Recent Progress in an Algebraic Analysis Approach to Linear Systems

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1 Introduction

This article summarizes recent progress in an algebraic analysis approach to systems of linear functional equations and emphasizes computational issues. Concentrating predominantly on work by a small group of people including the author of this article, the paper adopts a rather subjective viewpoint. Nevertheless, we hope that this article could also serve as an introduction to this subject.

We assume that a system of linear functional equations is given; e.g., a system of linear ordinary or partial differential equations, a linear differential time-delay system, or a multidimensional discrete linear system, etc. The origin of such a system could be mathematical physics or the engineering sciences, prominently control theory; e.g., the given equations could describe a finite-dimensional deterministic linear control system. The equations are assumed to be linear in the unknown functions and their derivatives, shifts, etc., but their coefficients may be non-constant (e.g., time-varying).

Employing the philosophy and the techniques of algebraic analysis and the theory of D-modules, i.e., the theory of modules over rings of partial differential operators, cf., e.g., [Kas95], [Bjö79], we associate to the system a module over a (not necessarily commutative) ring D which contains the operators that occur in the given description of the linear system. The system module reflects structural properties of the solution set of the given system in an appropriate signal space. In this article we mainly address certain degrees of parametrizability of this solution set, which is also referred to as a behavior. The favorable case is that the vector space of solutions can be realized as the image of a D-linear map. An even more desirable situation is given if such a linear map can be chosen to be injective. An example is provided by the de Rham complex on an open and convex subset of $\mathbb{R}^n$ (which we consider as a smooth manifold). The behavior defined by the divergence operator admits a parametrization in terms of the curl operator (not, however, an injective one), and the solutions of the system defined by the curl operator are parametrized by the gradient operator (cf. Example 4.2 below). Apart from the general interest in solving systems of linear functional equations, we note that the notion of parametrizability is equivalent to controllability in the context of linear control theory (cf. Section 4), which explains the impact of the above questions.

Methodically, this paper employs module theory over non-commutative rings, in particular, Ore algebras, as well as homological algebra. We owe much to work by many other authors, in particular, to the works in the following (not exhaustive) list: [Obe90, PQ99b, Qua99, Woo00, Zer00, Pom01, Zer01, PQ03, Qua10a, Qua10b] (in order of appearance).

For lack of space, more recent developments could not be included in this survey. For instance, recent progress in the study of autonomous systems using the technique of purity or grade filtration, cf., e.g., [Bar10], [Qua13]. In the present context, effective methods have been developed to simplify (e.g., factorize) systems of linear functional equations (cf., e.g., [CQ08]), in particular, the technique of "Serre's reduction" (cf. [BQ10], [CQ12]), which tries to reduce the number of equations and the number of unknowns of a system of linear functional equations. Moreover, recent work (cf. [QR13b, QR]) developing effective versions of Stafford's theorems (cf. [Sta78]) can only be mentioned here. (Subsection 5.4 presents only one of Stafford's theorems.) Specific results about (multidimensional) codes cannot be dealt with here either (cf., e.g., [LLO04]).

Section 2 introduces the point of view adopted in this paper, using module theory and homological algebra. We concentrate on a certain class of rings in what follows, which is introduced in Section 3. This choice allows to perform effectively the module-theoretic constructions which are necessary to study the structural properties of systems of linear functional equations, using Janet bases or Gröbner bases. The central Section 4 addresses the problem of deciding whether or not the solution set of a linear system can be parametrized. In the context of control theory this property amounts to controllability of the system. The more refined question about injective parametrizability, i.e., flatness of linear control systems, is dealt with in Section 5. Finally, Section 6 lists several software packages which have been developed in the context of investigating the topics of this article. We conclude in Section 7.

The following *notation* is used throughout this paper. If D is a ring and $R \in D^{q \times p}$, then we denote by $.R$ the homomorphism $D^{1 \times q} \to D^{1 \times p}$ of left D-modules which is defined by right multiplication with R. Similarly, and more generally, for any left D-module $\mathcal{F}$ we write $R.$ for the homomorphism $\mathcal{F}^{p \times 1} \to \mathcal{F}^{q \times 1}$ of abelian groups induced by the left action of R on column vectors with entries in $\mathcal{F}$, the reference of $R.$ to $\mathcal{F}$ being clear from the context. Finally, we denote by I_n the $(n \times n)$ identity matrix and by $\mathrm{GL}(n, D)$ the group of $(n \times n)$ matrices with entries in D which are invertible over D (i.e., the general linear group). All rings will be associative algebras with an identity element 1, all ring homomorphisms will preserve the identity elements, and all modules will be unital. We use the standard notations $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ for the sets of natural numbers, integers, rational, real, and complex numbers, respectively.

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2 The system module and behaviors

Inspired by conventions in the study of modules over rings of differential operators, we denote by D the ring of linear functional operators that allow to formulate a given system of linear functional equations as

$$R\,y = 0, \qquad R \in D^{q \times p}, \qquad y \text{ a vector of } p \text{ unknowns}. \tag{1}$$

We assume that D is a (not necessarily commutative) Noetherian domain, i.e., the product (composition) of two non-zero elements (operators) in D is non-zero, and every submodule of a finitely generated (left or right) D-module is finitely generated. This entails a straightforward generalization of the notion of rank from vector spaces to D-modules. For computational purposes we confine ourselves to certain iterated skew polynomial rings[1] as defined in the next section.

The set in which solutions y of (1) are to be found is assumed to be of the form $\mathcal{F}^{p\times 1}$, where $\mathcal{F}$ is a left D-module, the left action of D being chosen accordingly to the given equations in (1). We refer to $\mathcal{F}$ as a *signal space*.

All (linear) consequences of (1) are obtained by multiplying the equation by matrices with q columns and entries in D from the left. Hence, we study the cokernel of the homomorphism $.R : D^{1\times q} \to D^{1\times p}$ of left D-modules induced by R, which is in some sense an intrinsic representation of (1) (cf. Remark 2.10).

Definition 2.1. We refer to the left D-module

$$M := D^{1\times p}/D^{1\times q}R$$

as the *system module* defined by $R\,y = 0$.

We denote by $e_1 := (1, 0, \ldots, 0), \ldots, e_p := (0, \ldots, 0, 1)$ the standard basis vectors of $D^{1\times p}$.

Remark 2.2. The above description of M is a *finite presentation* of M in terms of generators and relations. We call R a *presentation matrix* of M. The residue classes $e_1 + D^{1\times q}\,R, \ldots, e_p + D^{1\times q}\,R$ of $e_1, \ldots, e_p$ form a generating set for M, and the rows of R form a generating set for the left D-linear relations that are satisfied by these generators (in the given order). If $\pi : D^{1\times p} \to M$ denotes the canonical projection, then every element of $\ker(\pi)$ is a left D-linear combination of the rows of R. Usually we express these facts by saying that

$$0 \longleftarrow M \xleftarrow{\ \pi\ } D^{1\times p} \xleftarrow{\ .R\ } D^{1\times q}$$

is an *exact sequence* of left D-modules, i.e., the kernel of each homomorphism in this sequence coincides with the image of the previous homomorphism if present. A sequence of left D-modules and homomorphisms as above which satisfies the weaker condition that the composition of each two consecutive homomorphisms is the zero map is called a *complex* of left D-modules.

Remark 2.3. A module-theoretic construction allows to relate the *set of solutions* or *behavior* of the linear system (1) in $\mathcal{F}^{p\times 1}$

$$\mathrm{Sol}_{\mathcal{F}}(R) := \{\, y \in \mathcal{F}^{p\times 1} \mid R\,y = 0 \,\}$$

to the system module $M = D^{1\times p}/D^{1\times q}R$. By definition of a solution y of (1) with components $y_1, \ldots, y_p$, the homomorphism of left D-modules

$$D^{1\times p} \longrightarrow \mathcal{F} : e_i \longmapsto y_i, \qquad i = 1, \ldots, p,$$

induces a homomorphism of left D-modules $M \to \mathcal{F}$. Conversely, any such homomorphism $\varphi : M \to \mathcal{F}$ defines a solution

$$(\varphi(e_1 + D^{1\times q}R), \ldots, \varphi(e_p + D^{1\times q}R))^T \in \mathrm{Sol}_{\mathcal{F}}(R).$$

[1] The assumption that D is a left Noetherian domain actually implies (cf. [MR00], Thm. 2.1.15) that D satisfies the *left Ore condition*, i.e., every pair of non-zero elements of D has a non-zero common left multiple, a property shared by all Ore algebras (cf. also Proposition 3.7).

This correspondence (observed by Malgrange in [Mal62, Subsect. 3.2]) establishes an isomorphism

$$\hom_D(M, \mathcal{F}) \cong \mathrm{Sol}_{\mathcal{F}}(R) \tag{2}$$

of abelian groups.

Example 2.4. Let Ω be an open and convex subset of $\mathbb{R}^2$, $\mathcal{F}$ the real vector space of all smooth real functions on Ω, and $D = \mathbb{R}[\partial_{x_1}, \partial_{x_2}]$ the commutative algebra of polynomials in ∂_{x_1}, ∂_{x_2} with real coefficients. Then $\mathcal{F}$ is a D-module, where ∂_{x_i} acts on functions by partial differentation with respect to x_i, $i = 1, 2$. Let us consider the linear system $R\,y = 0$, where

$$R := \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix} \in D^{2\times 1},$$

i.e., $p = 1$, $q = 2$, which can be understood as the gradient operator on the two-dimensional smooth manifold $\mathbb{R}^2$. The system module is defined by $M := D/D^{1\times 2}\,R$. For any solution $y \in \mathcal{F}$, the map

$$M \longrightarrow \mathcal{F} : d\,(e_1 + D^{1\times 2}\,R) \longmapsto d\,y, \qquad d \in D,$$

is well-defined (i.e., $d\,e_1 \in D^{1\times 2}\,R$ implies $d\,y = 0$), and is D-linear. On the other hand, given a homomorphism $\varphi : M \to \mathcal{F}$, we have

$$\partial_{x_i}\,\varphi(e_1 + D^{1\times 2}\,R) = \varphi(\partial_{x_i}\,e_1 + D^{1\times 2}\,R) = \varphi(0 + D^{1\times 2}\,R) = 0, \qquad i = 1, 2.$$

Certain manipulations of the system equations or the system module should be reflected by certain operations on the solutions of the system or the signal space, as exemplified next.

