```
                     SINGULAR                                 /
 A Computer Algebra System for Polynomial Computations       /   version 3-1-5
                                                           0<
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FB Mathematik der Universitaet, D-67653 Kaiserslautern        \
> // Let's compute the Bernstein-Sato polynomial of a Reiffen curve with
  // parameters (5,6)
. LIB "dmod.lib";         // load the D-module libraries
// ** loaded /usr/share/Singular/LIB/dmod.lib (14426,2011-11-11)
// ** loaded /usr/share/Singular/LIB/poly.lib (14852,2012-04-30)
// ** loaded /usr/share/Singular/LIB/ring.lib (15100,2012-07-10)
// ** loaded /usr/share/Singular/LIB/primdec.lib (14732,2012-03-30)
// ** loaded /usr/share/Singular/LIB/absfact.lib (14191,2011-05-04)
// ** loaded /usr/share/Singular/LIB/triang.lib (13499,2010-10-15)
// ** loaded /usr/share/Singular/LIB/random.lib (14661,2012-03-05)
// ** loaded /usr/share/Singular/LIB/inout.lib (13499,2010-10-15)
// ** loaded /usr/share/Singular/LIB/general.lib (14191,2011-05-04)
// ** loaded /usr/share/Singular/LIB/dmodapp.lib (14998,2012-06-15)
// ** loaded /usr/share/Singular/LIB/sing.lib (14840,2012-04-20)
// ** loaded /usr/share/Singular/LIB/bfun.lib (14203,2011-05-05)
// ** loaded /usr/share/Singular/LIB/presolve.lib (14203,2011-05-05)
// ** loaded /usr/share/Singular/LIB/control.lib (13733,2010-12-06)
// ** loaded /usr/share/Singular/LIB/homolog.lib (14661,2012-03-05)
// ** loaded /usr/share/Singular/LIB/deform.lib (13499,2010-10-15)
// ** loaded /usr/share/Singular/LIB/gmssing.lib (14194,2011-05-04)
// ** loaded /usr/share/Singular/LIB/linalg.lib (13733,2010-12-06)
// ** loaded /usr/share/Singular/LIB/gkdim.lib (12235,2009-11-03)
// ** loaded /usr/share/Singular/LIB/qhmoduli.lib (14203,2011-05-05)
// ** loaded /usr/share/Singular/LIB/rinvar.lib (13499,2010-10-15)
// ** loaded /usr/share/Singular/LIB/zeroset.lib (13499,2010-10-15)
// ** loaded /usr/share/Singular/LIB/primitiv.lib (13499,2010-10-15)
// ** loaded /usr/share/Singular/LIB/elim.lib (14661,2012-03-05)
// ** loaded /usr/share/Singular/LIB/nctools.lib (14246,2011-05-26)
// ** loaded /usr/share/Singular/LIB/matrix.lib (13658,2010-11-16)
> ring r = 0,(x,y),dp;    // Q[x,y] with degrevlex ordering
> poly f = x5 + xy5 + y6; // short notation for x^5 + x*y^5 + y^6
> printlevel = 1;         // let's see progress messages for computations
> bfctAnn(f);             // using annihilator based method
// starting computation of the s-parametric annihilator...
// ...done
// starting to intersect with subalgebra...
// ...done
[1]:
   _[1]=-11/30
   _[2]=-13/30
   _[3]=-7/15
   _[4]=-8/15
   _[5]=-17/30
   _[6]=-19/30
   _[7]=-7/10
   _[8]=-11/15
   _[9]=-23/30
   _[10]=-13/15
   _[11]=-9/10
   _[12]=-14/15
   _[13]=-29/30
   _[14]=-1
   _[15]=-31/30
   _[16]=-16/15
   _[17]=-11/10
   _[18]=-17/15
   _[19]=-37/30
   _[20]=-19/15
   _[21]=-13/10
[2]:
   1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
>
// Let's do it again step by step
. def A = Sannfs(f);        // Ann_D[s](f^s) using BM algorithm
                            // returns D_2[s] containing ideal LD
// -1-1- the ring @R(t,s,_x,_Dx) is ready
// -1-2- starting the elimination of t in @R
// -1-3- t is eliminated
//  -2-1- the ring @R2(_x,_Dx,s) is ready
. setring A; A;
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    x y Dx Dy
//        block   2 : ordering dp
//                  : names    s
//        block   3 : ordering C
//   noncommutative relations:
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
```

```
> LD;                         // the annihilator
LD[1]=5*x*y^4*Dx+6*y^5*Dx-y^5*Dy-5*x^4*Dy
LD[2]=5*x^2*Dx+6*x*y*Dx+4*x*y*Dy+5*y^2*Dy-25*x*s-30*y*s
LD[3]=5*x*y^3*Dx+y^4*Dx+4*y^4*Dy+36*x^3*Dx-x^3*Dy+30*x^2*y*Dy-25*y^3*s-180*x^2*s
LD[4]=125*x^3*y*Dx^2+25*x^2*y^2*Dx^2+200*x^2*y^2*Dx*Dy+20*x*y^3*Dx*Dy+80*x*y^3*Dy^2+1296*x^3*D
x^2-972*x^3*Dx*Dy+1080*x^2*y*Dx*Dy+x^3*Dy^2-780*x^2*y*Dy^2-625*x^2*y*Dx*s+150*x^2*y*Dx+625*x*y
^2*Dx*s+5*x*y^2*Dx-4*y^3*Dx-500*x*y^2*Dy*s+100*x*y^2*Dy+625*y^3*Dy*s-16*y^3*Dy-12960*x^2*Dx*s-
216*x^2*Dx+4860*x^2*Dy*s-774*x^2*Dy-5400*x*y*Dy*s-1260*x*y*Dy-125*x*y*s-3750*y^2*s^2+100*y^2*s
+32400*x*s^2+7560*x*s
LD[5]=625*x^2*y*Dx^2*s+275*x^2*y*Dx^2+125*x*y^2*Dx^2*s+35*x*y^2*Dx^2-4*y^3*Dx^2+1000*x*y^2*Dx*
Dy*s+440*x*y^2*Dx*Dy+100*y^3*Dx*Dy*s+28*y^3*Dx*Dy+400*y^3*Dy^2*s+176*y^3*Dy^2+6480*x^2*Dx^2*s+
3024*x^2*Dx^2-4860*x^2*Dx*Dy*s-2148*x^2*Dx*Dy+5400*x*y*Dx*Dy*s+2520*x*y*Dx*Dy+5*x^2*Dy^2*s-x^2
*Dy^2-3900*x*y*Dy^2*s-1720*x*y*Dy^2-6250*x*y*Dx*s^2-2000*x*y*Dx*s+330*x*y*Dx-625*y^2*Dx*s^2-50
*y^2*Dx*s+35*y^2*Dx-5000*y^2*Dy*s^2-1700*y^2*Dy*s+220*y^2*Dy-64800*x*Dx*s^2-31320*x*Dx*s-504*x
*Dx+24300*x*Dy*s^2+6870*x*Dy*s-1706*x*Dy-27000*y*Dy*s^2-18900*y*Dy*s-2940*y*Dy+15625*y*s^3+625
0*y*s^2-275*y*s+162000*s^3+113400*s^2+17640*s
> poly f = imap(r,f);         // map f from r to A
> ideal I = LD,f;             // add f to the annihilator
> // I = slimgb(I);           // compute a Groebner basis
. // at this point I have to cheat a little bit...
. setring r;
> def B = SannfsBFCT(f);  // directly compute a GB of Ann(f^s)+<f>
// -1-1- Starting the computation of syz(F,_Dx(F))
// -1-2- The module syz(F,_Dx(F)) has been computed
// -1-3- Starting GB computation of syz(F,_Dx(F))
// -1-4- GB computation finished
// -2-1- The ring D[s] is ready
// -2-2- Compute part of Ann(F^s)
// -2-3- GB computation finished
// -3-1- The ring D<t,s> is ready
// -3-2- Starting the elimination of t in D<t,s>
// -3-3- t is eliminated
// -4-1- Starting cosmetic Groebner computation
// -4-2- Finished cosmetic Groebner computation
// -4-3- Start GB computations for Ann f^s + <f>
// -4-4- Finished GB computations for Ann f^s + <f>
> setring B;
> B;
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    s
//        block   2 : ordering dp
//                  : names    x y Dx Dy
//        block   3 : ordering C
//   noncommutative relations:
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
> LD;
LD[1]=5*x*y^4*Dx+6*y^5*Dx-y^5*Dy-5*x^4*Dy
LD[2]=25*s*x+30*s*y-5*x^2*Dx-6*x*y*Dx-4*x*y*Dy-5*y^2*Dy
LD[3]=x*y^5+y^6+x^5
LD[4]=y^6*Dx-y^6*Dy-5*x^5*Dx-5*x^4*y*Dy-5*y^5-25*x^4
LD[5]=5*x^6*Dx+6*x^5*y*Dx+4*x^5*y*Dy+5*x^4*y^2*Dy+25*x^5+30*x^4*y
LD[6]=14105549537280*s*y^2+6103515625*x^2*y^5*Dx+6103515625*x*y^6*Dx-244140625*y^7*Dx+48828125
00*x*y^6*Dy+5126953125*y^7*Dy+6103515625*x^6*Dx-506250000*y^6*Dx+4882812500*x^5*y*Dy+244140625
*x^4*y^2*Dy-48828125*x^3*y^3*Dy+58593750*x^2*y^4*Dy-70312500*x*y^5*Dy+84375000*y^6*Dy+52488000
00*y^5*Dx+506250000*x^3*y^2*Dy-607500000*x^2*y^3*Dy+729000000*x*y^4*Dy-874800000*y^5*Dy-544195
58400*y^4*Dx-5248800000*x^3*y*Dy+6298560000*x^2*y^2*Dy-7558272000*x*y^3*Dy+9069926400*y^4*Dy-2
821109907456*x*y^2*Dx+54419558400*x^3*Dy-65303470080*x^2*y*Dy+78364164096*x*y^2*Dy-23509249228
80*y^3*Dy
LD[7]=625*x^5*y^2*Dx+500*x^4*y^3*Dy+25*x^3*y^4*Dy+150*y^7*Dy+7776*x^5*y*Dx+30*x^6*Dy-66*x^5*y*
Dy+6480*x^4*y^2*Dy+3125*x^4*y^2+150*x^2*y^4+1050*y^6+150*x^5+38880*x^4*y
LD[8]=91403961001574400*s^2*y-3814697265625*s*x^2*y^5*Dx-3814697265625*s*x*y^6*Dx+152587890625
*s*y^7*Dx-3051757812500*s*x*y^6*Dy-3204345703125*s*y^7*Dy-3814697265625*s*x^6*Dx+316406250000*
s*y^6*Dx-3051757812500*s*x^5*y*Dy-152587890625*s*x^4*y^2*Dy+30517578125*s*x^3*y^3*Dy-366210937
50*s*x^2*y^4*Dy+43945312500*s*x*y^5*Dy-52734375000*s*y^6*Dy-3280500000000*s*y^5*Dx-3164062500
0*s*x^3*y^2*Dy+379687500000*s*x^2*y^3*Dy-455625000000*s*x*y^4*Dy+546750000000*s*y^5*Dy+3401222
4000000*s*y^4*Dx+3280500000000*s*x^3*y*Dy-3936600000000*s*x^2*y^2*Dy+4723920000000*s*x*y^3*Dy-
5668704000000*s*y^4*Dy-1763193692160000*s*x*y^2*Dx-352638738432000*s*y^3*Dx-34012224000000*s*x
^3*Dy+40814668800000*s*x^2*y*Dy-48977602560000*s*x*y^2*Dy-1351781830656000*s*y^3*Dy-3656158440
