating and proving two Lemmas which in combination prove the theorem: Lemma 1 is the theorem under additional conditions and Lemma 2 states that these conditions are always met.

We start with Lemma 1. The statement as well as the proof is similar to [HHM15, Lemma 2.2] which itself is a worked out version of the application of [CR66, Theorem 45.2] in [DM74].

Lemma 1 *Assume that K is a splitting field of H. Let $J \subseteq S$ be a proper subset of S and M an H_J-module.*
If there exists an element $1 \neq r$ in X_{JJ} such that r commutes with every element of $J \cap J^r$, then $\mathrm{Ind}_J^S(M)$ is reducible.

Proof By Schur's Lemma we are done if we can show that the endomorphism algebra $\mathrm{End}_H\left(\mathrm{Ind}_J^S(M)\right)$ has K-dimension greater than 1, since K is a splitting field of H. By Frobenius Reciprocity we have

$$\begin{aligned} \mathrm{End}_H(\mathrm{Ind}_J^S(M)) &= \mathrm{Hom}_H(\mathrm{Ind}_J^S(M),\ \mathrm{Ind}_J^S(M)) && (1) \\ &\cong_K \mathrm{Hom}_{H_J}(M, \mathrm{Res}_J^S(\mathrm{Ind}_J^S(M))). && (2) \end{aligned}$$

By Mackey's formula we obtain a direct sum decomposition of the right-hand side:

$$\mathrm{Res}_J^S(\mathrm{Ind}_J^S(M)) \cong_{H_J} \bigoplus_{d \in X_{JJ}} \mathrm{Ind}_{J \cap J^d}^J(M \otimes T_d).$$

By hypothesis r and 1 are distinct elements of X_{JJ}. Therefore, there exists an H_J-module N such that

$$\mathrm{Res}_J^S(\mathrm{Ind}_J^S(M)) \cong_{H_J} \mathrm{Ind}_{J \cap J^1}^J(M \otimes T_1) \oplus \mathrm{Ind}_{J \cap J^r}^J(M \otimes T_r) \oplus N.$$

The H_J-homomorphisms from M to the first summand are well known because

$$\mathrm{Ind}_{J \cap J^1}^J(M \otimes T_1) \cong_{H_J} M.$$

Now consider the H_J-homomorphisms from M to the second direct summand, that is $\mathrm{Hom}_{H_J}(M, \mathrm{Ind}_{J \cap J^r}^S(M \otimes T_r))$. By the Nakayama relations

$$\mathrm{Hom}_{H_J}(M, \mathrm{Ind}_{J \cap J^r}^J(M \otimes T_r)) \cong_K \mathrm{Hom}_{H_{J \cap J^r}}(\mathrm{Res}_{J \cap J^r}^J(M),\ M \otimes T_r). \qquad (3)$$

We claim that $\varphi : \mathrm{Res}_{J \cap J^r}^J(M) \to M \otimes T_r : m \mapsto m \otimes T_r$ is an isomorphism of $H_{J \cap J^r}$-modules.
Clearly, this is an isomorphism of K-vector spaces so we only have to show the compatibility with multiplication with elements of $H_{J \cap J^r}$. To this end, let w be an element of $W_{J \cap J^r}$ and T_w the corresponding basis vector of $H_{J \cap J^r}$. Furthermore, let $m \in M$. Then we have

$$\begin{aligned} \varphi(mT_w) &= (mT_w) \otimes T_r \\ &= mT_{rwr^{-1}} \otimes T_r \\ &= (m \otimes T_r)T_w \\ &= \varphi(m)T_w. \end{aligned}$$

Here we use the fact that r commutes with every element of $J \cap J^r$ and therefore also with $W_{J\cap J^r}$, so that $T_w = T_{rwr^{-1}}$. Therefore, φ is an $H_{J\cap J^r}$-module isomorphism and from (3) it follows that $\mathrm{Hom}_{H_J}(M, \mathrm{Ind}_{J\cap J^r}(M \otimes T_r))$ has K-dimension at least 1. We apply this to (2) and get

$$\begin{aligned}
&\dim_K \left(\mathrm{End}_H(\mathrm{Ind}_J^S(M))\right) \\
=\ &\dim_K \left(\mathrm{Hom}_{H_J}(M, \mathrm{Res}_J^S(\mathrm{Ind}_J^S(M)))\right) \\
=\ &\dim_K \left(\mathrm{Hom}_{H_J}(M, M \oplus \mathrm{Ind}_{J\cap J^r}^J(M \otimes T_r) \oplus N)\right) \\
=\ &\dim_K \left(\mathrm{End}_{H_J}(M)\right) + \dim_K \left(\mathrm{Hom}_{H_J}(M, \mathrm{Ind}_{J\cap J^r}^J(M \otimes T_r))\right) \\
&+ \dim_K \left(\mathrm{Hom}_{H_J}(M, N)\right) \\
\geq\ &1 + 1 + 0 \\
>\ &1.
\end{aligned}$$

Therefore, $\mathrm{Ind}_J^S(M)$ is reducible, as K is a splitting field of H. ∎

Clearly, the theorem is proven once we show that there always exists an element satisfying the hypothesis of Lemma 1. This is done in Lemma 2:

Lemma 2 *Let J be a proper subset of S. If t is an element of $S\backslash J$, then t lies in X_{JJ}. Furthermore, t commutes with every element of $J \cap J^t \subseteq S$.*

Proof By definition $X_J = \{w \in W \mid \ell(sw) > \ell(w)\ \forall\ s \in J\}$. Since t is a generator of W that is not in J, it is an element of X_J and because it is an involution it is also in X_J^{-1}. This proves the first claim.
Now, suppose that $s \in J \cap J^t$ and let m_{st} be the order of st. If $m_{st} = 2$ then s and t commute, so we only have to consider the case that $m_{st} \geq 3$. In that case tst is a reduced expression of length 3, in particular $tst \notin J$ as all elements of S have length 1. But s is in $J \cap J^t$ if and only if tst is in $J \cap J^t$ because t is an involution and therefore $m_{st} \geq 3$ is a contradiction to the hypothesis $s \in J \cap J^t$. ∎

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