Example 2.5. In the context of the previous example, we consider the inhomogeneous linear system $R\,y = u$, where $u \in \mathcal{F}^{2\times 1}$ is given. Every matrix $S \in D^{r\times q}$ for some $r \in \mathbb{N}$ which satisfies $S\,R = 0$ yields a *compatibility condition* $S\,u = 0$ for $R\,y = u$ to be solvable. The fact that D is Noetherian implies that there exists $R_2 \in D^{p_2\times q}$ such that every matrix S as above is a left multiple of R_2. In other words, we have the exact sequence of D-modules

$$0 \longleftarrow M \longleftarrow D^{1\times p_0} \xleftarrow{.R_1} D^{1\times p_1} \xleftarrow{.R_2} D^{1\times p_2} \tag{3}$$

where $p_0 := p$, $p_1 := q$, and $R_1 := R$. We can choose (by a computation described in Example 3.10) $R_2 := \begin{pmatrix} \partial_{x_2} & -\partial_{x_1} \end{pmatrix} \in D^{1\times 2}$. The contravariant functor $\hom_D(-, \mathcal{F})$ transforms the exact sequence of D-modules (3) into the complex of $\mathbb{R}$-vector spaces

$$0 \longrightarrow \hom_D(M, \mathcal{F}) \longrightarrow \mathcal{F}^{p_0\times 1} \xrightarrow{(R_1).} \mathcal{F}^{p_1\times 1} \xrightarrow{(R_2).} \mathcal{F}^{p_2\times 1} \tag{4}$$

where canonical isomorphisms $\hom_D(D^{1\times m}, \mathcal{F}) \cong \mathcal{F}^{m\times 1}$ are used. The particular choice of $\mathcal{F}$ is irrelevant for showing that the complex (4) is exact at $\hom_D(M, \mathcal{F})$ and at $\mathcal{F}^{p_0\times 1}$. In the present context this general fact is consistent with Malgrange's isomorphism (2) because $\mathrm{Sol}_{\mathcal{F}}(R) \cong \hom_D(M, \mathcal{F})$ is a subset of $\mathcal{F}^{p_0\times 1}$ and is the kernel of the homomorphism $(R_1). : \mathcal{F}^{p_0\times 1} \to \mathcal{F}^{p_1\times 1}$. If (4) is exact at $\mathcal{F}^{p_1\times 1}$, i.e., we have $\ker((R_2).) = \mathrm{im}((R_1).)$, then $R_1\,y = u$ has a solution in $\mathcal{F}^{p_0\times 1}$ if and only if $R_2\,u = 0$, as expected. We would like that the construction of (3) and its meaning for system equations are translated by the functor $\hom_D(-, \mathcal{F})$ into the expected statement about solutions of the system as formulated in (4). For an appropriate choice of the signal space $\mathcal{F}$ this is true (cf. also Remark 4.12).

Definition 2.6. a) A left D-module $\mathcal{F}$ is said to be *injective* if the functor $\hom_D(-, \mathcal{F})$ is exact, i.e., if $\hom_D(-, \mathcal{F})$ transforms exact sequences into exact sequences.

b) A left D-module $\mathcal{F}$ is said to be a *cogenerator for the category of left D-modules* if for every left D-module M the element $0 \in M$ is the only element which is in the kernel of every homomorphism $M \to \mathcal{F}$.

Remark 2.7. If $\mathcal{F}$ is a cogenerator for the category of left D-modules, then the exactness of (4) implies the exactness of (3).

In the present context injective modules that are cogenerators for the category of left D-modules have a meaning that is analogous to algebraically closed fields in algebraic geometry.

Theorem 2.8 (cf., e.g., [Rot09], Lemma 5.49)**.** *For every ring D there exists an injective left D-module which is a cogenerator for the category of left D-modules.*

For the computational point of view of this article the abstract construction of the previous theorem is not useful. However, for certain rings of functional operators, which are relevant in this context, concrete modules satisfying the above properties are known.

Example 2.9. a) Let $k \in \{\mathbb{R}, \mathbb{C}\}$ and $D = k[\partial_1, \ldots, \partial_n]$. The following k-vector spaces $\mathcal{F}$ are injective D-modules that are cogenerators for the category of D-modules (cf. [Mal62, Thm. 3.2], [Ehr70, Thm. 5.20, Thm. 5.14], [Pal70, Thm. 3 in VII.8.2, Thm. 1 in VII.8.1], [Obe90, Thm. 2.54, Sections 3 and 4]):

(i) sequences $(a_j)_{j \in (\mathbb{Z}_{\geq 0})^n}$, where ∂_i acts by shifting the i-th index; equivalently, formal power series, where ∂_i acts by partial differentiation;

(ii) convergent power series in n variables, where ∂_i acts by partial differentiation;

(iii) k-valued smooth functions on an open and convex subset Ω of $\mathbb{R}^n$, where ∂_i acts by partial differentiation, i.e., $\mathcal{F} = C^\infty(\Omega)$;

(iv) k-valued distributions on an open and convex subset Ω of $\mathbb{R}^n$, where ∂_i acts by partial differentiation.

b) Let $D = B_1(\mathbb{R})$ be the algebra of differential operators with rational function coefficients (cf. Example 3.8 b)). Then the $\mathbb{R}$-valued functions on $\mathbb{R}$ which are smooth except in finitely many points form an injective cogenerator for the category of left D-modules (cf. [Zer06, Thm. 3]).

c) Let Ω be an open interval in $\mathbb{R}$, $A = \{\, f/g \mid f, g \in \mathbb{C}[t],\, g(\lambda) \neq 0 \text{ for all } \lambda \in \Omega \,\}$, and $D = A[\partial]$ the skew polynomial ring with the commutation rules that are implied by the product rule for $\partial = \frac{d}{dt}$ (cf. also Example 3.8 b)). Then Sato's hyperfunctions on Ω form an injective cogenerator for the category of left D-modules (cf. [FO98, Thm. 4]).

d) Let D be an Ore algebra as in Remark 3.9. Then $\hom_k(D, k)$ is an injective cogenerator for the category of left D-modules (cf. [Rob06, Thm. 4.4.7]; cf. also [Bou80, § 1.8, Prop. 11, Prop. 13], [Obe90, Cor. 3.12, Rem. 3.13]).

Remark 2.10. Let $\mathcal{F}$ be an injective cogenerator for the category of left D-modules. Let $S\, z = 0$ be a linear system with $S \in D^{s \times r}$ that is equivalent to $R\, y = 0$ in the sense that there exist matrices $T \in D^{p \times r}$ and $U \in D^{r \times p}$ such that $T. : \mathcal{F}^{r \times 1} \to \mathcal{F}^{p \times 1}$ and $U. : \mathcal{F}^{p \times 1} \to \mathcal{F}^{r \times 1}$

induce isomorphisms $\mathrm{Sol}_{\mathcal{F}}(S) \to \mathrm{Sol}_{\mathcal{F}}(R)$ and $\mathrm{Sol}_{\mathcal{F}}(R) \to \mathrm{Sol}_{\mathcal{F}}(S)$, respectively, which are inverse to each other. The assumption on $\mathcal{F}$ implies that this condition is equivalent to the condition that the homomorphisms

$$D^{1\times p} \longrightarrow D^{1\times r} : v \longmapsto v\,T, \qquad D^{1\times r} \longrightarrow D^{1\times p} : w \longmapsto w\,U$$

induce isomorphisms

$$D^{1\times p}/D^{1\times q}\,R \longrightarrow D^{1\times r}/D^{1\times s}\,S, \qquad D^{1\times r}/D^{1\times s}\,S \longrightarrow D^{1\times p}/D^{1\times q}\,R,$$

respectively, which are inverse to each other. Hence, equivalent behaviors correspond to isomorphic system modules, and if some property is recognized as being satisfied by each module of an isomorphism class of modules corresponding to a linear system $R\,y = 0$, then this property reflects some structural feature of the behavior that does not depend on the choice of the defining equations (e.g., the feature of admitting solutions, or, more interestingly, of being controllable, or flat, etc.).

We recall the following basic notions of module theory.

Definition 2.11. Let D be a left Noetherian domain[2]. A finitely generated left D-module M is said to be

a) *free* if there exists $r \in \mathbb{Z}_{\geq 0}$ such that $M \cong D^{1\times r}$,

b) *stably free* if there exist $r,\, s \in \mathbb{Z}_{\geq 0}$ such that $M \oplus D^{1\times s} \cong D^{1\times r}$,

c) *projective* if there exist a left D-module N and $r \in \mathbb{Z}_{\geq 0}$ such that $M \oplus N \cong D^{1\times r}$,

d) *reflexive* if $M \to \hom_D(\hom_D(M, D), D)$, defined by evaluation of homomorphisms $M \to D$ at the given element of M, is an isomorphism,

e) *torsion-free* if $d\,m = 0$ for $m \in M$ and $d \in D$ implies $d = 0$ or $m = 0$, i.e., the *torsion submodule* $\mathrm{t}(M)$ of M is trivial:

$$\mathrm{t}(M) := \{m \in M \mid \exists\, d \in D \setminus \{0\},\, d\,m = 0\} = \{0\},$$

f) *torsion* if $\mathrm{t}(M) = M$.

In cases a) and b) the (uniquely defined) integer r and $r - s$, respectively, is called the *rank* of M.

Proposition 2.12 (cf., e.g., [Bou80], [Rot09])**.** *Let M be a finitely generated left D-module. Then the following chain of implications holds:*

$$M \textit{ free} \Rightarrow M \textit{ stably free} \Rightarrow M \textit{ projective} \Rightarrow M \textit{ reflexive} \Rightarrow M \textit{ torsion-free.}$$

Remark 2.13. For certain classes of rings, some levels of the hierarchy of modules collapse.

[2] As mentioned earlier, every pair of non-zero elements of D has a non-zero common left multiple, which implies, e.g., that $\mathrm{t}(M)$ is a left D-module.

a) If D is a commutative principal ideal domain, then every finitely generated torsion-free D-module is free. More generally, if D is a right[3] principal ideal domain, i.e., every right ideal of the domain D is generated by one element, then every finitely generated torsion-free left D-module is free (cf. [Coh06, Prop. 1.4.1]).

b) Let k be a field or a commutative principal ideal domain and $D = k[\partial_1, \ldots, \partial_n]$ the commutative polynomial algebra over k. Then every finitely generated projective D-module is free (Quillen-Suslin Theorem, cf. [Lam06]; cf. also Subsection 5.3).

c) If $D = A_n(k)$ is a Weyl algebra over a field k of characteristic zero (cf. Example 3.8 b)), $n \in \mathbb{N}$, then every finitely generated projective left D-module is stably free (cf. Section 3) and every stably free left D-module of rank at least 2 is free (cf. Section 5).

Structural properties of behaviors that are addressed in this article can be characterized in terms of the properties of modules defined above. The unexplained terminology will be introduced and references will be given below.

Theorem 2.14. *Let D be an Ore algebra as in Remark 3.9, $R \in D^{q \times p}$, and $\mathcal{F}$ an injective left D-module which is a cogenerator for the category of left D-modules. Then the behavior of the linear system $R\,y = 0$ is*

a) flat if and only if M is free (cf. Section 5),

b) a projection in $\mathcal{F}^{(p+q)\times 1}$ of a flat behavior defined over D if and only if M is stably free (cf. [QR05b, Thm. 2]),

c) controllable, i.e., admits a parametrization, if and only if M is torsion-free (cf. Section 4),

d) autonomous (i.e., every left D-linear combination of the components of a solution of the system is annihilated by a non-trivial operator in D) if and only if M is torsion.

3 Module-theoretic constructions

Determining structural properties of linear systems defined over D in an effective way requires that certain constructions related to the system module M can be carried out. In particular, the possibility of deciding membership to a finitely presented D-module and, in the positive case, representing the element as linear combination of the generators is a least requirement.

Remark 3.1. Let R be a presentation matrix of the left D-module M. Row operations which are invertible over the coefficient ring D transform R into another presentation matrix of M. As opposed to Gaussian elimination when D is a field, for the more general setting it may be beneficial to adjoin new rows to the presentation matrix that are left D-linear combinations of already given rows, the new rows being, of course, redundant for a generating set of relations.

For instance, if $D = \mathbb{Q}[\partial_1, \partial_2]$ and $M = D/D^{1\times 2}\,R$, where

$$R := \begin{pmatrix} \partial_1^2 - \partial_2 \\ \partial_1\partial_2 - \partial_2 \end{pmatrix} \in D^{2\times 1},$$

[3] We thank Burt Totaro for a clarification of this statement and the reference [Coh06].

then the rows of R form a generating set for the D-linear relations that are satisfied by the residue class of 1 in M, which generates M. Neither can this generating set be reduced in size, nor can the generators be written in a simpler way. However, by appending the linear combination

$$(-\partial_2) \cdot (\partial_1^2 - \partial_2) + (\partial_1 + 1) \cdot (\partial_1 \partial_2 - \partial_2) = \partial_2^2 - \partial_2$$

to the above presentation matrix, we obtain a generating set of relations that allows an effective membership test. In fact, solving for the terms of highest degree defines a confluent and terminating rewriting system for the representatives of residue classes in M, in the sense that an element d of D represents the zero residue class in M if and only if iterated polynomial division of d modulo the above three generators eventually yields zero.