0629760*s*x*y*Dx+13710594150236160*s*x*y*Dy-15233993500262400*s*y^2*Dy-352638738432000*s*y^2-3
5038185050603520*s*x-18280792200314880*s*y+352638738432000*x^2*y^2*Dx^2+70527747686400*x*y^3*D
x^2+564221981491200*x*y^3*Dx*Dy+56422198149120*y^4*Dx*Dy+225688792596480*y^4*Dy^2+365615844006
2976*x^2*y*Dx^2-2742118830047232*x^2*y*Dx*Dy+3046798700052480*x*y^2*Dx*Dy+2821109907456*x^2*y*
Dy^2-2200465727815680*x*y^2*Dy^2+423166486118400*x*y^2*Dx+70527747686400*y^3*Dx+28211099074560
0*y^3*Dy+7007637010120704*x^2*Dx+7312316880125952*x*y*Dx-11284439629824*x^2*Dy+343611186728140
8*x*y*Dy+3046798700052480*y^2*Dy
LD[9]=125*x^4*y^3*Dx+100*x^3*y^4*Dy-25*y^7*Dy-1296*x^5*y*Dx-5*x^6*Dy+11*x^5*y*Dy-1080*x^4*y^2*
Dy+625*x^3*y^3-25*x^2*y^4-175*y^6-25*x^5-6480*x^4*y
LD[10]=5922976672902021120000*s^3+2384185791015625*s^2*x^2*y^5*Dx+2384185791015625*s^2*x*y^6*Dx
-95367431640625*s^2*y^7*Dx+1907348632812500*s^2*x*y^6*Dy+2002716064453125*s^2*y^7*Dy+238418579
1015625*s^2*x^6*Dx-197753906250000*s^2*y^6*Dx+1907348632812500*s^2*x^5*y*Dy+95367431640625*s^2
*x^4*y^2*Dy-19073486328125*s^2*x^3*y^3*Dy+22888183593750*s^2*x^2*y^4*Dy-27465820312500*s^2*x*y
^5*Dy+32958984375000*s^2*y^6*Dy+2050312500000000*s^2*y^5*Dx+197753906250000*s^2*x^3*y^2*Dy-237
```

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304687500000*s^2*x^2*y^3*Dy+284765625000000*s^2*x*y^4*Dy-341718750000000*s^2*y^5*Dy-2125764000
0000000*s^2*y^4*Dx-2050312500000000*s^2*x^3*y*Dy+2460375000000000*s^2*x^2*y^2*Dy-2952450000000
000*s^2*x*y^3*Dy+3542940000000000*s^2*y^4*Dy+1101996057600000000*s^2*x*y^2*Dx+2203992115200000
00*s^2*y^3*Dx+21257640000000000*s^2*x^3*Dy-25509168000000000*s^2*x^2*y*Dy+30611001600000000*s^
2*x*y^2*Dy+844863644160000000*s^2*y^3*Dy-2285099025039360000*s^2*y^2*Dx-856912134389760000*s^
2*x*y*Dy-875954626265088000*s^2*y^2*Dy+220399211520000000*s^2*y^2-236919066916080844800*s^2*x
*Dx+888446500935303168000*s^2*x*Dy-98716277881700352000*s^2*y*Dy+21898865656627200000*s^2*x+342
76485375590400000*s^2*y+414608367103141478400*s^2-220399211520000000*s*x^2*y^2*Dx^2-4407984230
4000000*s*x*y^3*Dx^2-352638738432000000*s*x*y^3*Dx*Dy-35263873843200000*s*y^4*Dx*Dy-1410554953
72800000*s*y^4*Dy^2+457019805007872000*s*x*y^2*Dx^2+171382426877952000*s*x^2*y*Dx*Dy+17519092
52530176000*s*x*y^2*Dx*Dy+365615844006297600*s*y^3*Dx*Dy-176319369216000*s*x^2*y*Dy^2+1375291
079884800000*s*x*y^2*Dy^2+1462463376025190400*s*y^3*Dy^2-2644790538240000000*s*x*y^2*Dx-4407984
2304000000*s*y^3*Dx+236919066916080844480*s*x^2*Dx^2-176319369216000000*s*y^3*Dy-17768930018706
063360*s*x^2*Dx*Dy+19743255576340070400*s*x*y*Dx*Dy+18280792200314880*s*x^2*Dy^2-1425901791624
5606400*s*x*y*Dy^2-4379773131325440000*s*x^2*Dx-11882514930204672000*s*x*y*Dx-1828079220031488
00*s*y^2*Dx+7052774768640000*s*x^2*Dy-2147569917050880000*s*x*y*Dy-8119718535639859200*s*y^2*D
y-114510882342772408320*s*x*Dx+25117808483232645120*s*x*Dy-69101394517190246400*s*y*Dy-1005443
571017318400*s*y+644946348827108966400*s+1005443571017318400*x^2*y*Dx^2+127965545402204160*x*y^
2*Dx^2-14624633760251904*y^3*Dx^2+160870971362770944 0*x*y^2*Dx*Dy+102372436321763328*y^3*Dx*Dy
+643483885451083776*y^3*Dy^2+11056223122750439424*x^2*Dx^2-78534283292552724448*x^2*Dx*Dy+92135
19268958699520*x*y*Dx*Dy-3656158440062976*x^2*Dy^2-62885925169083187 20*x*y*Dy^2+12065322852207
82080*x*y*Dx+127965545402204160*y^2*Dx+804354856813854720*y^2*Dy-1842703853791739904*x*Dx-6237
406298747437056*x*Dy-10749105813785149440*y*Dy
> vector v = pIntersect(s,LD); // intersect with Q[s]
// lower bound for the degree of the insection is 3
// Testing degree 1
// Testing degree 2
// Testing degree 3
// Testing degree 4
// Testing degree 5
// Testing degree 6
// Testing degree 7
// Testing degree 8
// Testing degree 9
// Testing degree 10
// Testing degree 11
// Testing degree 12
// Testing degree 13
// Testing degree 14
// Testing degree 15
// Testing degree 16
// Testing degree 17
// Testing degree 18
// Testing degree 19
// Testing degree 20
// Testing degree 21
// degree of the generator of the intersection is: 21
> v;
gen(22)+18*gen(21)+13813/90*gen(20)+370967/450*gen(19)+506894399/162000*gen(18)+1806943013/202
500*gen(17)+361668968887/18225000*gen(16)+2141909168347/60750000*gen(15)+66729180318668 27/1312
20000000*gen(14)+19739016362903981/328050000000*gen(13)+347082600153073 4353/59049000000000*gen
(12)+140371158963344878 27/295245000000000*gen(11)+3383999488861348718873/106288200000000000*ge
n(10)+973975492119135319 57/5535843750000000*gen(9)+1906042775344584844 12199/23914845000000000
00*gen(8)+175051473899120880679433/59787112500000000000*gen(7)+4617421365989719448719511/53808
40125000000000000*gen(6)+131700752897071318366 2923/672605015625000000000*gen(5)+1669241906063
41414169563/498225937500000000000*gen(4)+135927553031417080345 1117/336302507812500000000*g
en(3)+3219819471559964825041 49/105094533691406250000000*gen(2)+800564870425711601 21/729823150
6347656250000*gen(1)
> bFactor(vec2poly(v,1)); // factorize the BS poly to get roots and
                          // multiplicities
// found roots
// no irreducible factors found
[1]:
   _[1]=-11/30
   _[2]=-13/30
   _[3]=-7/15
   _[4]=-8/15
   _[5]=-17/30
   _[6]=-19/30
   _[7]=-7/10
   _[8]=-11/15
   _[9]=-23/30
   _[10]=-13/15
   _[11]=-9/10
   _[12]=-14/15
   _[13]=-29/30
   _[14]=-1
   _[15]=-31/30
   _[16]=-16/15
   _[17]=-11/10
   _[18]=-17/15
   _[19]=-37/30
   _[20]=-19/15
   _[21]=-13/10
[2]:
```

```
   1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
.
// Now let’s compute again using the initial ideal method
. setring r;
> // we could just use "bfct(f);"
. // or do it again step by step
. def C = initialMalgrange(f); // the initial ideal of the Malgrange
                               // ideal defined by f
// the (n+1)-th Weyl algebra :
//   characteristic : 0
//   number of vars : 6
//        block   1 : ordering dp
//                  : names    t x y Dt Dx Dy
//        block   2 : ordering C
//   noncommutative relations:
//    Dtt=t*Dt+1
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
// starting Groebner basis computation with engine: 0
// finished Groebner basis computation
// starting cosmetic Groebner basis computation
// finished cosmetic Groebner basis computation
. setring C;                   // D_3 with new vars t,Dt
> C;                           // containing ideal inF
//   characteristic : 0
//   number of vars : 6
//        block   1 : ordering dp
//                  : names    t x y Dt Dx Dy
//        block   2 : ordering C
//   noncommutative relations:
//    Dtt=t*Dt+1
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
> inF;
inF[1]=25*t*x*Dt+30*t*y*Dt+5*x^2*Dx+6*x*y*Dx+4*x*y*Dy+5*y^2*Dy+25*x+30*y
inF[2]=625*t*y^3*Dt+125*x*y^3*Dx+25*y^4*Dx+100*y^4*Dy+6480*t*y^2*Dt+1296*x*y^2*Dx-25*x^3*Dy+30
*x^2*y*Dy-36*x*y^2*Dy+1080*y^3*Dy+625*y^3+6480*y^2
inF[3]=5*x*y^4*Dx+6*y^5*Dx-y^5*Dy-5*x^4*Dy
inF[4]=15625*t^2*y^2*Dt^2-625*x^2*y^2*Dx^2+275*y^4*Dx^2-1000*x*y^3*Dx*Dy-250*y^4*Dx*Dy-400*y^4
*Dy^2+162000*t^2*y*Dt^2-6480*t*y^2*Dt*Dx-6480*x^2*y*Dx^2-1296*x*y^2*Dx^2+4320*t*y^2*Dt*Dy-275*
x^3*Dx*Dy+330*x^2*y*Dx*Dy-9900*x*y^2*Dx*Dy-1080*y^3*Dx*Dy+200*x^3*Dy^2-235*x^2*y*Dy^2+276*x*y^
2*Dy^2-3780*y^3*Dy^2+26250*t*y^2*Dt-4750*x*y^2*Dx-800*y^3*Dx-3700*y^3*Dy+304560*t*y*Dt-42768*x
*y*Dx-7776*y^2*Dx-1085*x^2*Dy+1278*x*y*Dy-29664*y^2*Dy-5000*y^2-19440*y
inF[5]=y^5*Dt+5*x^4*Dt
inF[6]=5*x*y^4*Dt+6*y^5*Dt
inF[7]=x^4*y*Dt
inF[8]=4*x^5*Dt+5*x^4*y*Dt
inF[9]=x*y^5+y^6+x^5
inF[10]=1350000*x^3*y^2*Dt*Dy-1620000*x^2*y^3*Dt*Dy+55987200*y^5*Dt*Dy+390625*x^3*y^2*Dt-62500
0*x^2*y^3*Dt+3627970560*t*y^3*Dt^2-69984000*x^4*Dt*Dx+725594112*x*y^3*Dt*Dx+303264000*x^4*Dt*D
y-13996800*x^3*y*Dt*Dy+16796160*x^2*y^2*Dt*Dy-20155392*x*y^3*Dt*Dy+604661760*y^4*Dt*Dy+1237500
0*x^4*Dt-7776000*x*y^3*Dt+291600000*y^4*Dt-279936000*x^3*Dt+7255941120*y^3*Dt
[...]
