Constructive D-module theory

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Experimental and constructive algebra

Introduction

A D-module is a module over a ring of differential operators. If $\mathbb{K}$ is a field of characteristic zero, the Weyl algebra

$$D_n := \mathbb{K}\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n \mid \partial_i x_j = x_j\partial_i + \delta_{ij} \text{ for } 1 \le i \le n\rangle$$

becomes the (non-commutative) ring of linear partial differential operators with polynomial coefficients by the action given by

$$x_i \bullet f := x_i \cdot f \quad \text{and} \quad \partial_i \bullet f := \frac{\partial f}{\partial x_i} \qquad \text{for } f \in \mathbb{K}[x] := \mathbb{K}[x_1,\dots,x_n].$$

The b-function of an ideal

For $0 \neq w \in \mathbb{R}^n_{\geq 0}$ and $p \in D_n$ define the initial form $\mathrm{in}_{(-w,w)}(p)$ of p with respect to the weight $(-w,w)$ to be the polynomial consisting of all terms of p, which have maximal weighted total degree with respect to the weight $-w_i$ for x_i and w_i for ∂_i. Moreover, for a left ideal $I \subset D_n$ define the initial ideal of I to be $\mathrm{in}_{(-w,w)}(I) := \langle \mathrm{in}_{(-w,w)}(p) \mid p \in I\rangle$.

Computing the initial ideal

For $\zeta,\eta \in \mathbb{R}^n_{>0}$ consider the n-th weighted homogenized Weyl algebra

$$D^{(h)}_{n,(\zeta,\eta)} := \mathbb{K}\langle x_1,\dots,x_n,\partial_1,\dots,\partial_n,h \mid \partial_j x_i = x_i\partial_j + \delta_{ij}h^{\zeta_i+\eta_j} \text{ for } 1 \le i \le n\rangle.$$

Moreover, for $p = \sum_{\alpha,\beta} c_{\alpha\beta}x^\alpha\partial^\beta \in D_n$ define the weighted homogenization of p to be

$$H_{(\zeta,\eta)}(p) := \sum_{\alpha,\beta} c_{\alpha\beta}x^\alpha\partial^\beta h^{\deg_{(\zeta,\eta)}(p)-(\zeta\alpha+\eta\beta)} \in D^{(h)}_{n,(\zeta,\eta)}.$$

Further let $I \subset D_n$ be an ideal, $\prec$ a global ordering on D_n and $\prec^{(h)}_{(-w,w)}$ the global ordering on $D^{(h)}_{n,(\zeta,\eta)}$ defined by $x^\alpha\partial^\beta \prec^{(h)}_{(-w,w)} x^\gamma\partial^\delta$

$$\begin{aligned}
&\text{if} && \zeta\alpha+\eta\beta < \zeta\gamma+\eta\delta\\
&\text{or} && \zeta\alpha+\eta\beta = \zeta\gamma+\eta\delta \text{ and } w(\beta-\alpha) < w(\delta-\gamma)\\
&\text{or} && \zeta\alpha+\eta\beta = \zeta\gamma+\eta\delta \text{ and } w(\beta-\alpha) = w(\delta-\gamma) \text{ and } x^\alpha\partial^\beta \prec x^\gamma\partial^\delta.
\end{aligned}$$

Theorem

If $G^{(h)}$ is a Gröbner basis of $H_{(\zeta,\eta)}(I)$ with respect to $\prec^{(h)}_{(u,v)}$, then $\mathrm{in}_{(-w,w)}(G^{(h)}|_{h=1})$ is a Gröbner basis of $\mathrm{in}_{(-w,w)}(I)$ with respect to $\prec$.

Put $s := \sum_{i=1}^n w_i x_i \partial_i$. If I is holonomic, then $\mathrm{in}_{(-w,w)}(I) \cap \mathbb{K}[s]$ is a non-trivial principal ideal in the subalgebra $\mathbb{K}[s] \subset D_n$. The monic generator $b_{I,w}$ of this intersection is called the (global) b-function of I with respect to w.

Intersecting an ideal with a principal subalgebra

Theorem

Let A be an associative $\mathbb{K}$-algebra, $J \subset A$ a left ideal and $s \in A$ satisfying $J \cdot s \subset J$ and $\dim_{\mathbb{K}}(\mathrm{End}_A(A/J)) < \infty$. Then $J \cap \mathbb{K}[s] \neq \{0\}$.

Note that both conditions of the previous theorem are fulfilled if $A = D_n$, J is a holonomic left ideal and $s = \sum_{i=1}^n w_i x_i \partial_i$ as in the setup for the b-function.