The concepts of *Gröbner basis* and *Janet basis* realize this idea for certain (not necessarily commutative) polynomial algebras. The original notions, developed in [Buc06] and [Jan29] (cf. also [Pom78]), respectively, have been generalized in recent years to more general algebras, e.g., to Ore algebras, G-algebras, and PBW extensions, cf., e.g., [KRW90], [Kre93], [Chy98], [Lev05], [Rob06], and [GL11].

In fact, the above polynomials $\partial_1^2 - \partial_2$, $\partial_1 \partial_2 - \partial_2$, $\partial_2^2 - \partial_2$ form a Gröbner basis and a Janet basis for the ideal of D they generate.

We are going to define the class of rings to be dealt with below (cf., e.g., [CS98], [CQR05]).

Definition 3.2. Let k be a field or $k = \mathbb{Z}$ and let A be a (not necessarily commutative) k-algebra which is a domain. Moreover, let $\sigma : A \to A$ be a k-algebra endomorphism and $\delta : A \to A$ a σ-derivation, i.e., a k-linear map which satisfies

$$\delta(a_1 \, a_2) = \sigma(a_1) \, \delta(a_2) + \delta(a_1) \, a_2, \qquad a_1, a_2 \in A.$$

Then we denote by $A[\partial; \sigma, \delta]$ the k-algebra generated by A and the indeterminate ∂ with commutation rules

$$\partial \, a = \sigma(a) \, \partial + \delta(a), \qquad a \in A,$$

and call it a *skew polynomial ring*. The commutation rule implies $A[\partial; \sigma, \delta] = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} A \, \partial^i$.

Remark 3.3. In what follows, we assume that σ is a monomorphism. Then the degree in ∂ of a non-zero element d of $A[\partial; \sigma, \delta]$ does not depend on the representation of d. Moreover, the degree of a product of non-zero skew polynomials equals the sum of their degrees. Hence, $A[\partial; \sigma, \delta]$ is a domain.

Definition 3.4. Iterating the definition of a skew polynomial ring, for $i = 1, \ldots, l$, let ∂_i be an indeterminate, σ_i a k-algebra monomorphism of $A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_{i-1}; \sigma_{i-1}, \delta_{i-1}]$ and δ_i a σ_i-derivation of $A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_{i-1}; \sigma_{i-1}, \delta_{i-1}]$ such that for all $1 \leq j < i \leq l$ we have

$$\sigma_i(\partial_j) = \partial_j, \qquad \delta_i(\partial_j) = 0,$$

and such that

$$\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i, \quad \delta_i \circ \delta_j = \delta_j \circ \delta_i, \quad \sigma_i \circ \delta_j = \delta_j \circ \sigma_i, \quad \sigma_j \circ \delta_i = \delta_i \circ \sigma_j$$

holds when these maps are restricted to $A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_{j-1}; \sigma_{j-1}, \delta_{j-1}]$. Then the iterated skew polynomial ring generated by A, ∂_1, ..., ∂_l with commutation rules

$$\partial_i \, d = \sigma_i(d) \, \partial_i + \delta_i(d), \qquad d \in A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_{i-1}; \sigma_{i-1}, \delta_{i-1}], \qquad i = 1, \ldots, l,$$

is called an *Ore algebra*. When the maps σ_i and δ_j are understood, we also denote this ring by $A\langle \partial_1, \ldots, \partial_l \rangle$.

Remark 3.5. Let $D = A[\partial_1; \sigma_1, \delta_1] \dots [\partial_l; \sigma_l, \delta_l]$ be an Ore algebra as defined above. The same argument as in Remark 3.3 shows that D is a domain.

Often the k-algebra A is chosen as a commutative polynomial ring $k[x_1, \dots, x_n]$. Then the monomials $x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \cdot \partial_1^{b_1} \cdot \ldots \cdot \partial_l^{b_l}$, where $a \in (\mathbb{Z}_{\geq 0})^n$, $b \in (\mathbb{Z}_{\geq 0})^l$, form a basis for D as a (left) k-vector space (or $\mathbb{Z}$-module if $k = \mathbb{Z}$).

Because of the following variant of Hilbert's Basis Theorem (cf., e.g., [Eis95]) we assume in what follows that $\sigma_1, \dots, \sigma_l$ are k-algebra automorphisms.

Theorem 3.6 (cf. [MR00], Thm. 1.2.9)**.** *Let A be a left (or right) Noetherian domain. If σ is an automorphism of A, then $A[\partial; \sigma, \delta]$ is left (right, respectively) Noetherian.*

The name "Ore algebra" is reminiscent of Ø. Ore's study of the existence of skew fields of fractions of certain non-commutative domains.

Proposition 3.7 (cf. [MR00], Cor. 2.1.14)**.** *Let D be a (not necessarily commutative) domain. A skew field of left fractions of D (i.e., whose elements are represented as $b^{-1}\, a$, where a, $b \in D$, $b \neq 0$) exists if and only if every pair of non-zero elements of D has a non-zero common left multiple. An analogous statement holds with "left" replaced with "right".*

In fact, every left Noetherian domain satisfies the left versions of the above equivalent conditions (cf. [MR00], Thm. 2.1.15), which ensure that a product $\widetilde{a} \cdot \widetilde{b}^{-1}$ has a representation of the form $b^{-1}\, a$, and similarly for right Noetherian domains.

Example 3.8. Let k be a field or $k = \mathbb{Z}$.

a) A commutative polynomial algebra $k[\partial_1, \dots, \partial_n]$ can be understood as an Ore algebra defined over $A = k$, where $\sigma_1, \dots, \sigma_n$ are the identity maps and $\delta_1, \dots, \delta_n$ are zero.

b) The *Weyl algebra*

$$A_n(k) := k[x_1, \dots, x_n][\partial_1; \sigma_1, \delta_1] \dots [\partial_n; \sigma_n, \delta_n], \qquad n \in \mathbb{N},$$

is defined to be the Ore algebra over $k[x_1, \dots, x_n]$, where $\sigma_1, \dots, \sigma_n$ are the identity maps and δ_i is partial differentiation with respect to x_i, $i = 1, \dots, n$. Hence, the commutation rules for the indeterminates of $A_n(k)$ are

$$x_i\, x_j = x_j\, x_i, \qquad \partial_i\, \partial_j = \partial_j\, \partial_i, \qquad \partial_i\, x_j = x_j\, \partial_i + \delta_{i,j}, \qquad 1 \leq i, j \leq n, \tag{5}$$

where $\delta_{i,j}$ is the Kronecker symbol. If $k = \mathbb{R}$ or $\mathbb{C}$, then these commutation rules are easily deduced from the differentiation rules for smooth (real or complex) functions $f(x_1, \dots, x_n)$; e.g., the product rule reads

$$\frac{\partial}{\partial x_i}\, (x_j\, f(x_1, \dots, x_n)) = (x_j\, \frac{\partial}{\partial x_i})\, f(x_1, \dots, x_n) + \delta_{i,j}\, f(x_1, \dots, x_n), \qquad 1 \leq i, j \leq n.$$

We may thus consider $A_n(k)$ as the ring of partial differential operators on k^n with polynomial coefficients. In an analogous way, we define the *ring of differential operators with rational function coefficients* $B_n(k) := k(x_1, \dots, x_n)[\partial_1; \sigma_1, \delta_1] \dots [\partial_n; \sigma_n, \delta_n]$, again with commutation rules (5).

More generally, if A is a differential ring with commuting derivations $\delta_1, \dots, \delta_n$, we define the *ring of differential operators* $A\langle \partial_1, \dots, \partial_l \rangle := A[\partial_1; \sigma_1, \delta_1] \dots [\partial_n; \sigma_n, \delta_n]$ with commutation rules

$$\partial_i\, \partial_j = \partial_j\, \partial_i, \qquad \partial_i\, a = a\, \partial_i + \delta_i(a), \qquad a \in A, \qquad 1 \leq i, j \leq n.$$

c) An *algebra of shift operators* is defined by $k[t][\partial; \sigma, \delta]$, where σ is the k-algebra automorphism mapping t to $t-1$ and δ is the zero map. Thus, the essential commutation rules read

$$\partial\, f(t) = f(t-1)\, \partial, \qquad f \in k[t],$$

which rephrase the "product rule" for the action of shift operators. Of course, 1 can be replaced with a different constant.

d) The types of Ore algebras described above and many more (cf., e.g., [CS98], [CQR05]) can obviously be combined, providing, e.g., algebras of operators for the study of differential systems with (point) delay (cf., e.g., Example 5.16), etc.

Remark 3.9. Let D be an Ore algebra which is generated over the commutative polynomial algebra $k[x_1, \ldots, x_n]$ by $\partial_1, \ldots, \partial_l$. We assume that D is a computable[4] ring in the sense that the arithmetic operations in D can be carried out effectively and that equality of elements in D can be decided.

Let $R \in D^{q \times p}$. The computation of a Gröbner basis or a Janet basis for the submodule $D^{1 \times q} R$ of $D^{1 \times p}$ considers every element of $D^{1 \times p}$ as a sum of terms, the sequence of summands being sorted with respect to a given *term ordering*. Every term is of the form $c \cdot m \cdot e_i$, where $c \in k \setminus \{0\}$, $m = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \cdot \partial_1^{b_1} \cdot \ldots \cdot \partial_l^{b_l}$ for some $a \in (\mathbb{Z}_{\geq 0})^n$, $b \in (\mathbb{Z}_{\geq 0})^l$, and e_i is a standard basis vector. The term ordering is a total ordering on the set of monomials $m \cdot e_i$ as above such that no infinitely decreasing sequence of monomials exists. Moreover, in what follows, we make the important assumption on both the commutation rules of D and the term ordering that left multiplication of an element of $D^{1 \times p}$ with greatest term $c \cdot m \cdot e_i$ by x_j or ∂_j yields elements with greatest terms $c \cdot x_j \cdot m \cdot e_i$ and $\widetilde{c} \cdot m \cdot \partial_j \cdot e_i$ for some $\widetilde{c} \in k \setminus \{0\}$, respectively. For every algebra of interest in our context (in particular, the ones in Example 3.8) a term ordering with this property can be chosen solving the computational tasks discussed below.

The term ordering singles out the greatest term in every non-zero element of $D^{1 \times p}$, which is called its *leading term*. A *Gröbner basis* or *Janet basis* for $D^{1 \times q} R$ with respect to the chosen term ordering is defined to be a finite subset of $D^{1 \times q} R \setminus \{0\}$ such that the leading term of every non-zero element of $D^{1 \times q} R$ is left divisible[5] by the leading term of some element of the basis. Then every element of $D^{1 \times q} R$ can be expressed as a left D-linear combination of the basis elements by iterated subtraction of left multiples of divisors.

A basis computation complements a generating set G for $D^{1 \times q} R$ with further elements of $D^{1 \times q} R$ whose leading terms have no divisors among the leading terms of elements of G. Suitable new elements are found as left D-linear combinations of elements of G in which the leading terms cancel; cf. also the example in Remark 3.1. Termination of such an algorithm follows essentially from Dickson's lemma, stating that a sequence of monomials in which no monomial divides any following monomial is finite.