> vector v = pIntersect(t*Dt,inF); // intersection with Q[t*Dt]
// lower bound for the degree of the insection is 5
// Testing degree 1
// Testing degree 2
// Testing degree 3
// Testing degree 4
// Testing degree 5
// Testing degree 6
// Testing degree 7
// Testing degree 8
// Testing degree 9
// Testing degree 10
// Testing degree 11
// Testing degree 12
// Testing degree 13
// Testing degree 14
// Testing degree 15
// Testing degree 16
// Testing degree 17
// Testing degree 18
// Testing degree 19
// Testing degree 20
// Testing degree 21
// degree of the generator of the intersection is: 21
> v;
gen(22)+3*gen(21)+313/90*gen(20)+128/75*gen(19)-361/162000*gen(18)-32993/90000*gen(17)-2311133
/18225000*gen(16)+553003/60750000*gen(15)+1772785547/131220000000*gen(14)+386454731/2187000000
00*gen(13)-26711812907/59049000000000*gen(12)-250191773/2050312500000*gen(11)+175082458673/106
288200000000000*gen(10)+470625016471/17714700000000000*gen(9)+2889009059579/23914845000000000
000*gen(8)-904947075667/398580750000000000000*gen(7)-2081023177321/134521003125000000000*gen(6
)+17574843451/249112968750000000000*gen(5)+23017967273/420378134765625000000*gen(4)-2327647/4
324877929687500000*gen(3)-15842827/3649115753173828125000*gen(2)
```

```
> bFactor(vec2poly(v,1));      // same as above...
// found roots
// no irreducible factors found
[1]:
   _[1]=3/10
   _[2]=4/15
   _[3]=7/30
   _[4]=2/15
   _[5]=1/10
   _[6]=1/15
   _[7]=1/30
   _[8]=0
   _[9]=-1/30
   _[10]=-1/15
   _[11]=-1/10
   _[12]=-2/15
   _[13]=-7/30
   _[14]=-4/15
   _[15]=-3/10
   _[16]=-11/30
   _[17]=-13/30
   _[18]=-7/15
   _[19]=-8/15
   _[20]=-17/30
   _[21]=-19/30
[2]:
   1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
> // ... but why is 0 a root and not -1?
. // Because we forgot to apply the Mellin transform.
. poly b = vec2poly(v,1);
> b = subst(b,var(1),-var(1)-1); // t*Dt |-> -t*Dt-1
> bFactor(b);
// found roots
// no irreducible factors found
[1]:
   _[1]=-11/30
   _[2]=-13/30
   _[3]=-7/15
   _[4]=-8/15
   _[5]=-17/30
   _[6]=-19/30
   _[7]=-7/10
   _[8]=-11/15
   _[9]=-23/30
   _[10]=-13/15
   _[11]=-9/10
   _[12]=-14/15
   _[13]=-29/30
   _[14]=-1
   _[15]=-31/30
   _[16]=-16/15
   _[17]=-11/10
   _[18]=-17/15
   _[19]=-37/30
   _[20]=-19/15
   _[21]=-13/10
[2]:
   1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
>
// Let's go back to the ring A
. setring A;
> A;
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    x y Dx Dy
//        block   2 : ordering dp
//                  : names    s
//        block   3 : ordering C
//   noncommutative relations:
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
> LD;                          // Ann_D[s](f^s)
LD[1]=5*x*y^4*Dx+6*y^5*Dx-y^5*Dy-5*x^4*Dy
LD[2]=5*x^2*Dx+6*x*y*Dx+4*x*y*Dy+5*y^2*Dy-25*x*s-30*y*s
LD[3]=5*x*y^3*Dx+y^4*Dx+4*y^4*Dy+36*x^3*Dx-x^3*Dy+30*x^2*y*Dy-25*y^3*s-180*x^2*s
LD[4]=125*x^3*y*Dx^2+25*x^2*y^2*Dx^2+200*x^2*y^2*Dx*Dy+20*x*y^3*Dx*Dy+80*x*y^3*Dy^2+1296*x^3*D
x^2-972*x^3*Dx*Dy+1080*x^2*y*Dx*Dy+x^3*Dy^2-780*x^2*y*Dy^2-625*x^2*y*Dx*s+150*x^2*y*Dx+625*x*y
^2*Dx*s+5*x*y^2*Dx-4*y^3*Dx-500*x*y^2*Dy*s+100*x*y^2*Dy+625*y^3*Dy*s-16*y^3*Dy-12960*x^2*Dx*s-
216*x^2*Dx+4860*x^2*Dy*s-774*x^2*Dy-5400*x*y*Dy*s-1260*x*y*Dy-125*x*y*s-3750*y^2*s^2+100*y^2*s
+32400*x*s^2+7560*x*s
LD[5]=625*x^2*y*Dx^2*s+275*x^2*y*Dx^2+125*x*y^2*Dx^2*s+35*x*y^2*Dx^2-4*y^3*Dx^2+1000*x*y^2*Dx*
Dy*s+440*x*y^2*Dx*Dy+100*y^3*Dx*Dy*s+28*y^3*Dx*Dy+400*y^3*Dy^2*s+176*y^3*Dy^2+6480*x^2*Dx^2*s+
3024*x^2*Dx^2-4860*x^2*Dx*Dy*s-2148*x^2*Dx*Dy+5400*x*y*Dx*Dy*s+2520*x*y*Dx*Dy+5*x^2*Dy^2*s-x^2
*Dy^2-3900*x*y*Dy^2*s-1720*x*y*Dy^2-6250*x*y*Dx*s^2-2000*x*y*Dx*s+330*x*y*Dx-625*y^2*Dx*s^2-50
```

```
*y^2*Dx*s+35*y^2*Dx-5000*y^2*Dy*s^2-1700*y^2*Dy*s+220*y^2*Dy-64800*x*Dx*s^2-31320*x*Dx*s-504*x
*Dx+24300*x*Dy*s^2+6870*x*Dy*s-1706*x*Dy-27000*y*Dy*s^2-18900*y*Dy*s-2940*y*Dy+15625*y*s^3+625
0*y*s^2-275*y*s+162000*s^3+113400*s^2+17640*s
> // What is Ann_D(f^s)?
. eliminate(LD,s);
_[1]=5*x*y^4*Dx+6*y^5*Dx-y^5*Dy-5*x^4*Dy
>
// Let's now compute Ann_D(1/f)
. ideal J = annfspecial(LD,f,-1,-1);
// the first  -1 is the min integral root of b_f
// the second -1 is the power of f
// -1-1- d<=0, no syzygy computations needed
// -2-1- starting final Groebner basis
// -2-2- finished final Groebner basis
. J;
J[1]=5*x^2*Dx+6*x*y*Dx+4*x*y*Dy+5*y^2*Dy+25*x+30*y
J[2]=165*x*y^2*Dx^2-2*y^3*Dx^2-140*x*y^2*Dx*Dy+139*y^3*Dx*Dy-112*y^3*Dy^2-1728*x^2*Dx^2+1356*x
^2*Dx*Dy-1440*x*y*Dx*Dy-3*x^2*Dy^2+1090*x*y*Dy^2-735*x*y*Dx+990*y^2*Dx-1400*y^2*Dy-16992*x*Dx+
7862*x*Dy-5520*y*Dy-3675*y-33120
J[3]=5*x*y^3*Dx+y^4*Dx+4*y^4*Dy+36*x^3*Dx-x^3*Dy+30*x^2*y*Dy+25*y^3+180*x^2
J[4]=35*y^4*Dx^2+140*x*y^3*Dx*Dy-7*y^4*Dx*Dy+112*y^4*Dy^2+1188*x^3*Dx^2+1728*x^2*y*Dx^2-33*x^3
*Dx*Dy-366*x^2*y*Dx*Dy+1440*x*y^2*Dx*Dy+3*x^2*y*Dy^2-1090*x*y^2*Dy^2+735*x*y^2*Dx+1400*y^3*Dy+
9504*x^2*Dx+16992*x*y*Dx-99*x^2*Dy-5882*x*y*Dy+5520*y^2*Dy+3675*y^2+11880*x+33120*y
J[5]=5*y^5*Dx-5*y^5*Dy-36*x^3*y*Dx-5*x^4*Dy+x^3*y*Dy-30*x^2*y^2*Dy-25*y^4-180*x^2*y
J[6]=25*x*y^5*Dy+25*y^6*Dy+180*x^4*y*Dx+216*x^3*y^2*Dx+25*x^5*Dy+144*x^3*y^2*Dy+180*x^2*y^3*Dy
+125*x*y^4+150*y^5+900*x^3*y+1080*x^2*y^2
> // Is J holonomic?
. A;
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    x y Dx Dy
//        block   2 : ordering dp
//                  : names    s
//        block   3 : ordering C
//   noncommutative relations:
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
> def W2 = makeWeyl(2); setring W2; W2;
//   characteristic : 0
//   number of vars : 4
//        block   1 : ordering dp
//                  : names    x(1) x(2) D(1) D(2)
//        block   2 : ordering C
//   noncommutative relations:
//    D(1)x(1)=x(1)*D(1)+1
//    D(2)x(2)=x(2)*D(2)+1
> ideal J = imap(A,J); J;
J[1]=0
J[2]=-33120
J[3]=0
J[4]=0
J[5]=0
J[6]=0
> // oops...