Algorithm (principalIntersect)

Input: $s \in A$, $J \subset A$ a left ideal such that $J \cap \mathbb{K}[s] \neq \{0\}$.
Output: $b \in \mathbb{K}[s]$ monic such that $J \cap \mathbb{K}[s] = \langle b\rangle$.

$G :=$ a finite left Gröbner basis of J
$i := 1$
loop
  if $\exists\, a_0,\dots,a_{i-1} \in \mathbb{K}$ such that $\mathrm{NF}(s^i,G) + \sum_{j=0}^{i-1} a_j\,\mathrm{NF}(s^j,G) = 0$
  then
   return $b := s^i + \sum_{j=0}^{i-1} a_j s^j$
  else
   $i := i+1$
  end if
end loop

Note that this approach requires a Gröbner basis with respect to an arbitrary well ordering, avoiding the use of (expensive) elimination orderings.

The b-function of a polynomial

Consider a non-constant polynomial $f \in \mathbb{K}[x]$.

Applying the b-function of an ideal

Consider the Malgrange ideal of f defined by

$$I_f := \langle t - f, \partial_i + \frac{\partial f}{\partial x_i}\partial_t, i = 1,\dots,n\rangle \subset D_{n+1} = D_n\langle t,\partial_t \mid \partial_t t = t\partial_t + 1\rangle.$$

Chosing $w = (1,0,\dots,0) \in \mathbb{R}^{n+1}$ such that the weight of ∂_t is 1,

$$b_f(s) := (-1)^{\deg(b_{I_f,w})} b_{I_f,w}(-s-1)$$

is called the (global) b-function or the (global) Bernstein-Sato polynomial of f.

The annihilator based approach

Let s be a new indeterminate and put $D_n[s] := D_n \otimes_{\mathbb{K}} \mathbb{K}[s]$. Consider the commutative ring $R_f := \mathbb{K}[x,s,f^{-1}]$. The free R_f-module $R_f \cdot f^s$ generated by the formal symbol f^s becomes a left $D_n[s]$-module via

$$s \bullet g \cdot f^{s+j} := s \cdot g \cdot f^{s+j}, \qquad x_i \bullet g \cdot f^{s+j} := x_i \cdot g \cdot f^{s+j} \qquad \text{and}$$

$$\partial_i \bullet g \cdot f^{s+j} := \frac{\partial g}{\partial x_i} \cdot f^{s+j} + g \cdot \frac{\partial f}{\partial x_i} \cdot (s+j) \cdot f^{s+j-1}$$

for $g \in \mathbb{K}[x,s]$ and $f^{s+j} := f^j \cdot f^s$, $j \in \mathbb{Z}$.

Theorem (Bernstein)

The Bernstein-Sato polynomial is the uniquely determined monic element of minimal degree in $\mathbb{K}[s]$ satisfying $P \bullet f^{s+1} = b_f \cdot f^s$ for some $P \in D_n[s]$.

The theorem provides another option to compute the Bernstein-Sato polynomial. Again `principalIntersect` is applicable.

Corollary

Denote $\mathrm{Ann}_{D_n[s]}(f^s) := \{p \in D_n[s] \mid p \bullet f^s = 0\}$. Then

$$\langle b_f\rangle = (\mathrm{Ann}_{D_n[s]}(f^s) + \langle f\rangle) \cap \mathbb{K}[s].$$

Applications

Annihilators of powers of polynomials

Note that -1 is always a root of b_f.

Theorem

Let $\lambda \in \mathbb{C}$ and λ_0 be the minimal integral root of b_f.
If $\lambda \in \mathbb{C} \setminus \{\lambda_0 + k \mid k \in \mathbb{N}\}$, then $\mathrm{Ann}_{D_n}(f^\lambda) = \mathrm{Ann}_{D_n[s]}(f^s)|_{s=\lambda}$.
Otherwise, let $G := \{g_1,\dots,g_r\}$ be a Gröbner basis of $\mathrm{Ann}_{D_n[s]}(f^s)$ and $S := \mathrm{Syz}(f^{\lambda-\lambda_0}, g_1|_{s=\lambda_0},\dots,g_r|_{s=\lambda_0})$. Then

$$\mathrm{Ann}_{D_n}(f^\lambda) = \mathrm{Ann}_{D_n[s]}(f^s)|_{s=\lambda} + \langle c_0 \mid (c_0,c_1,\dots,c_r) \in S\rangle.$$

Other applications of b-functions include the following problems.

Other applications

restriction	localization	de Rham cohomology
integration	Weyl closure	...

Implementation

SINGULAR plural

http://www.singular.uni-kl.de
bfun.lib dmod.lib dmodapp.lib dmodvar.lib