These techniques were developed in the given setting in [Chy98] and [Rob06] (cf. also [Rob07]). Further properties of the resulting Gröbner basis or Janet basis may be realized

[4] In concrete examples we may assume that the matrix R is defined over a computable subalgebra of D.

[5] For Janet bases the divisibility relation of terms is actually a restriction of the usual divisibility relation. The concept of Janet division (or, more generally, of an involutive division) determines for each monomial the set of indeterminates which may be multiplied from the left to the monomial when it is used for reduction of other terms. As a consequence, every element of $D^{1 \times q} R$ has a *unique* representation as left D-linear combination of the Janet basis elements taking their so-called multiplicative variables into account. For a survey on the algorithmic development of this efficient alternative to Buchberger's algorithm we refer to [Ger05].

by an appropriate choice of the term ordering. This possibility makes such computations a versatile method, e.g., for elimination purposes, cf. also, e.g., [Rob12, Sects. 3.1.1 and 3.1.3].

Example 3.10. Let $R \in D^{q \times p}$ be as above. We would like to compute a generating set for the kernel of the homomorphism $.R : D^{1 \times q} \to D^{1 \times p}$ of left D-modules. To this end we compute a Janet basis J for the submodule $D^{1 \times q} \, (R \quad I_q)$ of $D^{1 \times (p+q)}$ with respect to a term ordering which ranks $m_1 \cdot e_i$ higher than $m_2 \cdot e_j$ if $1 \le i \le p$ and $p+1 \le j \le p+q$. For every $\lambda \in \ker(.R)$ we have

$$\lambda \cdot (R \quad I_q) = (0 \quad \lambda).$$

The elements of J whose first p components are zero form a generating set G for $\ker(.R)$ after removing the first p components. In fact, if $\lambda \in \ker(.R)$, then iterated subtraction of left D-linear combinations of elements of G from $(0 \quad \lambda)$ eventually results in 0 by definition of a Janet basis and the property of the term ordering which guarantees that the leading term of any intermediate element in this reduction process is in a component with index greater than p. An element of $\ker(.R)$ is also called a *syzygy* of the rows of R.

As another application one can decide whether or not R admits a *left inverse* with entries in D. This is the case if and only if the (minimal) Janet basis computed above is given by the rows of a matrix

$$\begin{pmatrix} 0 & * \\ I_p & S \end{pmatrix},$$

where $S \in D^{p \times q}$. Then S satisfies $S \cdot R = I_p$ (where necessarily $p \le q$).

Remark 3.11. If D admits an *involution* θ, i.e., a map $\theta : D \to D$ satisfying

$$\theta(d_1 + d_2) = \theta(d_1) + \theta(d_2), \quad \theta(d_1 \cdot d_2) = \theta(d_2) \cdot \theta(d_1), \quad \theta(\theta(d)) = d, \quad \text{for all } d_1, d_2, d \in D,$$

then the computation of Gröbner bases and Janet bases for the submodule $R \, D^{p \times 1}$ of the right D-module $D^{q \times 1}$ can be reduced to the one for the submodule $D^{1 \times p} \, \theta(R)$ of the left D-module $D^{1 \times q}$, where

$$\theta(R) := (\theta((R^T)_{i,j}))_{1 \le i \le p, \, 1 \le j \le q} \in D^{p \times q}.$$

In particular, the computation of a *right inverse* of R with entries in D (if it exists) can be reduced to the computation of a left inverse of $\theta(R)$ (and vice versa).

Example 3.12. Let $D = A_n(k) = k[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ be the Weyl algebra. The most common involution of D is defined by extending

$$\theta(x_i) := x_i, \qquad \theta(\partial_i) := -\partial_i, \qquad i = 1, \ldots, n,$$

to a map $\theta : D \to D$ using the definition of an involution. Then $\theta(R)$ is the formal adjoint of the differential operator $R \in D^{q \times p}$, which is also obtained by integration by parts.

Remark 3.13. By iterating the computation of syzygies a *free resolution* of the left D-module M with presentation matrix $R_1 \in D^{q \times p}$ can be constructed, i.e., an exact sequence of left D-modules of the form

$$0 \longleftarrow M \longleftarrow D^{1 \times p_0} \xleftarrow{.R_1} D^{1 \times p_1} \xleftarrow{.R_2} D^{1 \times p_2} \longleftarrow \ldots \tag{6}$$

in which each module is a free left D-module except possibly the module M that we study. If only finitely many modules in the above exact sequence are non-zero, i.e., (6) is of the form

$$0 \longleftarrow M \longleftarrow D^{1\times p_0} \xleftarrow{.R_1} D^{1\times p_1} \xleftarrow{.R_2} \dots \xleftarrow{.R_m} D^{1\times p_m} \longleftarrow 0 \tag{7}$$

then (7) is said to be a *finite free resolution* of M. The *length* of the free resolution (7) is defined to be m, and the modules in (7) are numbered consecutively such that $D^{1\times p_i}$ is in *homological degree* i. Similarly, an exact sequence

$$0 \longleftarrow M \longleftarrow P_0 \xleftarrow{\alpha_1} P_1 \xleftarrow{\alpha_2} P_2 \longleftarrow \dots \tag{8}$$

in which each module is a projective (or stably free) left D-module except possibly M is called a *projective* (*stably free*, respectively) *resolution* of M.

The concept of free resolution was elaborated in the commutative algebra context by D. Hilbert, who proved the following celebrated result about the existence of finite free resolutions in this setting. Below we deal with the more general case of Ore algebras as defined above.

Theorem 3.14 (Hilbert's Syzygy Theorem; cf., e.g., [Eis95])**.** *Let $D = k[\partial_1, \dots, \partial_n]$ be a commutative polynomial algebra, where k is a field. Then for every finitely generated D-module M there exists a finite free resolution of M of length at most n.*

We recall a technique to reduce the length of a free resolution (cf. [QR07]). It is essential for deciding whether or not a finitely generated left D-module is stably free.

Remark 3.15. Let (7) be a finite free resolution of the left D-module M, and let us assume that R_m admits a right inverse $S \in D^{p_{m-1}\times p_m}$. If $m \geq 3$, then a shorter free resolution of M is obtained by replacing the three non-trivial homomorphisms in the highest homological degrees in (7) with

$$D^{1\times p_{m-3}} \xleftarrow{.\begin{pmatrix} R_{m-2} \\ 0 \end{pmatrix}} D^{1\times p_{m-2}} \oplus D^{1\times p_m} \xleftarrow{.(R_{m-1} \quad S)} D^{1\times p_{m-1}} \longleftarrow 0.$$

The resulting complex is exact at $D^{1\times p_{m-2}} \oplus D^{1\times p_m}$ because

$$\begin{aligned} \ker .\begin{pmatrix} R_{m-2} \\ 0 \end{pmatrix} &= \operatorname{im} .\begin{pmatrix} R_{m-1} & 0 \\ 0 & I_{p_m} \end{pmatrix} = \operatorname{im} .\begin{pmatrix} R_{m-1} & S \\ 0 & I_{p_m} \end{pmatrix} \\ &= \operatorname{im} .\begin{pmatrix} I_{p_{m-1}} \\ R_m \end{pmatrix} \cdot (R_{m-1} \quad S) = \operatorname{im} .(R_{m-1} \quad S). \end{aligned}$$

In order to show the exactness at $D^{1\times p_{m-1}}$ we note that both homomorphisms $\pi_1 := .(S\,R_m)$ and $\pi_2 := .(I_{p_{m-1}} - S\,R_m)$ are projections of $D^{1\times p_{m-1}}$ onto their images. More specifically, we have

$$\pi_i \circ \pi_j = \delta_{i,j}\,\pi_i \qquad \text{for } i, j \in \{1, 2\}, \qquad \text{and} \quad \mathrm{id}_{D^{1\times p_{m-1}}} = \pi_1 + \pi_2,$$

where $\delta_{i,j}$ is the Kronecker symbol. This implies $D^{1\times p_{m-1}} = \operatorname{im}(\pi_1) \oplus \operatorname{im}(\pi_2)$. Since we have $\operatorname{im}(\pi_1) = \operatorname{im}(.R_m) = \ker(.R_{m-1})$, the restriction of $.R_{m-1}$ to $\operatorname{im}(\pi_2)$ is injective. Hence, there exists $T \in D^{p_{m-2}\times p_{m-1}}$ such that $I_{p_{m-1}} - S\,R_m = (I_{p_{m-1}} - S\,R_m)\,R_{m-1}\,T = R_{m-1}\,T$, and we have $\operatorname{im}(\pi_2) = \operatorname{im}(.T)$. Solving the last matrix equation for $I_{p_{m-1}}$ shows that $.(R_{m-1} \quad S)$ is injective. The exactness of the shorter complex at $D^{1\times p_{m-3}}$ is obvious.

If $m = 2$, then the same technique applies with $D^{1\times p_{m-3}}$ replaced with M and multiplication by R_{m-2} on $D^{1\times p_{m-2}}$ replaced with the canonical projection $D^{1\times p_0} \to M$.

The situation of the previous remark arises whenever M is projective.

Remark 3.16. Let M be projective and let $\psi_0 : F_0 \to M$ be a surjective homomorphism, where F_0 is a finitely generated left D-module. Then there exists a left D-module N such that $M \oplus N$ is isomorphic to $D^{1\times r}$ for some $r \in \mathbb{Z}_{\geq 0}$. With respect to a basis of $M \oplus N$ a homomorphism $M \oplus N \to F_0$ can be defined whose restriction σ_0 to M satisfies $\psi_0 \circ \sigma_0 = \mathrm{id}_M$, i.e., σ_0 is a right inverse of ψ_0. Then we have $F_0 = \mathrm{im}(\sigma_0) \oplus \ker(\psi_0)$. If F_0 is free and $\psi_1 : F_1 \to F_0$ is a homomorphism with image $\ker(\psi_0)$, then this construction can be applied again to $\ker(\psi_0)$.

Hence, in a finite free resolution (7) of a (finitely generated) projective left D-module M, for every i, we have $D^{1\times p_i} = \mathrm{im}(\sigma_i) \oplus \ker(.R_i)$ for some homomorphism $\sigma_i : \mathrm{im}(.R_i) \to D^{1\times p_i}$. In particular, the matrix representing the homomorphism $D^{1\times p_{m-1}} \to D^{1\times p_m}$ whose restriction to $\mathrm{im}(.R_m)$ defines σ_m is a right inverse of R_m.

Even if F_0 is not necessarily free, the direct summands $M \cong \mathrm{im}(\sigma_0)$ and $C = \ker(\psi_0)$, the module $L := F_0$, the canonical projection ψ, and the canonical injection ϕ form an exact sequence with a special property.

Definition 3.17. An exact sequence

$$0 \longleftarrow M \overset{\psi}{\longleftarrow} L \overset{\phi}{\longleftarrow} C \longleftarrow 0$$

is said to be *split* if there exist homomorphisms $\sigma : M \to L$ and $\rho : L \to C$ such that we have $\psi \circ \sigma = \mathrm{id}_M$ and $\rho \circ \phi = \mathrm{id}_C$. (In fact, the existence of σ implies that of ρ and vice versa.)

Definition 3.18. The *left projective dimension* of a left D-module M, denoted by $\mathrm{lpd}(M)$, is defined as the minimal length of a finite projective resolution of M and as ∞ if no such resolution of M exists. The *left global dimension* of D, denoted by $\mathrm{lgld}(D)$, is defined to be the supremum of the left projective dimensions of left D-modules.

Two left D-modules M_1 and M_2 are said to be *projectively equivalent* if there exist projective left D-modules Q_1 and Q_2 such that we have $M_1 \oplus Q_1 \cong M_2 \oplus Q_2$.