. J = fetch(A,J); J;
J[1]=5*x(1)^2*D(1)+6*x(1)*x(2)*D(1)+4*x(1)*x(2)*D(2)+5*x(2)^2*D(2)+25*x(1)+30*x(2)
J[2]=165*x(1)*x(2)^2*D(1)^2-2*x(2)^3*D(1)^2-140*x(1)*x(2)^2*D(1)*D(2)+139*x(2)^3*D(1)*D(2)-112
*x(2)^3*D(2)^2-1728*x(1)^2*D(1)^2+1356*x(1)^2*D(1)*D(2)-1440*x(1)*x(2)*D(1)*D(2)-3*x(1)^2*D(2)
^2+1090*x(1)*x(2)*D(2)^2-735*x(1)*x(2)*D(1)+990*x(2)^2*D(1)-1400*x(2)^2*D(2)-16992*x(1)*D(1)+7
862*x(1)*D(2)-5520*x(2)*D(2)-3675*x(2)-33120
J[3]=5*x(1)*x(2)^3*D(1)+x(2)^4*D(1)+4*x(2)^4*D(2)+36*x(1)^3*D(1)-x(1)^3*D(2)+30*x(1)^2*x(2)*D(
2)+25*x(2)^3+180*x(1)^2
J[4]=35*x(2)^4*D(1)^2+140*x(1)*x(2)^3*D(1)*D(2)-7*x(2)^4*D(1)*D(2)+112*x(2)^4*D(2)^2+1188*x(1)
^3*D(1)^2+1728*x(1)^2*x(2)*D(1)^2-33*x(1)^3*D(1)*D(2)-366*x(1)^2*x(2)*D(1)*D(2)+1440*x(1)*x(2)
^2*D(1)*D(2)+3*x(1)^2*x(2)*D(2)^2-1090*x(1)*x(2)^2*D(2)^2+735*x(1)*x(2)^2*D(1)+1400*x(2)^3*D(2
)+9504*x(1)^2*D(1)+16992*x(1)*x(2)*D(1)-99*x(1)^2*D(2)-5882*x(1)*x(2)*D(2)+5520*x(2)^2*D(2)+36
75*x(2)^2+11880*x(1)+33120*x(2)
J[5]=5*x(2)^5*D(1)-5*x(2)^5*D(2)-36*x(1)^3*x(2)*D(1)-5*x(1)^4*D(2)+x(1)^3*x(2)*D(2)-30*x(1)^2*
x(2)^2*D(2)-25*x(2)^4-180*x(1)^2*x(2)
J[6]=25*x(1)*x(2)^5*D(2)+25*x(2)^6*D(2)+180*x(1)^4*x(2)*D(1)+216*x(1)^3*x(2)^2*D(1)+25*x(1)^5*
D(2)+144*x(1)^3*x(2)^2*D(2)+180*x(1)^2*x(2)^3*D(2)+125*x(1)*x(2)^4+150*x(2)^5+900*x(1)^3*x(2)+
1080*x(1)^2*x(2)^2
> def CV = charVariety(J);
// Starting Groebner basis computation...
// ... done.
> setring CV; CV;
//   characteristic : 0
//   number of vars : 4
//        block   1 : ordering dp
//                  : names    x(1) x(2) D(1) D(2)
//        block   2 : ordering C
> charVar;
charVar[1]=5*x(1)^2*D(1)+6*x(1)*x(2)*D(1)+4*x(1)*x(2)*D(2)+5*x(2)^2*D(2)
charVar[2]=825*x(1)*x(2)^2*D(1)^2-10*x(2)^3*D(1)^2-700*x(1)*x(2)^2*D(1)*D(2)+695*x(2)^3*D(1)*D
```

```
(2)-560*x(2)^3*D(2)^2+10368*x(1)*x(2)*D(1)^2-8424*x(1)*x(2)*D(1)*D(2)+8640*x(2)^2*D(1)*D(2)-15
*x(1)^2*D(2)^2+26*x(1)*x(2)*D(2)^2-6780*x(2)^2*D(2)^2
charVar[3]=125*x(1)*x(2)^3*D(1)+25*x(2)^4*D(1)+100*x(2)^4*D(2)+1296*x(1)*x(2)^2*D(1)-25*x(1)^3
*D(2)+30*x(1)^2*x(2)*D(2)-36*x(1)*x(2)^2*D(2)+1080*x(2)^3*D(2)
charVar[4]=34375*x(2)^4*D(1)^2-34375*x(2)^4*D(1)*D(2)-4320*x(2)^3*D(1)^2-213300*x(1)*x(2)^2*D(
1)*D(2)+3240*x(2)^3*D(1)*D(2)+27500*x(1)^3*D(2)^2-4125*x(1)^2*x(2)*D(2)^2+4675*x(1)*x(2)^2*D(2
)^2-175920*x(2)^3*D(2)^2+4478976*x(1)*x(2)*D(1)^2-3639168*x(1)*x(2)*D(1)*D(2)+3732480*x(2)^2*D
(1)*D(2)-6480*x(1)^2*D(2)^2+11232*x(1)*x(2)*D(2)^2-2928960*x(2)^2*D(2)^2
charVar[5]=15625*x(2)^5*D(1)-15625*x(2)^5*D(2)+32400*x(2)^4*D(1)-15625*x(1)^4*D(2)+3125*x(1)^3
*x(2)*D(2)-3750*x(1)^2*x(2)^2*D(2)+4500*x(1)*x(2)^3*D(2)-5400*x(2)^4*D(2)+1679616*x(1)*x(2)^2*
D(1)-32400*x(1)^3*D(2)+38880*x(1)^2*x(2)*D(2)-46656*x(1)*x(2)^2*D(2)+1399680*x(2)^3*D(2)
charVar[6]=x(1)*x(2)^5*D(2)+x(2)^6*D(2)+x(1)^5*D(2)
> dim(groebner(charVar));
2
> // Hence J is holonomic
. // While we're here, let's compute the primary decomposition of charVar
. primdecGTZ(charVar);
[1]:
   [1]:
      _[1]=D(2)
      _[2]=D(1)
   [2]:
      _[1]=D(2)
      _[2]=D(1)
[2]:
   [1]:
      _[1]=625*x(2)^2*D(1)^5-625*x(2)^2*D(1)^4*D(2)+7776*x(2)*D(1)^5-7560*x(2)*D(1)^4*D(2)+30*
x(2)*D(1)^3*D(2)^2+10*x(2)*D(1)^2*D(2)^3-256*x(2)*D(2)^5-3125*D(2)^5
      _[2]=-125*x(2)^2*D(1)^4+125*x(2)^2*D(1)^3*D(2)+1296*x(1)*D(1)^4-324*x(1)*D(1)^3*D(2)+108
0*x(2)*D(1)^3*D(2)-179*x(1)*D(1)^2*D(2)^2-240*x(2)*D(1)^2*D(2)^2-104*x(1)*D(1)*D(2)^3-135*x(2)
*D(1)*D(2)^3-64*x(1)*D(2)^4-80*x(2)*D(2)^4
      _[3]=875000*x(2)^3*D(1)^4-875000*x(2)^3*D(1)^3*D(2)+86400*x(2)^2*D(1)^4+419875*x(2)^2*D(
1)^3*D(2)-14875*x(2)^2*D(1)^2*D(2)^2-21000*x(2)^2*D(1)*D(2)^3+448000*x(1)*x(2)*D(2)^4-22400*x(
2)^2*D(2)^4+111974400*x(1)*D(1)^4-30107376*x(1)*D(1)^3*D(2)+93312000*x(2)*D(1)^3*D(2)-16461252
*x(1)*D(1)^2*D(2)^2-22497480*x(2)*D(1)^2*D(2)^2-9473843*x(1)*D(1)*D(2)^3-12542640*x(2)*D(1)*D(
2)^3-214304*x(1)*D(2)^4-7377255*x(2)*D(2)^4
      _[4]=2625*x(2)^2*D(1)^3-625*x(2)^2*D(1)^2*D(2)+8000*x(1)*x(2)*D(1)*D(2)^2-400*x(2)^2*D(1
)*D(2)^2+6400*x(2)^2*D(2)^3-27216*x(1)*D(1)^3-13932*x(1)*D(1)^2*D(2)-22680*x(2)*D(1)^2*D(2)+91
887*x(1)*D(1)*D(2)^2-12240*x(2)*D(1)*D(2)^2-1864*x(1)*D(2)^3+75795*x(2)*D(2)^3
      _[5]=105*x(1)*x(2)*D(1)^2+x(2)^2*D(1)^2-80*x(1)*x(2)*D(1)*D(2)+88*x(2)^2*D(1)*D(2)-64*x(
2)^2*D(2)^2+1296*x(1)*D(1)^2-972*x(1)*D(1)*D(2)+1080*x(2)*D(1)*D(2)+x(1)*D(2)^2-780*x(2)*D(2)^
2
      _[6]=4375*x(2)^3*D(1)^2+17500*x(1)*x(2)^2*D(1)*D(2)-875*x(2)^3*D(1)*D(2)+14000*x(2)^3*D(
2)^2+432*x(2)^2*D(1)^2+158220*x(1)*x(2)*D(1)*D(2)+216*x(2)^2*D(1)*D(2)+3675*x(1)^2*D(2)^2-4445
*x(1)*x(2)*D(2)^2+131952*x(2)^2*D(2)^2+559872*x(1)*D(1)^2-419904*x(1)*D(1)*D(2)+466560*x(2)*D(
1)*D(2)+432*x(1)*D(2)^2-336960*x(2)*D(2)^2
      _[7]=5*x(1)^2*D(1)+6*x(1)*x(2)*D(1)+4*x(1)*x(2)*D(2)+5*x(2)^2*D(2)
      _[8]=-125*x(1)*x(2)^3*D(1)-25*x(2)^4*D(1)-100*x(2)^4*D(2)-1296*x(1)*x(2)^2*D(1)+25*x(1)^
3*D(2)-30*x(1)^2*x(2)*D(2)+36*x(1)*x(2)^2*D(2)-1080*x(2)^3*D(2)
      _[9]=x(1)*x(2)^5+x(2)^6+x(1)^5
   [2]:
      _[1]=625*x(2)^2*D(1)^5-625*x(2)^2*D(1)^4*D(2)+7776*x(2)*D(1)^5-7560*x(2)*D(1)^4*D(2)+30*
x(2)*D(1)^3*D(2)^2+10*x(2)*D(1)^2*D(2)^3-256*x(2)*D(2)^5-3125*D(2)^5
      _[2]=-125*x(2)^2*D(1)^4+125*x(2)^2*D(1)^3*D(2)+1296*x(1)*D(1)^4-324*x(1)*D(1)^3*D(2)+108
0*x(2)*D(1)^3*D(2)-179*x(1)*D(1)^2*D(2)^2-240*x(2)*D(1)^2*D(2)^2-104*x(1)*D(1)*D(2)^3-135*x(2)
*D(1)*D(2)^3-64*x(1)*D(2)^4-80*x(2)*D(2)^4
