The $b$-function of a polynomial

Consider a non-constant polynomial $f \in \mathbb{K}[x]$. Consider the Malgrange ideal of $f$ defined by

$$I_r := \{ t - f, \partial_i \frac{\partial f}{\partial x_i} \mid i = 1, \ldots, n \} \subset D_{n+1} = D_n(t, \partial_i \mid \partial t = t \partial h + 1).$$

Chosing $w = (1, \ldots, 0) \in \mathbb{R}^{n+1}$ such that the weight of $\partial_1$ is 1,

$$b_i(s) := (-1)^{\deg_{\partial_1} b_i} b_i(w \langle -s - 1 \rangle)$$

is called the (global) $b$-function or the (global) Bernstein-Sato polynomial of $f$.

The annihilator based approach

Let $s$ be a new indeterminate and put $D_s[s] := D_s \otimes_{\mathbb{K}} \mathbb{K}[s]$. Consider the commutative ring $R_s := \mathbb{K}[x, s, f^{-1}]$. The free $R_s$-module $R_s \cdot f^i$ generated by the formal symbol $t^i$ becomes a left $D_s[s]$-module via

$$s \cdot g \cdot f^i := s \cdot g \cdot f^i \quad x_i \cdot g \cdot f^i := x_i \cdot g \cdot f^i \quad \partial_i \cdot g \cdot f^i := \partial_i g \cdot f^{i+1} + \partial g \cdot f^i.$$  

Theorem (Bernstein)

The Bernstein-Sato polynomial is the uniquely determined monic element of minimal degree in $\mathbb{K}[x]$ satisfying $P \cdot f^{i+1} = b_i \cdot f^i$ for some $P \in D_s[s]$. The theorem provides another option to compute the Bernstein-Sato polynomial. Again prinicipalIntersect is applicable.

Corollary

Denote $\text{Ann}_{D_s[s]}(f^i) := \{ p \in D_s[s] \mid p \cdot f^i = 0 \}$. Then

$$\langle b_i \rangle = (\text{Ann}_{D_s[s]}(f^i) + \langle f \rangle) \cap \mathbb{K}[x].$$

Applications

Annullators of powers of polynomials

Note that $-1$ is always a root of $b_f$.

Theorem

Let $\lambda_i \in \mathbb{C}$ and $\lambda_0$ be the minimal integral root of $b_f$. If $\lambda_i \in \mathbb{C} \setminus \{ \lambda_0 + k \mid k \in \mathbb{N} \}$, then $\text{Ann}_{D_{s'}}(f^{i+1}) = \text{Ann}_{D_{s'}}(f^i) + \langle \lambda_i \rangle$.

Otherwise, let $G := (g_1, \ldots, g_r)$ be a Gröbner basis of $\text{Ann}_{D_{s'}}(f^i)$ and $S := \text{Syz}(f^{i+1}, g_1, \ldots, g_r)$. Then

$$\text{Ann}_{D_{s'}}(f^{i+1}) = \text{Ann}_{D_{s'}}(f^i) + \langle \lambda_0 \rangle + \langle g_1 \rangle + \langle g_2 \rangle + \ldots + \langle g_r \rangle.$$