Corresponding notions for right D-modules M are defined in a similar way, the *right projective dimension* of M being denoted by $\mathrm{rpd}(M)$ and the *right global dimension* of D by $\mathrm{rgld}(D)$.

Proposition 3.19 (cf. [MR00], Subsect. 7.1.11)**.** *If D is Noetherian then we have* $\mathrm{lgld}(D) = \mathrm{rgld}(D)$.

Remark 3.20. We have $\mathrm{lpd}(M) = 0$ if and only if M is projective.

Remark 3.21. Schanuel's lemma (cf., e.g., [Lam99, Cor. 5.5]) states that, for any left D-module M and each two exact sequences of left D-modules

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \ldots \longleftarrow P_r \longleftarrow L \longleftarrow 0$$

$$0 \longleftarrow M \longleftarrow \widetilde{P}_0 \longleftarrow \widetilde{P}_1 \longleftarrow \ldots \longleftarrow \widetilde{P}_r \longleftarrow \widetilde{L} \longleftarrow 0$$

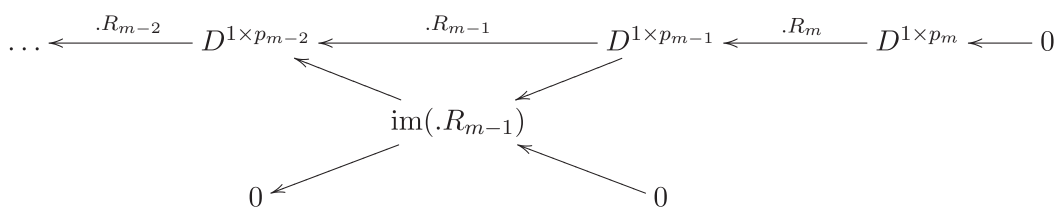
where every P_i and $\widetilde{P}_j$ is projective, the left D-modules L and $\widetilde{L}$ are projectively equivalent.

Proposition 3.22 (cf. [MR00], Thm. 7.5.3)**.** *Let $D = A[\partial; \sigma, \delta]$ be an Ore algebra, where σ is an automorphism of A and* $\mathrm{lgld}(A)$ *is finite. Then we have* $\mathrm{lgld}(A) \leq \mathrm{lgld}(D) \leq \mathrm{lgld}(A) + 1$.

In what follows we assume that D is an Ore algebra as in Remark 3.9, i.e., which admits Gröbner basis or Janet basis computations. Since the construction of D starts with either a field or a commutative polynomial ring, an iteration of Proposition 3.22 (and use of Theorem 3.14) shows that $\mathrm{lgld}(D)$ is finite.

Theorem 3.23. *Let $M = D^{1\times p}/D^{1\times q}\,R$ be a left D-module, where $R \in D^{q\times p}$. Then a finite free resolution (7) of M can be computed such that either $m = 1$ and R_1 admits a right inverse, or $m \geq 1$ and R_m does not admit a right inverse. In the former case M is stably free, in the latter case M is not projective. Analogous statements hold for finitely generated right D-modules.*

Proof. The assumptions on D imply that a finite free resolution of M can be constructed by iteratively computing syzygies (cf. [Rob12, Cor. 3.1.46], or [Eis95, Sect. 15.5] for the case of a commutative polynomial algebra D). The technique discussed in Remark 3.15 reduces any finite free resolution of M to one as described in the theorem. If $m = 1$ and R_1 admits a right inverse, then the exact sequence is split, which implies that we have $D^{1\times p_0} \cong M \oplus D^{1\times p_1}$. Hence, M is stably free. If $m \geq 1$ and R_m does not admit a right inverse, then $\mathrm{im}(.R_{m-1})$ is not projective because the following short exact sequence bending down to the left is not split:



Let us assume that there exists a projective resolution of M of length $m-1$. By Remark 3.21, the two modules in the highest homological degree of this projective resolution and of the above (long) exact sequence of length $m-1$ including the upper left arrows are projectively equivalent. This implies that $\mathrm{im}(.R_{m-1})$ is projective, which is a contradiction. Therefore, we have $\mathrm{lpd}(M) = m$. In particular, M is not projective. □

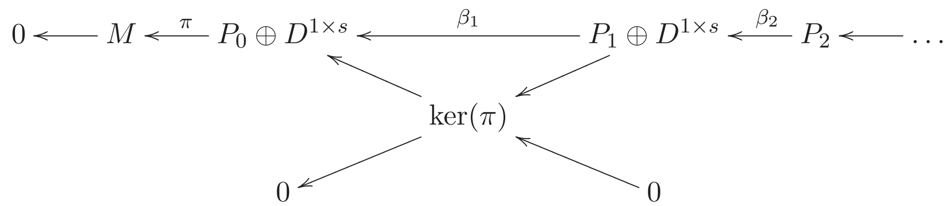
We obtain the following corollary which generalizes a theorem of J.-P. Serre [Ser58, Prop. 10] about projective modules over commutative polynomial algebras $k[\partial_1, \ldots, \partial_n]$ to Ore algebras D as above.

Corollary 3.24. *Every finitely generated projective left D-module is stably free.*

Finally, we are in a position to prove an analogon to Hilbert's Syzygy Theorem for the class of Ore algebras which is relevant in what follows.

Corollary 3.25 (cf. [MR00], Cor. 12.3.3, [Rot09], Lemma 8.42, [CQR05], Prop. 8)**.** *Let M be a finitely generated left module over an Ore algebra D as above. Then a finite free resolution of M of length less than or equal to* $\mathrm{lgld}(D) + 1$ *can be computed.*

Proof. Recall that $\mathrm{lgld}(D)$ is finite. We prove the assertion that a free resolution of M of length at most $\mathrm{lpd}(M) + 1$ exists by induction on the left projective dimension. Let (8) be a projective resolution of M of length $m = \operatorname{lpd}(M)$. By Corollary 3.24, each P_i is stably free. In particular, there exist $r, s \in \mathbb{Z}_{\geq 0}$ such that $P_0 \oplus D^{1\times s} \cong D^{1\times r}$. We obtain exact sequences



where β_1 is defined componentwise by α_1 and the identity on $D^{1\times s}$, β_2 is defined by α_2, and π is defined by $P_0 \to M$. If $m = 0$, then we have $\ker(\pi) \cong D^{1\times s}$ and the assertion follows. Otherwise we observe that $\ker(\pi)$ has a projective resolution of length $m - 1$. By induction, $\ker(\pi)$ has a resolution of length at most m with free modules F_j. Composing $F_0 \to \ker(\pi)$ with $\ker(\pi) \to P_0 \oplus D^{1\times s}$ yields a free resolution of M of length at most $m+1$. The techniques discussed in this section (in particular, Example 3.10 and Remark 3.15) allow to compute such a finite free resolution from any finite presentation of M. □

4 Parametrizability of the behavior

We continue to consider a linear system which is defined over an Ore algebra D as in Remark 3.9. In this section the possibility of identifying its set of solutions or behavior as the image of a D-linear map is investigated. In many control-theoretic situations this possibility corresponds to controllability of the system.

Definition 4.1. Let $\mathcal{F}$ be a signal space, $R \in D^{q\times p}$ as above, and $P \in D^{p\times m}$ for some $m \in \mathbb{N}$. We call the homomorphism $P.: \mathcal{F}^{m\times 1} \to \mathcal{F}^{p\times 1}$ or simply the matrix P a *parametrization of the linear system* $R\,y = 0$ *in* $\mathcal{F}^{p\times 1}$, if

$$\mathcal{F}^{m\times 1} \xrightarrow{P.} \mathcal{F}^{p\times 1} \xrightarrow{R.} \mathcal{F}^{q\times 1} \tag{9}$$

is an exact sequence of abelian groups, i.e., if $\ker(R.) = \operatorname{im}(P.)$.

Example 4.2. Let us consider the system of linear partial differential equations

$$\nabla \cdot y := \partial_1\, y_1 + \partial_2\, y_2 + \partial_3\, y_3 = 0 \tag{10}$$

for an unknown vector $y \in C^\infty(\Omega)^{3\times 1}$ of smooth functions, where Ω is an open and convex subset of $\mathbb{R}^3$ with coordinates x_1, x_2, x_3 and ∂_i denotes the partial differential operator with respect to x_i, $i = 1, 2, 3$. It is well-known (Poincaré's lemma) that (10) is equivalent to

$$\exists\, z \in C^\infty(\Omega)^{3\times 1} : \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & \partial_3 & -\partial_2 \\ -\partial_3 & 0 & \partial_1 \\ \partial_2 & -\partial_1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} =: \nabla \times z.$$

In other words, the curl operator is a parametrization of the linear system $\nabla \cdot y = 0$ in $C^\infty(\Omega)^{3\times 1}$. Clearly, the linear map $C^\infty(\Omega)^{3\times 1} \to C^\infty(\Omega)^{3\times 1}$ defined by the curl operator is not injective. In fact, the gradient operator is a parametrization of the linear system $\nabla \times z = 0$ in $C^\infty(\Omega)^{3\times 1}$. Finally, the gradient operator defines a linear system $\nabla\, u = 0$ for which no parametrization in $C^\infty(\Omega)$ exists. The set of solutions of $\nabla\, u = 0$ in $C^\infty(\Omega)$ is actually $\{u : \Omega \to \mathbb{R} \mid u \text{ constant}\}$, which is not the image of any operator $P \in D^{1\times m}$, where $D = \mathbb{R}[\partial_1, \partial_2, \partial_3]$, $m \in \mathbb{N}$. (This follows from Theorem 4.4 below.)

The following remark outlines the relationship of parametrizability in the above sense and some notions of controllability.

Remark 4.3. A parametrization as defined above is also called an *image representation* in the behavioral approach to systems theory, cf. [PW98], [Woo00], [Zer00], where the signal space $\mathcal{F}$ is chosen appropriately, e.g., from the list given in Example 2.9. This notion has been studied in different settings by many authors. In particular, several notions of primeness of a matrix with entries in a commutative polynomial algebra of operators were developed which characterize structural properties as discussed here (cf., e.g., [Obe90, Sect. 7], [Zer00]).

For one-dimensional linear differential systems (i.e., linear ODEs) and for multidimensional discrete linear systems, each with constant coefficients, it was shown that the existence of an image representation is equivalent to controllability of the system, in the sense that the restrictions of each two trajectories of the behavior to some regions with positive distance in the domain of definition can be concatenated by a trajectory of the behavior (cf. [PW98, Thm. 6.6.1] for the one-dimensional case; [Woo00, Sect. 4], [Zer00, Ch. 1] and the references therein). The corresponding equivalence for systems of linear partial differential equations with constant coefficients was proved in [PS98, Prop. 2, Thm. 2, Thm. 3]. The case of one-dimensional linear differential time-delay equations with constant coefficients was settled in [GL02, Sect. 4.3] ($\mathcal{F}$ consisting of complex-valued smooth functions on $\mathbb{R}$). For other notions of controllability in the context of linear differential time-delay equations, cf. also [FM98].

The equivalence of controllability and torsion-freeness of the system module was first established by J.-F. Pommaret [Pom95] and M. Fliess [Fli91], cf. also [Woo00] and the references therein. This suggested the following definition of controllability for more general kinds of systems. An *observable* of a linear system is defined to be any element of the system module M. For each element y of the behavior an observable corresponds to a certain left D-linear combination of the *signals* $y_1, \ldots, y_p$ via Malgrange's isomorphism (2). The linear system is said to be *controllable* if every observable is *free*, i.e., is not annihilated by a non-zero element of D.