      _[3]=875000*x(2)^3*D(1)^4-875000*x(2)^3*D(1)^3*D(2)+86400*x(2)^2*D(1)^4+419875*x(2)^2*D(
1)^3*D(2)-14875*x(2)^2*D(1)^2*D(2)^2-21000*x(2)^2*D(1)*D(2)^3+448000*x(1)*x(2)*D(2)^4-22400*x(
2)^2*D(2)^4+111974400*x(1)*D(1)^4-30107376*x(1)*D(1)^3*D(2)+93312000*x(2)*D(1)^3*D(2)-16461252
*x(1)*D(1)^2*D(2)^2-22497480*x(2)*D(1)^2*D(2)^2-9473843*x(1)*D(1)*D(2)^3-12542640*x(2)*D(1)*D(
2)^3-214304*x(1)*D(2)^4-7377255*x(2)*D(2)^4
      _[4]=2625*x(2)^2*D(1)^3-625*x(2)^2*D(1)^2*D(2)+8000*x(1)*x(2)*D(1)*D(2)^2-400*x(2)^2*D(1
)*D(2)^2+6400*x(2)^2*D(2)^3-27216*x(1)*D(1)^3-13932*x(1)*D(1)^2*D(2)-22680*x(2)*D(1)^2*D(2)+91
887*x(1)*D(1)*D(2)^2-12240*x(2)*D(1)*D(2)^2-1864*x(1)*D(2)^3+75795*x(2)*D(2)^3
      _[5]=105*x(1)*x(2)*D(1)^2+x(2)^2*D(1)^2-80*x(1)*x(2)*D(1)*D(2)+88*x(2)^2*D(1)*D(2)-64*x(
2)^2*D(2)^2+1296*x(1)*D(1)^2-972*x(1)*D(1)*D(2)+1080*x(2)*D(1)*D(2)+x(1)*D(2)^2-780*x(2)*D(2)^
2
      _[6]=4375*x(2)^3*D(1)^2+17500*x(1)*x(2)^2*D(1)*D(2)-875*x(2)^3*D(1)*D(2)+14000*x(2)^3*D(
2)^2+432*x(2)^2*D(1)^2+158220*x(1)*x(2)*D(1)*D(2)+216*x(2)^2*D(1)*D(2)+3675*x(1)^2*D(2)^2-4445
*x(1)*x(2)*D(2)^2+131952*x(2)^2*D(2)^2+559872*x(1)*D(1)^2-419904*x(1)*D(1)*D(2)+466560*x(2)*D(
1)*D(2)+432*x(1)*D(2)^2-336960*x(2)*D(2)^2
      _[7]=5*x(1)^2*D(1)+6*x(1)*x(2)*D(1)+4*x(1)*x(2)*D(2)+5*x(2)^2*D(2)
      _[8]=-125*x(1)*x(2)^3*D(1)-25*x(2)^4*D(1)-100*x(2)^4*D(2)-1296*x(1)*x(2)^2*D(1)+25*x(1)^
3*D(2)-30*x(1)^2*x(2)*D(2)+36*x(1)*x(2)^2*D(2)-1080*x(2)^3*D(2)
      _[9]=x(1)*x(2)^5+x(2)^6+x(1)^5
[3]:
   [1]:
      _[1]=x(2)^3
      _[2]=10368*x(1)*x(2)*D(1)^3-8424*x(1)*x(2)*D(1)^2*D(2)+8640*x(2)^2*D(1)^2*D(2)+44*x(1)*x
(2)*D(1)*D(2)^2-6780*x(2)^2*D(1)*D(2)^2+12*x(1)*x(2)*D(2)^3+15*x(2)^2*D(2)^3
      _[3]=x(1)*x(2)^2
      _[4]=-10368*x(1)*x(2)*D(1)^2+8424*x(1)*x(2)*D(1)*D(2)-8640*x(2)^2*D(1)*D(2)+15*x(1)^2*D(
2)^2-26*x(1)*x(2)*D(2)^2+6780*x(2)^2*D(2)^2
      _[5]=5*x(1)^2*D(1)+6*x(1)*x(2)*D(1)+4*x(1)*x(2)*D(2)+5*x(2)^2*D(2)
      _[6]=x(1)^2*x(2)
```

```
      _[7]=x(1)^3
   [2]:
      _[1]=x(2)
      _[2]=x(1)
> // But let's get a second opinion about the holonomicity of J
. setring W2;
> W2;
//   characteristic : 0
//   number of vars : 4
//        block   1 : ordering dp
//                  : names    x(1) x(2) D(1) D(2)
//        block   2 : ordering C
//   noncommutative relations:
//    D(1)x(1)=x(1)*D(1)+1
//    D(2)x(2)=x(2)*D(2)+1
> gkdim(J);
2
> // and a third one
. isHolonomic(J);
1
>
// Next consider
. ideal K = J[1],J[3..6];
> NF(J,slimgb(K));
_[1]=0
_[2]=165*x(1)*x(2)^2*D(1)^2-2*x(2)^3*D(1)^2-140*x(1)*x(2)^2*D(1)*D(2)+139*x(2)^3*D(1)*D(2)-112
*x(2)^3*D(2)^2+10368/5*x(1)*x(2)*D(1)^2-8424/5*x(1)*x(2)*D(1)*D(2)+1728*x(2)^2*D(1)*D(2)-3*x(1
)^2*D(2)^2+26/5*x(1)*x(2)*D(2)^2-1356*x(2)^2*D(2)^2-735*x(1)*x(2)*D(1)+990*x(2)^2*D(1)-1400*x(
2)^2*D(2)-32616/5*x(1)*D(1)+62208/5*x(2)*D(1)-14/5*x(1)*D(2)-74928/5*x(2)*D(2)-3675*x(2)-32616
_[3]=0
_[4]=0
_[5]=0
_[6]=0
> // Can we recover J from K?
. LIB "dmodloc.lib"; // not released yet
// ** loaded /usr/share/Singular/LIB/dmodloc.lib
// ** loaded /usr/share/Singular/LIB/intprog.lib (12231,2009-11-02)
> holonomicRank(K);
// Computing characteristic variety...
// ...done.
// Starting GB computation...
// ...done.
1
> printlevel = 2;    // let's see even more progress messages
> ideal KK = WeylClosure(K); KK;
// Computing characteristic variety...
// ...done.
// Starting GB computation...
// ...done.
// Starting to compute singular locus...
// Computing characteristic variety...
// ...done
// Computing saturation...
// ...done
// Computing elimination...
// ...done
// Computing radical...
// ...done
// ...done.
// x(1)*x(2)^5+x(2)^6+x(1)^5
// Found poly vanishing on singular locus: x(1)*x(2)^5+x(2)^6+x(1)^5
// Starting to compute localization...
// found GB wrt weight -1,0,0
// found b-function
// maximal integral root is 0
// found right normalforms
// found GB of free module of rank 1
// ...done.
// 5*x(1)^2*D(1)+6*x(1)*x(2)*D(1)+4*x(1)*x(2)*D(2)+5*x(2)^2*D(2)+75*x(1)+90*x(2),125*x(1)*x(2)
^3*D(1)+25*x(2)^4*D(1)+100*x(2)^4*D(2)+1296*x(1)*x(2)^2*D(1)-25*x(1)^3*D(2)+30*x(1)^2*x(2)*D(2
)-36*x(1)*x(2)^2*D(2)+1080*x(2)^3*D(2)+1875*x(2)^3+19440*x(2)^2,125*x(2)^4*D(1)^2-125*x(2)^4*D
(1)*D(2)-1296*x(1)*x(2)^2*D(1)^2+324*x(1)*x(2)^2*D(1)*D(2)-1080*x(2)^3*D(1)*D(2)+100*x(1)^3*D(
2)^2-15*x(1)^2*x(2)*D(2)^2+17*x(1)*x(2)^2*D(2)^2+240*x(2)^3*D(2)^2+7375*x(1)*x(2)^2*D(1)-400*x
(2)^3*D(1)+5900*x(2)^3*D(2)+77220*x(1)*x(2)*D(1)-20736*x(2)^2*D(1)+1460*x(1)^2*D(2)-1730*x(1)*
x(2)*D(2)+68946*x(2)^2*D(2)+110625*x(2)^2+1158300*x(2),15625*x(2)^5*D(1)-15625*x(2)^5*D(2)+324
00*x(2)^4*D(1)-15625*x(1)^4*D(2)+3125*x(1)^3*x(2)*D(2)-3750*x(1)^2*x(2)^2*D(2)+4500*x(1)*x(2)^
3*D(2)-5400*x(2)^4*D(2)-234375*x(2)^4+1679616*x(1)*x(2)^2*D(1)-32400*x(1)^3*D(2)+38880*x(1)^2*
x(2)*D(2)-46656*x(1)*x(2)^2*D(2)+1399680*x(2)^3*D(2)+25194240*x(2)^2,1975*x(1)*x(2)^2*D(1)^2-5
*x(2)^3*D(1)^2-1600*x(1)*x(2)^2*D(1)*D(2)+1660*x(2)^3*D(1)*D(2)-1280*x(2)^3*D(2)^2+24624*x(1)*
x(2)*D(1)^2-19332*x(1)*x(2)*D(1)*D(2)+20520*x(2)^2*D(1)*D(2)-20*x(1)^2*D(2)^2+43*x(1)*x(2)*D(2
)^2-15540*x(2)^2*D(2)^2-28400*x(1)*x(2)*D(1)+31600*x(2)^2*D(1)-48000*x(2)^2*D(2)-281988*x(1)*D
(1)+393984*x(2)*D(1)-52*x(1)*D(2)-530934*x(2)*D(2)-426000*x(2)-4229820,x(1)*x(2)^5*D(2)+x(2)^6
*D(2)+x(1)^5*D(2)+15*x(1)*x(2)^4+18*x(2)^5
// Starting to compute kernel of localization map...
// ...done.