On the other hand, a corresponding notion of parametrization for non-linear systems is more difficult to approach. The problem of expressing the solutions of a system of (not necessarily linear) partial differential equations in terms of arbitrary functions and constants is also known as *Monge's problem*, cf., e.g., [Zer32], [Jan71], or the introduction to [QR07]. The special case in which the correspondence between solutions and (tuples of) parameter values is one-to-one is referred to by the notion of *(differential) flatness*, cf., e.g., [FLMR95], and in particular, Section 5 for the linear case. The problem of deciding flatness and computing, if possible, an injective parametrization is not solved in general up to the present day, but cf., e.g., [AP07], [Lev11], [LHR13] for some approaches.

The torsion submodule $\mathrm{t}(M) := \{m \in M \mid \exists\, d \in D \setminus \{0\},\, d\, m = 0\}$ of the system module M plays a crucial role for the (non-) parametrizability of the given linear system (cf., e.g., [PQ99b], [Pom01]).

Theorem 4.4. *Let M be the system module defined by $R\, y = 0$, and let $\mathcal{F}$ be an injective left D-module which is a cogenerator for the category of left D-modules. There exists a parametrization $P \in D^{p \times m}$ of $R\, y = 0$ in $\mathcal{F}^{p \times 1}$ for some $m \in \mathbb{N}$ if and only if $\mathrm{t}(M) = \{0\}$.*

Since we obtain an algorithm which computes a parametrization, if one exists, we include here a

Proof. If $P \in D^{p\times m}$ is a parametrization of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$, then, by definition,

$$\hom_D(D^{1\times m}, \mathcal{F}) \xrightarrow{P.} \hom_D(D^{1\times p}, \mathcal{F}) \xrightarrow{R.} \hom_D(D^{1\times q}, \mathcal{F})$$

is an exact sequence of abelian groups. Since $\mathcal{F}$ is a cogenerator for the category of left D-modules, the complex

$$D^{1\times m} \xleftarrow{.P} D^{1\times p} \xleftarrow{.R} D^{1\times q} \tag{11}$$

is an exact sequence of left D-modules. Hence, the homomorphism $\iota : M \to D^{1\times m}$ which is induced by P is injective. Now, $\mathrm{t}(D^{1\times m}) = \{0\}$ implies $\mathrm{t}(M) = \{0\}$.

$$\begin{array}{ccccc} D^{1\times m} & \xleftarrow{.P} & D^{1\times p} & \xleftarrow{.R} & D^{1\times q} \\ \uparrow{\scriptstyle \iota} & \swarrow & & & \\ M & & & & \end{array} \tag{12}$$

Conversely, let us assume that $\mathrm{t}(M) = \{0\}$. Since D is Noetherian, there exist $m \in \mathbb{N}$ and $P \in D^{p\times m}$ such that

$$D^{m\times 1} \xrightarrow{P.} D^{p\times 1} \xrightarrow{R.} D^{q\times 1} \tag{13}$$

is an exact sequence of right D-modules. If $\lambda \in D^{1\times p}$ satisfies $\lambda\, P = 0$, then there exists a unique epimorphism $\varphi : R\, D^{p\times 1} \to \lambda\, D^{p\times 1}$ of right D-modules such that $\varphi \circ (R.) = (\lambda.)$, as is easily checked on the following commutative diagram:

$$\begin{array}{ccccccccc} D^{m\times 1} & \xrightarrow{P.} & D^{p\times 1} & \xrightarrow{R.} & R\, D^{p\times 1} & = & \mathrm{im}(R.) & \subseteq & D^{q\times 1} \\ & & & {\scriptstyle \lambda.}\searrow & \downarrow{\scriptstyle \varphi} & & & & \\ & & & & \lambda\, D^{p\times 1} & & & & \end{array}$$

Let K be the skew field of fractions of D (which exists due to Proposition 3.7). Then φ induces an epimorphism $\varphi \otimes_D K : (R\, D^{p\times 1}) \otimes_D K \to (\lambda\, D^{p\times 1}) \otimes_D K$ of right K-vector spaces. We choose a complement of $(R\, D^{p\times 1}) \otimes_D K$ in $K^{q\times 1}$ and extend $\varphi \otimes_D K$ to a K-linear map $K^{q\times 1} \to K$ with image $(\lambda\, D^{p\times 1}) \otimes_D K$ in an arbitrary way. The latter map is represented with respect to the standard bases by a matrix $\rho \in K^{1\times q}$, and we have $\rho\, R = \lambda$. Denoting by $d \in D \setminus \{0\}$ a common left denominator of the entries of ρ, we may write $\frac{1}{d}\, \mu\, R = \lambda$ with a certain matrix $\mu \in D^{1\times q}$.

Now we claim that (11) is an exact sequence of left D-modules. Let $\lambda \in \ker(.P) \setminus \{0\}$. By the above reasoning, there exist $d \in D \setminus \{0\}$ and $\mu \in D^{1\times q}$ such that $d\, \lambda = \mu\, R$. If d is invertible in D, then we have $\lambda \in D^{1\times q}\, R$. Otherwise, we have $\lambda + D^{1\times q}\, R \in \mathrm{t}(M) = \{0\}$, i.e., again $\lambda \in D^{1\times q}\, R$. This shows that (11) is exact. Since $\mathcal{F}$ is injective, (9) is exact. □

Remark 4.5. We conclude from Theorem 4.4 that the obstructions to parametrizing a linear system are given by the *autonomous elements* of the behavior $\mathrm{Sol}_{\mathcal{F}}(R)$, i.e., the left D-linear combinations $T\, y$ of the signals or system variables, $T \in D^{1\times p}$, for which there exists $d \in D \setminus \{0\}$ such that $d\, T\, \eta = 0$ for every $\eta \in \mathrm{Sol}_{\mathcal{F}}(R)$. For more details, cf. [CQR05].

Example 4.6. In the context of Example 4.2 we have $D = \mathbb{R}[\partial_1, \partial_2, \partial_3]$. The system module associated with the linear system $\nabla\, u = 0$ defined by the gradient operator is given by $M = D/D^{1\times 3}\, R$, where $R := (\partial_1, \partial_2, \partial_3)^T$. Clearly, M is torsion, i.e., $\mathrm{t}(M) = M$, because every element of M is annihilated by ∂_1, ∂_2, and ∂_3. Hence, there exists no parametrization of $\nabla\, u = 0$ in $C^\infty(\Omega)$.

Remark 4.7. The part of the proof of Theorem 4.4 which shows that a parametrization of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$ exists if $\mathrm{t}(M) = \{0\}$ holds actually provides an algorithm which decides parametrizability and constructs a parametrization if possible. More precisely, a computation of syzygies (cf. Example 3.10) yields $P \in D^{p\times m}$ in (13). By construction, (13) is part of a free resolution of the right D-module $N = D^{q\times 1}/R\,D^{p\times 1}$. The check whether or not the dual complex (11) is exact in fact computes the cohomology group $\mathrm{ext}^1_D(N, D)$. This standard construction of homological algebra is recalled below. Now, P is a parametrization of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$ if and only if $\mathrm{ext}^1_D(N, D) = \{0\}$; cf. Example 4.15 for an illustration.

Definition 4.8. The *Auslander transpose* of the left D-module with finite presentation $D^{1\times p}/D^{1\times q}\,R$ is defined to be the right D-module with finite presentation $D^{q\times 1}/R\,D^{p\times 1}$.

Remark 4.9. If the construction of the Auslander transpose is applied to two finite presentations of the same left D-module M, then the results are projectively equivalent right D-modules (cf. Def. 3.18, cf. also [PQ00] and the references therein).

Definition 4.10. Let M_1 and M_2 be left D-modules. We apply the functor $\hom_D(-, M_2)$ to a projective resolution

$$0 \longleftarrow M_1 \longleftarrow P_0 \xleftarrow{\alpha_1} P_1 \xleftarrow{\alpha_2} P_2 \longleftarrow \ldots \tag{14}$$

of M_1, where we replace the module M_1 with the zero module, and obtain a complex of abelian groups

$$0 \xrightarrow{\alpha_0^*} \hom_D(P_0, M_2) \xrightarrow{\alpha_1^*} \hom_D(P_1, M_2) \xrightarrow{\alpha_2^*} \hom_D(P_2, M_2) \longrightarrow \ldots$$

where $\alpha_i^* := \hom_D(\alpha_i, M_2)$ composes homomorphisms $P_{i-1} \to M_2$ with α_i for $i \in \mathbb{N}$ and $\alpha_0^* := 0$. For $n \in \mathbb{Z}_{\geq 0}$, the abelian group $\mathrm{ext}^n_D(M_1, M_2)$ is defined to be the (co-)homology of the above (co-)complex at $\hom_D(P_n, M_2)$, i.e., the factor group

$$\mathrm{ext}^n_D(M_1, M_2) := \ker(\alpha^*_{n+1}) / \operatorname{im}(\alpha^*_n),$$

and is called the *n-th extension group of M_1 with coefficients in M_2*. A corresponding notion is defined in a similar way for right D-modules M_1 and M_2.

Remark 4.11.
a) Standard techniques of homological algebra (cf., e.g., [Rot09]) show that every choice of the projective resolution (14) of M_1 yields the same abelian group $\mathrm{ext}^n_D(M_1, M_2)$ up to isomorphism. Moreover, it can be shown that, for each n, the n-th extension group of a direct sum of left D-modules with coefficients in M_2 is isomorphic to the direct product of the respective n-th extension groups, and that, if M_1 is projective, we have $\mathrm{ext}^n_D(M_1, M_2) = \{0\}$ for all left D-modules M_2 and all $n \in \mathbb{N}$ (cf., e.g., [Rot09, Propositions 7.21 and 8.6]). It follows that we have $\mathrm{ext}^n_D(N, M_2) \cong \mathrm{ext}^n_D(N', M_2)$ for all $n \in \mathbb{N}$ if N and N' are projectively equivalent.

b) If M_1 is a right D-module and $M_2 = D$, then $\mathrm{ext}^n_D(M_1, M_2)$ inherits a left D-module structure from the D-bimodule structure of D, a remark which is relevant for Theorem 4.13 below.

c) The name "extension group" originates from the fact that the elements of $\mathrm{ext}^1_D(M_1, M_2)$ are in bijection with the equivalence classes of extensions of M_2 by M_1, i.e., the left D-modules M realizing, up to isomorphism, M_2 as a submodule and M_1 as a factor module M/M_2, where an equivalence relation is defined by isomorphisms of modules which respect the submodule M_2 and the factor module M_1. The binary operation of the group $\mathrm{ext}^1_D(M_1, M_2)$ can be translated into a construction of extensions called Baer sum (cf., e.g., [Rot09, Subsect. 7.2.1] and also [QR08] for system theoretic interpretations).

Remark 4.12. If $M_1 = M$ is the system module associated with a linear system $R\,y = 0$ and $M_2 = \mathcal{F}$ is the chosen signal space, we have $\mathrm{ext}^0_D(M, \mathcal{F}) \cong \mathrm{Sol}_{\mathcal{F}}(R)$ by Malgrange's isomorphism (2), and $\mathrm{ext}^1_D(M, \mathcal{F}) = \{0\}$ implies that compatibility conditions of inhomogeneous systems $R\,y = u$ are all found by computing syzygies of the rows of $R_1 := R$, cf. Example 2.5.