```

```
KK[1]=5*x(1)^2*D(1)+6*x(1)*x(2)*D(1)+4*x(1)*x(2)*D(2)+5*x(2)^2*D(2)+25*x(1)+30*x(2)
KK[2]=223367315085112275*x(1)*x(2)^2*D(1)^2-2707482607092270*x(2)^3*D(1)^2-2541010473467081154
5*x(1)^3*D(1)*D(2)-42924942265467477534*x(1)^2*x(2)*D(1)*D(2)-15108903683131463316*x(1)*x(2)^2
*D(1)*D(2)+188170041192912765*x(2)^3*D(1)*D(2)-20328083787736649236*x(1)^2*x(2)*D(2)^2-3535635
8001760814489*x(1)*x(2)^2*D(2)^2-12584435609859670800*x(2)^3*D(2)^2-69499391710064460480*x(1)^
2*D(1)^2-80592152085044087040*x(1)*x(2)*D(1)^2+65275097484463840260*x(1)^2*D(1)*D(2)+204498202
65090511680*x(1)*x(2)*D(1)*D(2)-67160126737536739200*x(2)^2*D(1)*D(2)-4061223910638405*x(1)^2*
D(2)^2+52227117442349512110*x(1)*x(2)*D(2)^2+63439424276855281200*x(2)^2*D(2)^2-24651298209187
611429*x(1)^2*D(1)-8905386791840578731*x(1)*x(2)*D(1)+1340203890510673650*x(2)^2*D(1)-14737860
7461090706961*x(1)^2*D(2)-270718512085835123764*x(1)*x(2)*D(2)-95516943023447256015*x(2)^2*D(2
)-416996350260386762880*x(1)*D(1)-483552912510264522240*x(2)*D(1)+378591774934240344330*x(1)*D
(2)+446314640829238193040*x(2)*D(2)-123256491045938057145*x(1)-44526933959202893655*x(2)
KK[3]=108299304283690800*x(2)^4*D(1)^2+838533456244136780985*x(1)^4*D(1)*D(2)+2432927284147259
220422*x(1)^3*x(2)*D(1)*D(2)+2215591512162037390788*x(1)^2*x(2)^2*D(1)*D(2)+604043433084139375
455*x(1)*x(2)^3*D(1)*D(2)-155680249907805525*x(2)^4*D(1)*D(2)+670826764995309424788*x(1)^3*x(2
)*D(2)^2+1979883165567572847577*x(1)^2*x(2)^2*D(2)^2+1829540695195801715960*x(1)*x(2)^3*D(2)^2
+503377424394386832000*x(2)^4*D(2)^2+2293479926432127195840*x(1)^3*D(1)^2+54395166872090332915
20*x(1)^2*x(2)*D(1)^2+3223686083401763481600*x(1)*x(2)^2*D(1)^2-2154078216987306728580*x(1)^3*
D(1)*D(2)-3285847968126540495840*x(1)^2*x(2)*D(1)*D(2)+1398291371735091926400*x(1)*x(2)^2*D(1)
*D(2)+2686405069501469568000*x(2)^3*D(1)*D(2)+134020389051067365*x(1)^3*D(2)^2-172333242664110
8363430*x(1)^2*x(2)*D(2)^2-4182585698830204764000*x(1)*x(2)^2*D(2)^2-2537576971074211248000*x(
2)^3*D(2)^2+813492840903191177157*x(1)^3*D(1)+1279929692498243555283*x(1)^2*x(2)*D(1)+36358659
3071431854315*x(1)*x(2)^2*D(1)-536081556204269460*x(2)^3*D(1)+4863494046215993329713*x(1)^3*D(
2)+14828855197276187362652*x(1)^2*x(2)*D(2)+13980799603207164399055*x(1)*x(2)^2*D(2)+382657461
8056137204660*x(2)^3*D(2)+13760879558592763175040*x(1)^2*D(1)+32637100123254199749120*x(1)*x(2
)*D(1)+19342116500410580889600*x(2)^2*D(1)-12493528572829931362890*x(1)^2*D(2)-298720541447344
74143520*x(1)*x(2)*D(2)-17852585633169527721600*x(2)^2*D(2)+4067464204515955885785*x(1)^2+6399
648462491217776415*x(1)*x(2)+1817932965357159271575*x(2)^2
KK[4]=98127565*x(1)*x(2)^3*D(1)+19625513*x(2)^4*D(1)+78502052*x(2)^4*D(2)+60040172193*x(1)^3*D
(1)+124297757590*x(1)^2*x(2)*D(1)+63716847744*x(1)*x(2)^2*D(1)-19625513*x(1)^3*D(2)+4805568837
0*x(1)^2*x(2)*D(2)+101811552221*x(1)*x(2)^2*D(2)+53097373120*x(2)^3*D(2)+490637825*x(2)^3+3002
00860965*x(1)^2+621488787950*x(1)*x(2)+318584238720*x(2)^2
KK[5]=19625513*x(2)^5*D(1)-19625513*x(2)^5*D(2)+60040172193*x(1)^4*D(1)+184337929783*x(1)^3*x(
2)*D(1)+188014605334*x(1)^2*x(2)^2*D(1)+63716847744*x(1)*x(2)^3*D(1)-19625513*x(1)^4*D(2)+4803
6062857*x(1)^3*x(2)*D(2)+149867240591*x(1)^2*x(2)^2*D(2)+154908925341*x(1)*x(2)^3*D(2)+5309737
3120*x(2)^4*D(2)-98127565*x(2)^4+300200860965*x(1)^3+921689648915*x(1)^2*x(2)+940073026670*x(1
)*x(2)^2+318584238720*x(2)^3
KK[6]=98127565*x(1)*x(2)^5*D(2)+98127565*x(2)^6*D(2)-300200860965*x(1)^5*D(1)-981729821108*x(1
)^4*x(2)*D(1)-1064370784260*x(1)^3*x(2)^2*D(1)-382301086464*x(1)^2*x(2)^3*D(1)+98127565*x(1)^5
*D(2)-240160688772*x(1)^4*x(2)*D(2)-797391891325*x(1)^3*x(2)^2*D(2)-876356178926*x(1)^2*x(2)^3
*D(2)-318584238720*x(1)*x(2)^4*D(2)+490637825*x(1)*x(2)^4+588765390*x(2)^5-1501004304825*x(1)^
4-4908649105540*x(1)^3*x(2)-5321853921300*x(1)^2*x(2)^2-1911505432320*x(1)*x(2)^3
> NF(J,slimgb(KK));
_[1]=0
_[2]=0
_[3]=0
_[4]=0
_[5]=0
_[6]=0
> NF(KK,slimgb(J));
_[1]=0
_[2]=0
_[3]=0
_[4]=0
_[5]=0
_[6]=0
>
// Let’s have a look at varieties
. LIB "dmodvar.lib";
// ** loaded /usr/share/Singular/LIB/dmodvar.lib (13492,2010-10-14)
> ring rr = 0,(x,y,z),dp;
> ideal F = x2+y3, xyz+z-1;
> bfctVarAnn(F);
// Have not found smaller generating set of the given variety.
// The codim of the given variety is 2.
// -1-1- the ring @R(_Dt,_s,_x,_Dx) is ready
//   characteristic : 0
//   number of vars : 12
//        block   1 : ordering a
//                  : names    Dt(1) Dt(2)
//                  : weights      1     1
//        block   2 : ordering a
//                  : names    Dt(1) Dt(2) s(1)(1) s(1)(2) s(2)(1) s(2)(2)
//                  : weights      1       1       1       1       1       1
//        block   3 : ordering dp
//                  : names    Dt(1) Dt(2) s(1)(1) s(1)(2) s(2)(1) s(2)(2) x y z Dx Dy Dz
//        block   4 : ordering C
//   noncommutative relations:
//    s(1)(1)Dt(1)=Dt(1)*s(1)(1)+Dt(1)
//    s(2)(1)Dt(1)=Dt(1)*s(2)(1)+Dt(2)
//    s(1)(2)Dt(2)=Dt(2)*s(1)(2)+Dt(1)
//    s(2)(2)Dt(2)=Dt(2)*s(2)(2)+Dt(2)
//    s(1)(2)s(1)(1)=s(1)(1)*s(1)(2)-s(1)(2)
//    s(2)(1)s(1)(1)=s(1)(1)*s(2)(1)+s(2)(1)
//    s(2)(1)s(1)(2)=s(1)(2)*s(2)(1)-s(1)(1)+s(2)(2)
```

```
//    s(2)(2)s(1)(2)=s(1)(2)*s(2)(2)-s(1)(2)
//    s(2)(2)s(2)(1)=s(2)(1)*s(2)(2)+s(2)(1)
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
//    Dzz=z*Dz+1
// -1-2- starting the elimination of Dt(i) in @R
Dt(1)*y^3+Dt(1)*x^2+s(1)(1),
Dt(1)*x*y*z+Dt(1)*z-Dt(1)+s(1)(2),
Dt(2)*y^3+Dt(2)*x^2+s(2)(1),
Dt(2)*x*y*z+Dt(2)*z-Dt(2)+s(2)(2),
Dt(2)*y*z+2*Dt(1)*x+Dx,
3*Dt(1)*y^2+Dt(2)*x*z+Dy,
Dt(2)*x*y+Dt(2)+Dz
// -1-3- all Dt(i) are eliminated
5*s(2)(2)*z+6*s(1)(1)-5*z^2*Dz-3*x*Dx-2*y*Dy+5*z*Dz,
s(2)(2)*x*y+s(2)(2)-x*y*z*Dz-z*Dz+Dz,
6*s(1)(1)*x*y+6*s(1)(1)-3*x^2*y*Dx-2*x*y^2*Dy+5*x*y*z*Dz-3*x*Dx-2*y*Dy,
s(2)(2)*y^3+s(2)(2)*x^2+s(2)(1)-y^3*z*Dz-x^2*z*Dz,
10*s(1)(2)*x-6*s(1)(1)*y-5*x*y*z*Dx+5*y*z^2*Dz+3*x*y*Dx+2*y^2*Dy-5*y*z*Dz-5*z*Dx+5*Dx,
2*s(1)(1)*x^2-s(2)(1)*z-x*y^3*Dx+y^3*z*Dz-x^3*Dx+x^2*z*Dz,
s(2)(2)*x^3-s(2)(2)*y^2+s(2)(1)*x-x^3*z*Dz+y^2*z*Dz-y^2*Dz,
10*s(1)(1)*s(2)(2)*x-6*s(1)(1)*s(2)(1)*y+3*s(2)(1)*x*y*Dx+2*s(2)(1)*y^2*Dy-10*s(1)(1)*x*z*Dz-5
*s(2)(1)*y*z*Dz+10*s(2)(2)*x-6*s(2)(1)*y+5*s(2)(1)*Dx-10*x*z*Dz,
[...]