Theorem 4.13 (cf. [PQ03], Cor. 2; [CQR05], Thm. 5)**.** *Let $N = D^{q\times 1}/R\,D^{p\times 1}$ be the Auslander transpose of $M = D^{1\times p}/D^{1\times q}\,R$. Then we have $\mathrm{t}(M) \cong \mathrm{ext}^1_D(N, D)$.*

Corollary 4.14. *Let M be the system module defined by $R\,y = 0$, and let $\mathcal{F}$ be an injective left D-module which is a cogenerator for the category of left D-modules. There exists a parametrization $P \in D^{p\times m}$ of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$ for some $m \in \mathbb{N}$ if and only if $\mathrm{ext}^1_D(N, D) = \{0\}$, where $N = D^{q\times 1}/R\,D^{p\times 1}$ is the Auslander transpose of $M = D^{1\times p}/D^{1\times q}\,R$.*

We illustrate the meaning of this corollary on an example taken from [Fre71, p. 23]. At the same time we demonstrate that the parametrizability problem can sometimes be solved even though the injective cogenerator $\mathcal{F}$ is not specified. The following example deals with differential operators with power series coefficients, a situation arising, e.g., in a local study of a singularity at the origin. In this case we do not know a space of functions $\mathcal{F}$ meeting the requirements. More remarks about systems of linear ordinary differential equations with power series coefficients are given in Subsection 5.5. The following computations were performed using an extension of the `Janet` package (cf. Section 6).

Example 4.15. Let $A = k\{t\}$ be the ring of convergent power series in t, where $k \in \{\mathbb{R}, \mathbb{C}\}$, and $D = A\langle\partial\rangle$ the skew polynomial ring generated by A and ∂ with the commutation rules that are implied by the product rule for $\partial = \frac{d}{dt}$. Since A is a Noetherian domain, so is D (cf. Theorem 3.6). Let us consider the system of linear ordinary differential equations

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} 1 & -\sin(2t) & \cos(2t) \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} \sin(t) \\ \cos(t) \\ \sin(t) \end{pmatrix} u(t). \tag{15}$$

The system may be written as $R\,y = 0$, where

$$R := \begin{pmatrix} \partial - 1 & \sin(2t) & -\cos(2t) & -\sin(t) \\ 0 & \partial & 2 & -\cos(t) \\ 0 & -2 & \partial & -\sin(t) \end{pmatrix} \in D^{3\times 4}$$

and $y = (x_1, x_2, x_3, u)^T$. The system module M is defined by the following finite presentation:

$$0 \longleftarrow M \longleftarrow D^{1\times 4} \xleftarrow{\ .R\ } D^{1\times 3}.$$

In order to decide whether or not a parametrization of $R\,y = 0$ in $\mathcal{F}^{4\times 1}$ for an appropriate signal space $\mathcal{F}$ exists and to construct one if possible, we compute $\mathrm{ext}^1_D(N, D)$. In fact, using the involution θ from Example 3.12 allows to avoid dealing with right D-modules. Hence, we are going to determine $\operatorname{ext}^1_D(\widetilde{N}, D)$, where $\widetilde{N} := D^{1\times 3}/D^{1\times 4}\, R_1$ and

$$R_1 := \theta(R) = \begin{pmatrix} -\partial - 1 & 0 & 0 \\ \sin(2t) & -\partial & -2 \\ -\cos(2t) & 2 & -\partial \\ -\sin(t) & -\cos(t) & -\sin(t) \end{pmatrix}.$$

A syzygy computation (cf. Example 3.10) yields the free (and hence projective) resolution

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 4} \xrightarrow{.R_1} D^{1\times 3} \longrightarrow \widetilde{N} \longrightarrow 0 \tag{16}$$

of $\widetilde{N}$, where

$$R_2 = \begin{pmatrix} \sin(t)\,\partial + 2\cos(t) & \cos(t)\,\partial & \sin(t)\,\partial & -\partial^2 - 1 \end{pmatrix}.$$

By applying the functor $\hom_D(-, D)$ to (16) we obtain a complex of right D-modules, which we again transform into left D-modules by using the involution θ. Hence, defining

$$\widetilde{R}_2 := \theta(R_2) = \begin{pmatrix} -\sin(t)\,\partial + \cos(t) \\ -\cos(t)\,\partial + \sin(t) \\ -\sin(t)\,\partial - \cos(t) \\ -\partial^2 - 1 \end{pmatrix},$$

we obtain the horizontal complex of left D-modules

$$\begin{array}{ccccc} D & \xleftarrow{.\widetilde{R}_2} & D^{1\times 4} & \xleftarrow{.R} & D^{1\times 3} \\ & & & \nwarrow^{.S} & \\ & & & & D^{1\times 3} \end{array} \tag{17}$$

and another syzygy computation yields

$$S := \begin{pmatrix} \partial & 2 & -\partial & 0 \\ \sin(2t)\,\partial + 2 & 0 & -\sin(2t)\,\partial + 2\cos(2t) & 0 \\ \partial & 0 & 0 & -\sin(t) \end{pmatrix}$$

such that the bended complex in (17) is exact. This allows to compute the factor module $\ker(.\widetilde{R}_2)/\operatorname{im}(.R) = \operatorname{im}(.S)/\operatorname{im}(.R)$. By performing a reduction of the rows of S modulo a Janet basis for $D^{1\times 3}\, R$ and by another syzygy computation we obtain the finite presentation

$$\operatorname{t}(M) = (D^{1\times 3}\, S)/(D^{1\times 3}\, R) = (D\,(1 \;\; -\sin(2t) \;\; \cos(2t) \;\; 0))/(D^{1\times 3}\, R) \cong D/D\,(\partial - 1).$$

Hence, $a(t) := x_1(t) - \sin(2t)\, x_2(t) + \cos(2t)\, x_3(t)$ is a non-trivial autonomous element satisfying $\dot{a}(t) = a(t)$ whenever $x_1(t)$, $x_2(t)$, $x_3(t)$ are the first three components of a solution of $R\, y = 0$. The given linear system does not admit any parametrization in $\mathcal{F}^{4\times 1}$, but clearly for the linear system $R'\, y = 0$, where

$$R' := \begin{pmatrix} 1 & -\sin(2t) & \cos(2t) & 0 \\ 0 & \partial & 2 & -\cos(t) \\ 0 & -2 & \partial & -\sin(t) \end{pmatrix} \in D^{3\times 4},$$

whose system module is isomorphic to $M' := M/\,\mathrm{t}(M)$ and is therefore torsion-free, a parametrization in $\mathcal{F}^{4\times 1}$ for an appropriate signal space $\mathcal{F}$ exists. By construction, $P := \widetilde{R}_2$ is a parametrization:

$$\left\{\begin{array}{rcl} x_1(t) & = & -\sin(t)\,\dot{\phi}(t) + \cos(t)\,\phi(t), \\ x_2(t) & = & -\cos(t)\,\dot{\phi}(t) + \sin(t)\,\phi(t), \\ x_3(t) & = & -\sin(t)\,\dot{\phi}(t) - \cos(t)\,\phi(t), \\ u(t) & = & -\ddot{\phi}(t) - \phi(t) \end{array}\right.$$

for a suitable function ϕ. In fact, we can compute

$$Q' := \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{2}\cos(2t) & -\frac{1}{2}\sin(2t) \\ 0 & \frac{1}{2}\sin(2t) & -\frac{1}{2} - \frac{1}{2}\cos(2t) \\ 0 & \frac{1}{2} - \frac{1}{2}\cos(2t) & -\frac{1}{2}\sin(2t) \\ 0 & \sin(t)\,\partial & -\cos(t)\,\partial \end{pmatrix} \in D^{4\times 3}, \quad Q_2 := \begin{pmatrix} 0 \\ \sin(t) \\ -\cos(t) \\ 0 \end{pmatrix}^T \in D^{1\times 4},$$

Q' being a right inverse of R' and Q_2 a left inverse of $\widetilde{R}_2$ satisfying $Q_2\,Q' = 0$. This shows that the short exact sequence

$$0 \longleftarrow D \underset{.Q_2}{\overset{.\widetilde{R}_2}{\rightleftarrows}} D^{1\times 4} \underset{.Q'}{\overset{.R'}{\rightleftarrows}} D^{1\times 3} \longleftarrow 0$$

is split. The functor $\hom_D(-,\mathcal{F})$ transforms this exact sequence into the split exact sequence

$$0 \longrightarrow \mathcal{F} \underset{(Q_2).}{\overset{(\widetilde{R}_2).}{\rightleftarrows}} \mathcal{F}^{4\times 1} \underset{(Q').}{\overset{(R').}{\rightleftarrows}} \mathcal{F}^{3\times 1} \longrightarrow 0,$$

no assumption on $\mathcal{F}$ being necessary. We have $\mathrm{Sol}_{\mathcal{F}}(R') = \ker((R').) = \mathrm{im}((\widetilde{R}_2).)$.

Remark 4.16. Let us assume that P is a parametrization of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$. Applying the algorithm outlined in Remark 4.7 again to the linear system $P\,z = 0$ is actually a computation of $\mathrm{ext}^2_D(N,D)$, where N is the Auslander transpose of M. Since the isomorphism type of $\mathrm{ext}^n_D(N,D)$ does not depend on the choice of projective resolution of N, the existence of a *chain of parametrizations* of length r can be decided by checking whether we have $\mathrm{ext}^n_D(N,D) = \{0\}$ for all $n = 1, \ldots, r$ (cf. Example 4.18 below).

The existence of a chain of parametrizations of $R\,y = 0$ is a structural property of the system, i.e., is reflected by the system module M.

Theorem 4.17 (cf. [PQ99a], Corollaries 2, 3, 4; [CQR05], Theorems 5, 6, 7)**.** *Let D be an Ore algebra as in Remark 3.9, i.e., which admits Gröbner basis or Janet basis computations. Let $N = D^{q\times 1}/R\,D^{p\times 1}$ be the Auslander transpose of $M = D^{1\times p}/D^{1\times q}\,R$. Then we have:*

a) M is torsion-free if and only if $\mathrm{ext}^1_D(N,D) = \{0\}$.

b) M is reflexive if and only if $\mathrm{ext}^n_D(N,D) = \{0\}$ for $n = 1, 2$.

c) M is projective if and only if $\mathrm{ext}^n_D(N,D) = \{0\}$ for $n = 1, 2, \ldots, \mathrm{rgld}(D) = \mathrm{lgld}(D)$.

Example 4.18. Resuming Example 4.2, the following part of the de Rham complex for $\mathcal{F} = C^\infty(\Omega)$, where Ω is an open and convex subset of $\mathbb{R}^3$, is known to be an exact sequence:

$$\mathcal{F} \xrightarrow{\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}.} \mathcal{F}^{3\times 1} \xrightarrow{\begin{pmatrix} 0 & \partial_3 & -\partial_2 \\ -\partial_3 & 0 & \partial_1 \\ \partial_2 & -\partial_1 & 0 \end{pmatrix}.} \mathcal{F}^{3\times 1} \xrightarrow{\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \end{pmatrix}.} \mathcal{F}.$$

In other words, the linear systems defined by the divergence operator and the curl operator, respectively, are parametrizable under the above assumptions. Defining $D := \mathbb{R}[\partial_1, \partial_2, \partial_3]$ and $N := D/(\partial_1 \quad \partial_2 \quad \partial_3)\, D^{3\times 1}$, this is equivalent to $\mathrm{ext}^1_D(N, D) = \{0\}$ and $\mathrm{ext}^2_D(N, D) = \{0\}$.

For lack of space, we only mention here that direct sum decompositions of the system module M as $\mathrm{t}(M) \oplus (M/\,\mathrm{t}(M))$ were studied in the context of multidimensional linear systems in [QR05a] (cf. also [ZL01]). This study was again refined by the technique of purity or grade filtration (cf., e.g., [Bar10], [Qua13]). Moreover, parametrizations of linear differential systems can be applied to solve quadratic variational problems for multidimensional linear systems (even uncontrollable ones), arising, e.g., in optimal control, as shown in [PQ04], [QR06b].