// -2-1- the ring @R(_s,_x,_Dx) is ready
//   characteristic : 0
//   number of vars : 10
//        block   1 : ordering a
//                  : names    s(1)(1) s(1)(2) s(2)(1) s(2)(2)
//                  : weights        1       1       1       1
//        block   2 : ordering dp
//                  : names    s(1)(1) s(1)(2) s(2)(1) s(2)(2) x y z Dx Dy Dz
//        block   3 : ordering C
//   noncommutative relations:
//    s(1)(2)s(1)(1)=s(1)(1)*s(1)(2)-s(1)(2)
//    s(2)(1)s(1)(1)=s(1)(1)*s(2)(1)+s(2)(1)
//    s(2)(1)s(1)(2)=s(1)(2)*s(2)(1)-s(1)(1)+s(2)(2)
//    s(2)(2)s(1)(2)=s(1)(2)*s(2)(2)-s(1)(2)
//    s(2)(2)s(2)(1)=s(2)(1)*s(2)(2)+s(2)(1)
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
//    Dzz=z*Dz+1
// -2-2- starting cosmetic Groebner basis computation
5*s(2)(2)*z+6*s(1)(1)-5*z^2*Dz-3*x*Dx-2*y*Dy+5*z*Dz,
s(2)(2)*x*y+s(2)(2)-x*y*z*Dz-z*Dz+Dz,
6*s(1)(1)*x*y+6*s(1)(1)-3*x^2*y*Dx-2*x*y^2*Dy+5*x*y*z*Dz-3*x*Dx-2*y*Dy,
s(2)(2)*y^3+s(2)(2)*x^2+s(2)(1)-y^3*z*Dz-x^2*z*Dz,
10*s(1)(2)*x-6*s(1)(1)*y-5*x*y*z*Dx+5*y*z^2*Dz+3*x*y*Dx+2*y^2*Dy-5*y*z*Dz-5*z*Dx+5*Dx,
2*s(1)(1)*x^2-s(2)(1)*z-x*y^3*Dx+y^3*z*Dz-x^3*Dx+x^2*z*Dz,
s(2)(2)*x^3-s(2)(2)*y^2+s(2)(1)*x-x^3*z*Dz+y^2*z*Dz-y^2*Dz,
10*s(1)(1)*s(2)(2)*x-6*s(1)(1)*s(2)(1)*y+3*s(2)(1)*x*y*Dx+2*s(2)(1)*y^2*Dy-10*s(1)(1)*x*z*Dz-5
*s(2)(1)*y*z*Dz+10*s(2)(2)*x-6*s(2)(1)*y+5*s(2)(1)*Dx-10*x*z*Dz,
[...]
// -2-3- the cosmetic Groebner basis has been computed
3*x*y^3*Dx-3*y^3*z*Dz-2*x^2*y*Dy+2*x^2*z*Dz+3*y^2*Dx-2*x*Dy,
s(1)(2)*s(2)(1)-s(1)(1)*s(2)(2)-s(1)(1),
5*s(2)(2)*z+6*s(1)(1)-5*z^2*Dz-3*x*Dx-2*y*Dy+5*z*Dz,
10*s(1)(2)*x-6*s(1)(1)*y-5*x*y*z*Dx+5*y*z^2*Dz+3*x*y*Dx+2*y^2*Dy-5*y*z*Dz-5*z*Dx+5*Dx,
s(2)(1)*y*z+2*s(1)(1)*x-y^3*Dx-x^2*Dx,
15*s(1)(2)*y^2-6*s(1)(1)*x-5*x*y*z*Dy+5*x*z^2*Dz+3*x^2*Dx+2*x*y*Dy-5*x*z*Dz-5*z*Dy+5*Dy,
3*s(1)(1)*y^2+s(2)(1)*x*z-y^3*Dy-x^2*Dy,
s(2)(2)*x*y+s(2)(2)-x*y*z*Dz-z*Dz+Dz,
[...]
// -3-1- starting Groebner basis of ann F^s + F
-3*x^3*Dx-2*x^2*y*Dy+5*x^2*z*Dz+3*y^2*Dx-6*x^2-2*x*Dy,
s(1)(2)*s(2)(1)-s(1)(1)*s(2)(2)-s(1)(1),
5*s(2)(2)*z+6*s(1)(1)-5*z^2*Dz-3*x*Dx-2*y*Dy+5*z*Dz,
10*s(1)(2)*x-6*s(1)(1)*y+5*y*z^2*Dz+3*x*y*Dx+2*y^2*Dy-5*y*z*Dz+5*y*z,
s(2)(1)*y*z+2*s(1)(1)*x+2*x,
15*s(1)(2)*y^2-6*s(1)(1)*x+5*x*z^2*Dz+3*x^2*Dx+2*x*y*Dy-5*x*z*Dz+5*x*z,
3*s(1)(1)*y^2+s(2)(1)*x*z+3*y^2,
s(2)(2)*x*y+s(2)(2)+x*y+1,
[...]
// -3-2- finished Groebner basis of ann F^s + F
x*y*z+z-1,
s(2)(1),
y^3+x^2,
s(2)(2)+1,
6*s(1)(1)-5*z^2*Dz-3*x*Dx-2*y*Dy+5*z*Dz-5*z,
5*z^3*Dz+3*x*z*Dx+2*y*z*Dy-10*z^2*Dz+5*z^2-3*x*Dx-2*y*Dy+5*z*Dz+z-6,
5*x*z^2*Dz+3*x^2*Dx+2*x*y*Dy-5*x*z*Dz+5*x*z+6*x,
3*x^2*y*Dx+2*x*y^2*Dy-5*x*y*z*Dz-5*z^2*Dz+6*x*y+5*z*Dz-5*z,
3*x^3*Dx+2*x^2*y*Dy-5*x^2*z*Dz-3*y^2*Dx+6*x^2+2*x*Dy,
s(1)(2)*x,
s(1)(2)*y^2,
```

```
s(1)(2)*z-s(1)(2),
x^3*z-y^2*z+y^2,
5*y^2*z^2*Dz+3*x*y^2*Dx-5*y^2*z*Dz+5*y^2*z-2*x^2*Dy
// -4-1- computing the minimal polynomial of S
S = s(1)(1)+s(2)(2)
// lower bound for the degree of the insection is 1
// Testing degree 1
// Testing degree 2
// Testing degree 3
// degree of the generator of the intersection is: 3
// -4-2- the minimal polynomial has been computed
s3+6s2+431/36s+143/18
// -5-1- codimension of the variety
2
// -5-2- shifting BS(s)=minpoly(s-codim+1)
s3+3s2+107/36s+35/36
// -5-3- factorization of the minimal polynomial
// found roots
// no irreducible factors found
[1]:
   _[1]=-5/6
   _[2]=-1
   _[3]=-7/6
[2]:
   1,1,1
> bfctVarIn(F);
// Have not found smaller generating set of the given variety.
// Computing in 5-th Weyl algebra:
//   characteristic : 0
//   number of vars : 10
//        block   1 : ordering dp
//                  : names    t(1) t(2) x y z Dt(1) Dt(2) Dx Dy Dz
//        block   2 : ordering C
//   noncommutative relations:
//    Dt(1)t(1)=t(1)*Dt(1)+1
//    Dt(2)t(2)=t(2)*Dt(2)+1
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
//    Dzz=z*Dz+1
// The Malgrange ideal:
-y^3-x^2+t(1),
-x*y*z+t(2)-z+1,
y*z*Dt(2)+2*x*Dt(1)+Dx,
3*y^2*Dt(1)+x*z*Dt(2)+Dy,
x*y*Dt(2)+Dt(2)+Dz
// Computing the b-function of the Malgrange ideal...
// ... done.
// The b-function:
[1]:
   _[1]=1/6
   _[2]=0
   _[3]=-1/6
[2]:
   1,1,1
[1]:
   _[1]=-5/6
   _[2]=-1
   _[3]=-7/6
[2]:
   1,1,1
>
// my last topic for today: Bernstein-Sato ideals
. setring r; r;
//   characteristic : 0
//   number of vars : 2
//        block   1 : ordering dp
//                  : names    x y
//        block   2 : ordering C
> ideal F = x3+y2, y3+x2;
> // Ucha, Castro (1994): "At the moment it
  // seems to be intractable with any method."
. def AA = annfsBMI(F); setring AA; BS;
// -1-1- the ring @R(_t,_s,_x,_Dx) is ready
//   characteristic : 0
//   number of vars : 8
//        block   1 : ordering lp
//                  : names    t(1) t(2) s(1) s(2)
//        block   2 : ordering dp
//                  : names    x y Dx Dy
//        block   3 : ordering C
//   noncommutative relations:
//    s(1)t(1)=t(1)*s(1)+t(1)
//    s(2)t(2)=t(2)*s(2)+t(2)
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
```

```
// -1-2- starting the elimination of t(1)t(2) in @R
t(1)*x^3+t(1)*y^2+s(1),
t(2)*y^3+t(2)*x^2+s(2),
3*t(1)*x^2+2*t(2)*x+Dx,
2*t(1)*y+3*t(2)*y^2+Dy
// -1-3- all t(i) are eliminated
18*s(1)*x*y^2-8*s(1)*y+27*s(2)*x*y^2-12*s(2)*y-6*x^2*y^2*Dx-9*x*y^3*Dy-5*x^3*Dy+6*x*y*Dx+4*y^2
*Dy,
6*s(1)*y^3+6*s(1)*x^2+9*s(2)*y^3+4*s(2)*x^2-2*x*y^3*Dx-3*y^4*Dy-2*x^3*Dx-3*x^2*y*Dy,
27*s(1)*x^2*y-12*s(1)*x+18*s(2)*x^2*y-8*s(2)*x-9*x^3*y*Dx-6*x^2*y^2*Dy-5*y^3*Dx+4*x^2*Dx+6*x*y
*Dy,
9*s(1)*x^3+4*s(1)*y^2+6*s(2)*x^3+6*s(2)*y^2-3*x^4*Dx-2*x^3*y*Dy-3*x*y^2*Dx-2*y^3*Dy,
9*s(2)*x^2*y^2-4*s(2)*x*y-3*x^2*y^3*Dy+2*y^4*Dx-3*x^4*Dy+2*x^2*y*Dx,
54*s(1)^2*x*y-24*s(1)^2+117*s(1)*s(2)*x*y-52*s(1)*s(2)-36*s(1)*x^2*y*Dx-39*s(1)*x*y^2*Dy+26*s(
1)*x*Dx+24*s(1)*y*Dy+54*s(2)^2*x*y-24*s(2)^2-39*s(2)*x^2*y*Dx-36*s(2)*x*y^2*Dy+24*s(2)*x*Dx+26
*s(2)*y*Dy+6*x^3*y*Dx^2+13*x^2*y^2*Dx*Dy+6*x*y^3*Dy^2+6*x^2*y*Dx-6*x^2*Dx^2+6*x*y^2*Dy-13*x*y*
Dx*Dy-6*y^2*Dy^2-6*x*Dx-6*y*Dy
// -2-1- the ring @R2(_x,_Dx,_s) is ready
//   characteristic : 0
//   number of vars : 6
//        block   1 : ordering dp
//                  : names    x y Dx Dy
//        block   2 : ordering dp
//                  : names    s(1) s(2)
//        block   3 : ordering C
//   noncommutative relations:
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
// -2-1-1 computing the ideal B from Castro-Ucha
// -2-2- starting the elimination of _x,_Dx in @R2
-6*x^2*y^2*Dx-9*x*y^3*Dy-5*x^3*Dy+18*x*y^2*s(1)+27*x*y^2*s(2)+6*x*y*Dx+4*y^2*Dy-8*y*s(1)-12*y*
s(2),
-2*x*y^3*Dx-3*y^4*Dy-2*x^3*Dx-3*x^2*y*Dy+6*y^3*s(1)+9*y^3*s(2)+6*x^2*s(1)+4*x^2*s(2),
-9*x^3*y*Dx-6*x^2*y^2*Dy-5*y^3*Dx+27*x^2*y*s(1)+18*x^2*y*s(2)+4*x^2*Dx+6*x*y*Dy-12*x*s(1)-8*x*
s(2),
-3*x^4*Dx-2*x^3*y*Dy-3*x*y^2*Dx-2*y^3*Dy+9*x^3*s(1)+6*x^3*s(2)+4*y^2*s(1)+6*y^2*s(2),
-3*x^2*y^3*Dy+2*y^4*Dx-3*x^4*Dy+9*x^2*y^2*s(2)+2*x^2*y*Dx-4*x*y*s(2),
6*x^3*y*Dx^2+13*x^2*y^2*Dx*Dy+6*x*y^3*Dy^2-36*x^2*y*Dx*s(1)-39*x^2*y*Dx*s(2)+6*x^2*y*Dx-6*x^2*
Dx^2-39*x*y^2*Dy*s(1)-36*x*y^2*Dy*s(2)+6*x*y^2*Dy-13*x*y*Dx*Dy-6*y^2*Dy^2+54*x*y*s(1)^2+117*x*
y*s(1)*s(2)+54*x*y*s(2)^2+26*x*Dx*s(1)+24*x*Dx*s(2)-6*x*Dx+24*y*Dy*s(1)+26*y*Dy*s(2)-6*y*Dy-24
*s(1)^2-52*s(1)*s(2)-24*s(2)^2,
x^3*y^3+x^5+y^5+x^2*y^2
// -2-3- _x,_Dx are eliminated in @R2
7962624*s(1)^11*s(2)+86261760*s(1)^10*s(2)^2+413614080*s(1)^9*s(2)^3+1154949120*s(1)^8*s(2)^4+
2078423040*s(1)^7*s(2)^5+2517578752*s(1)^6*s(2)^6+2078423040*s(1)^5*s(2)^7+1154949120*s(1)^4*s
(2)^8+413614080*s(1)^3*s(2)^9+86261760*s(1)^2*s(2)^10+7962624*s(1)*s(2)^11+7962624*s(1)^11+243
523584*s(1)^10*s(2)+1943101440*s(1)^9*s(2)^2+7664947200*s(1)^8*s(2)^3+17991874560*s(1)^7*s(2)^
4+27148590592*s(1)^6*s(2)^5+27148590592*s(1)^5*s(2)^6+17991874560*s(1)^4*s(2)^7+7664947200*s(1
)^3*s(2)^8+1943101440*s(1)^2*s(2)^9+243523584*s(1)*s(2)^10+7962624*s(2)^11+157261824*s(1)^10+2
918522880*s(1)^9*s(2)+18515589120*s(1)^8*s(2)^2+60704916480*s(1)^7*s(2)^3+118952120320*s(1)^6*
s(2)^4+148003178752*s(1)^5*s(2)^5+118952120320*s(1)^4*s(2)^6+60704916480*s(1)^3*s(2)^7+1851558
9120*s(1)^2*s(2)^8+2918522880*s(1)*s(2)^9+157261824*s(2)^10+1389035520*s(1)^9+19248399360*s(1)
^8*s(2)+99728870400*s(1)^7*s(2)^2+271132637440*s(1)^6*s(2)^3+437001057280*s(1)^5*s(2)^4+437001
057280*s(1)^4*s(2)^5+271132637440*s(1)^3*s(2)^6+99728870400*s(1)^2*s(2)^7+19248399360*s(1)*s(2
)^8+1389035520*s(2)^9+7242808320*s(1)^8+79709235072*s(1)^7*s(2)+339102319168*s(1)^6*s(2)^2+755
319074560*s(1)^5*s(2)^3+977371125760*s(1)^4*s(2)^4+755319074560*s(1)^3*s(2)^5+339102319168*s(1
)^2*s(2)^6+79709235072*s(1)*s(2)^7+7242808320*s(2)^8+24771829632*s(1)^7+220641065440*s(1)^6*s(
2)+764625985888*s(1)^5*s(2)^2+1364374119040*s(1)^4*s(2)^3+1364374119040*s(1)^3*s(2)^4+76462598
5888*s(1)^2*s(2)^5+220641065440*s(1)*s(2)^6+24771829632*s(2)^7+58350275232*s(1)^6+419961051088
*s(1)^5*s(2)+1162003538560*s(1)^4*s(2)^2+1600875970240*s(1)^3*s(2)^3+1162003538560*s(1)^2*s(2)
^4+419961051088*s(1)*s(2)^5+58350275232*s(2)^6+96586093680*s(1)^5+552457792960*s(1)^4*s(2)+117
8105113360*s(1)^3*s(2)^2+1178105113360*s(1)^2*s(2)^3+552457792960*s(1)*s(2)^4+96586093680*s(2)
^5+112343110080*s(1)^4+493940668624*s(1)^3*s(2)+763986500692*s(1)^2*s(2)^2+493940668624*s(1)*s
(2)^3+112343110080*s(2)^4+89977426944*s(1)^3+286546685206*s(1)^2*s(2)+286546685206*s(1)*s(2)^2
+89977426944*s(2)^3+47255546994*s(1)^2+97245732337*s(1)*s(2)+47255546994*s(2)^2+14645706075*s(
1)+14645706075*s(2)+2029052025
// -3-1- the ring @R3(_s) is ready
// -3-2- the Bernstein ideal is principal
// -3-2-1- factorization
// -4-1- the ring @R4i(_x,_Dx,_s) is ready
//   characteristic : 0
//   number of vars : 6
//        block   1 : ordering dp
//                  : names    x y Dx Dy s(1) s(2)
//        block   2 : ordering C
//   noncommutative relations:
//    Dxx=x*Dx+1
//    Dyy=y*Dy+1
// -4-2- the final cosmetic std
[1]:
   _[1]=s(1)+1
   _[2]=s(2)+1
   _[3]=6*s(1)+4*s(2)+7
   _[4]=1327104*s(1)^9+13492224*s(1)^8*s(2)+59940864*s(1)^7*s(2)^2+152530944*s(1)^6*s(2)^3+244
716544*s(1)^5*s(2)^4+256452096*s(1)^4*s(2)^5+175435776*s(1)^3*s(2)^6+75534336*s(1)^2*s(2)^7+18
```

```
579456*s(1)*s(2)^8+1990656*s(2)^9+23334912*s(1)^8+209240064*s(1)^7*s(2)+806639616*s(1)^6*s(2)^
2+1744037888*s(1)^5*s(2)^3+2310570240*s(1)^4*s(2)^4+1919190528*s(1)^3*s(2)^5+975614976*s(1)^2*
s(2)^6+277530624*s(1)*s(2)^7+33841152*s(2)^8+179398656*s(1)^7+1396681728*s(1)^6*s(2)+457697945
6*s(1)^5*s(2)^2+8174403840*s(1)^4*s(2)^3+8584802560*s(1)^3*s(2)^4+5298177024*s(1)^2*s(2)^5+177
8844672*s(1)*s(2)^6+250712064*s(2)^7+791213568*s(1)^6+5238902272*s(1)^5*s(2)+14186997120*s(1)^
4*s(2)^2+20089437440*s(1)^3*s(2)^3+15675678720*s(1)^2*s(2)^4+6387748992*s(1)*s(2)^5+1062021888
*s(2)^6+2205043776*s(1)^5+12070543200*s(1)^4*s(2)+25923904640*s(1)^3*s(2)^2+27276357120*s(1)^2
*s(2)^3+14049900160*s(1)*s(2)^4+2833651104*s(2)^5+4024368528*s(1)^4+17477988448*s(1)^3*s(2)+27
895831488*s(1)^2*s(2)^2+19372741568*s(1)*s(2)^3+4936489968*s(2)^4+4805666064*s(1)^3+1551448204
8*s(1)^2*s(2)+16343141312*s(1)*s(2)^2+5612286576*s(2)^3+3616478592*s(1)^2+7706746216*s(1)*s(2)
+4013131992*s(2)^2+1553923800*s(1)+1636742250*s(2)+289864575
[2]:
   1,1,1,1
> size(BS[1][4]);
55
> printlevel=0;                // show no progress messages
> setring r;
def BB = annfsBMI(F,0,-1);   // B_Sigma
setring BB; BS;
BS[1]=576*s(1)^4*s(2)+2496*s(1)^3*s(2)^2+3856*s(1)^2*s(2)^3+2496*s(1)*s(2)^4+576*s(2)^5+576*s(
1)^4+5376*s(1)^3*s(2)+12976*s(1)^2*s(2)^2+11616*s(1)*s(2)^3+3456*s(2)^4+2880*s(1)^3+14396*s(1)
^2*s(2)+19968*s(1)*s(2)^2+8156*s(2)^3+5276*s(1)^2+15048*s(1)*s(2)+9476*s(2)^2+4200*s(1)+5425*s
(2)+1225
BS[2]=576*s(1)^5+1920*s(1)^4*s(2)+1360*s(1)^3*s(2)^2-1360*s(1)^2*s(2)^3-1920*s(1)*s(2)^4-576*s
(2)^5+2880*s(1)^4+6240*s(1)^3*s(2)-6240*s(1)*s(2)^3-2880*s(2)^4+5276*s(1)^3+5572*s(1)^2*s(2)-5
572*s(1)*s(2)^2-5276*s(2)^3+4200*s(1)^2-4200*s(2)^2+1225*s(1)-1225*s(2)
> setring r;
def B1 = annfsBMI(F,0,1);    // B_1
setring B1; BS;
[1]:
   _[1]=s(1)+1
   _[2]=4*s(1)+6*s(2)+7
   _[3]=864*s(1)^4+3024*s(1)^3*s(2)+3744*s(1)^2*s(2)^2+1984*s(1)*s(2)^3+384*s(2)^4+4104*s(1)^3
+10728*s(1)^2*s(2)+8832*s(1)*s(2)^2+2336*s(2)^3+7212*s(1)^2+12476*s(1)*s(2)+5112*s(2)^2+5550*s
(1)+4750*s(2)+1575
[2]:
   1,1,1
> setring r;
def B2 = annfsBMI(F,0,2);    // B_2
setring B2; BS;
[1]:
   _[1]=s(2)+1
   _[2]=4*s(1)+6*s(2)+7
   _[3]=576*s(1)^4+2496*s(1)^3*s(2)+3856*s(1)^2*s(2)^2+2496*s(1)*s(2)^3+576*s(2)^4+3168*s(1)^3
+9936*s(1)^2*s(2)+9824*s(1)*s(2)^2+3072*s(2)^3+6212*s(1)^2+12576*s(1)*s(2)+6012*s(2)^2+5200*s(
1)+5100*s(2)+1575
[2]:
   1,1,1
>
```