5 Flatness and injective parametrizations

The case that a parametrization as defined in the previous section may be chosen as an injective map from the set of parameter values to the set of solutions is particularly interesting. This structural property of the system, referred to here by *flatness*, is equivalent to the freeness of the system module. In this section the notion of flatness is discussed for linear systems in general, focussing later on particular classes of linear systems, i.e., on particular classes of rings D of functional operators. After recalling the notion of stable range of a ring, linear systems with constant coefficients, linear differential systems with polynomial coefficients and coefficients of a more general kind will be investigated.

5.1 Free modules

The particular role played by free modules F in algebra is due to the universal property which allows to specify a well-defined homomorphism $F \to X$ unambiguously by any choice of values in a given module X for the elements of a basis of F. As it turns out, the meaning of freeness of a system module M is the possibility to specify any solution of the linear system unambiguously by a unique tuple of parameter values (cf. also Remark 4.3 for references to the non-linear case).

Definition 5.1. Let D be a Noetherian domain, $R \in D^{q\times p}$, and $\mathcal{F}$ a left D-module.

a) A parametrization $P \in D^{p\times m}$ of the linear system $R\,y = 0$ in $\mathcal{F}^{p\times 1}$ is said to be *injective* if the homomorphism $P. : \mathcal{F}^{m\times 1} \to \mathcal{F}^{p\times 1}$ is injective.

b) The linear system $R\,y = 0$ is said to be *flat* (over $\mathcal{F}$) if it admits an injective parametrization (in $\mathcal{F}^{p\times 1}$).

Remark 5.2. Flatness of a control system allows for a control paradigm called "open-loop control" (i.e., without feedback). Given a desired trajectory (which satisfies the governing equations of the system) and an injective parametrization, the tuple of parameter values corresponding to this solution is obtained by inverting the parametrization. Observables which express the parameters are also referred to as a *flat output* of the system. By construction, arbitrary trajectories may be assigned to the flat output. Now, the open-loop control law is obtained by substituting the given trajectory into the expressions defining the flat output. For a concrete example, cf. Example 5.16 below.

We recall how freeness can be characterized for finitely presented modules. To this end, let D be a (not necessarily commutative) Noetherian domain.

Remark 5.3. Let us assume that the left D-module $M = D^{1\times p}/D^{1\times q}\,R$ is free. Hence, we may choose a basis $(b_1, \ldots, b_r)$ of M, where

$$b_i = \widehat{b}_i + D^{1\times q}\,R, \qquad \widehat{b}_i \in D^{1\times p}, \qquad i = 1, \ldots, r.$$

If the rows $R_{1,-}$, …, $R_{q,-}$ of R are D-linearly independent, then $(R_{1,-}, \ldots, R_{q,-}, \widehat{b}_1, \ldots, \widehat{b}_r)$ is a basis of $D^{1\times p}$, and we have $r = p - q$. By stacking the rows $R_{1,-}$, …, $R_{q,-}$, $\widehat{b}_1$, …, $\widehat{b}_{p-q}$ we obtain a matrix $T \in \mathrm{GL}(p, D)$, and we have

$$R\,T^{-1} = \left(\begin{array}{cccc|cccc} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \end{array}\right). \tag{18}$$

Note that, if M is a finitely generated stably free left module over an Ore algebra as in Remark 3.9, we can compute a presentation matrix R of M whose rows are D-linearly independent by using the technique leading to Theorem 3.23; cf. Example 5.16 for an illustration.

Conversely, let us assume that there exists $T \in \mathrm{GL}(p, D)$ such that (18) holds. Then, in particular, R admits a right inverse with entries in D. Using the rows of T as a basis for $D^{1\times p}$ reveals that the residue classes of the last $p - q$ rows of T form a basis for $M \cong D^{1\times(p-q)}$.

Summing up the previous remark, we obtain the following proposition.

Proposition 5.4. *Let $R \in D^{q\times p}$ be a matrix which admits a right inverse with entries in D. Then the left D-module $M = D^{1\times p}/D^{1\times q}\,R$ is free if and only if there exists $T \in \mathrm{GL}(p, D)$ such that (18) holds. If this is the case, then the residue classes of the last $p - q$ rows of T form a basis for M.*

Finally, the module-theoretic characterization of flatness for linear systems can be stated as follows. We assume that D is an Ore algebra as in Remark 3.9.

Proposition 5.5. *Let M be the system module defined by $R\,y = 0$, where $R \in D^{q\times p}$, and let $\mathcal{F}$ be an injective left D-module which is a cogenerator for the category of left D-modules. There exists an injective parametrization $P \in D^{p\times m}$ of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$ if and only if M is free.*

Proof. If M is free, we may assume that R admits a right inverse (by Proposition 2.12 and Theorem 3.23). With respect to a basis of M the canonical projection $D^{1\times p} \to D^{1\times p}/D^{1\times q}\,R$ is represented by a matrix $P \in D^{p\times(p-q)}$. By applying the functor $\hom_D(-, \mathcal{F})$ to the split short exact sequence

$$0 \longleftarrow D^{1\times(p-q)} \xleftarrow{.P} D^{1\times p} \xleftarrow{.R} D^{1\times q} \longleftarrow 0 \tag{19}$$

we obtain the split short exact sequence

$$0 \longrightarrow \mathcal{F}^{(p-q)\times 1} \xrightarrow{P.} \mathcal{F}^{p\times 1} \xrightarrow{R.} \mathcal{F}^{q\times 1} \longrightarrow 0.$$

Conversely, if $P \in D^{p\times m}$ is an injective parametrization of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$, then the homomorphism $.P$ in (12) is surjective because $\mathcal{F}$ is a cogenerator for the category of left D-modules. Since ι in (12) is injective with the same image as $.P$, we have $M \cong D^{1\times m}$. □

Remark 5.6. Let $M = D^{1\times p}/D^{1\times q}\,R$ be free. Using the notation of Remark 5.3, the matrix whose rows are $\widehat{b}_1, \ldots, \widehat{b}_{p-q}$ admits a right inverse $P \in D^{p\times(p-q)}$ defining the split short exact sequence (19). Note that the first part of the proof of Proposition 5.5 does not depend on the assumption that $\mathcal{F}$ is a cogenerator. Hence, the computation of a basis of M (cf. the next subsections) yields an injective parametrization of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$ for any signal space $\mathcal{F}$.

Conversely, if P is an injective parametrization of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$, where $\mathcal{F}$ is a cogenerator for the category of left D-modules, then P admits a left inverse whose rows represent a basis of M.

5.2 The stable range of a ring

Let D be a (not necessarily commutative) ring. In this subsection we recall a method to split off a free direct summand of rank one from a given left or right D-module. Clearly, this is fundamental for a study of stably free D-modules.

Definition 5.7. Let V be either the left D-module $D^{1\times r}$ or the right D-module $D^{r\times 1}$, where $r \in \mathbb{N}$. Moreover, let $V = V' \oplus V''$ be the decomposition into submodules of V, where V' and V'' are generated by the first $r-1$ standard basis vectors and the last standard basis vector, respectively. For each $v \in V$ let $v = v' + v''$ be the corresponding decomposition of v.

a) A vector $v \in V$ is said to be *unimodular* if there exists $\varphi \in \hom_D(V, D)$ such that $\varphi(v) = 1$.

b) A unimodular vector $v \in V$ is said to be *stable* if there exists $\psi \in \hom_D(D, V')$ such that $v' + \psi(v'')$ is unimodular (as an element of V').

c) A positive integer r is said to be *in the stable range* of D, considered as a left or right D-module, if for every $s > r$, every unimodular vector in $D^{1\times s}$ or in $D^{s\times 1}$, respectively, is stable.

d) The *stable rank* of D, considered as a left or right D-module, is defined as the least positive integer in the stable range of D and as ∞ if no such integer exists.

Proposition 5.8 (cf. [MR00], Prop. 11.3.4)**.** *The stable rank of D considered as a left D-module is equal to the one of D considered as a right D-module.*

Remark 5.9. Let $v \in D^{1\times r}$. Then v is unimodular if and only if $v_1\,D + v_2\,D + \ldots + v_r\,D = D$. Let us assume that v is stable. Then there exist $u_1, \ldots, u_{r-1} \in D$ such that

$$(v_1 + v_r\,u_1,\ v_2 + v_r\,u_2,\ \ldots,\ v_{r-1} + v_r\,u_{r-1}) \in D^{1\times(r-1)} \tag{20}$$

is unimodular. Hence, there exist $a_1, \ldots, a_{r-1} \in D$ such that

$$(v_1 + v_r\,u_1)\,a_1 + (v_2 + v_r\,u_2)\,a_2 + \ldots + (v_{r-1} + v_r\,u_{r-1})\,a_{r-1} = 1. \tag{21}$$

Defining the vector $\widetilde{v} \in D^{1\times r}$ whose first $r-1$ entries are given by (20) and whose last entry is v_r and using (21), a suitable right D-linear combination of the first $r-1$ entries of $\widetilde{v}$ can be added to the last entry of $\widetilde{v}$ to get $\widetilde{v}_1 - 1$. By subtracting the last entry of the modified vector from the first one, we obtain 1 as first entry, which now allows to eliminate the other entries. These transformations can be realized by right multiplication of v by a product of certain lower or upper triangular matrices with entries in D whose diagonal entries are equal to 1. The following lemma summarizes the above discussion (cf. also [QR07] for more details).

Lemma 5.10. *Let $v \in D^{1\times r}$ be stable. Then there exists $T \in \mathrm{GL}(r, D)$ which is a product of matrices of the form $I_r + d\, E^{(i,j)}$, where $d \in D$, $i \neq j$, and $E^{(i,j)} \in D^{r\times r}$ is defined by $E^{(i,j)}_{k,l} := \delta_{i,k}\, \delta_{j,l}$, such that $v\, T = (1, 0, \ldots, 0)$.*

Hence, if the first row of a matrix $R \in D^{q\times p}$ is stable, then the right D-module which is generated by the columns of R is recognized as a direct sum of the free right D-module D and a right D-module which is generated by the suitably reduced last $p-1$ columns of R. We perform this splitting in the context where R is a presentation matrix. In what follows we assume that D is an Ore algebra as in Remark 3.9 and we assume that Lemma 5.10 is constructive, i.e., that suitable $u_1, \ldots, u_{r-1} \in D$ as in Remark 5.9 can be computed (cf. the following subsections).

Remark 5.11. Let $M = D^{1\times p}/D^{1\times q}\, R$ be a stably free right D-module with presentation matrix $R \in D^{q\times p}$. Using the technique discussed in Remark 3.15 and leading to Theorem 3.23, we may assume that R admits a right inverse $S \in D^{p\times q}$. Let $v_1, \ldots, v_q$ be the rows of R. Then, in particular, v_1 is a unimodular vector. If p is greater than the stable rank of D, then v_1 is stable and, by Lemma 5.10, there exists $T \in \mathrm{GL}(p, D)$ such that $v_1\, T = (1, 0, \ldots, 0)$. Using T to define a change of basis for $D^{1\times p}$, we conclude

$$D^{1\times q}\, R \;\cong\; D^{1\times q}\, R\, T \;=\; D\, v_1\, T + D\, v_2\, T + \ldots + D\, v_q\, T.$$

Using the fact that $v_1\, T$ is the first standard basis vector of $D^{1\times p}$, subtraction of a suitable left D-multiple of $v_1\, T$ from $v_i\, T$ yields a vector of the form $(0, w_i)$, where $w_i \in D^{1\times(p-1)}$, $i = 2, \ldots, q$. Hence, we have

$$M \;\cong\; (D \oplus D^{1\times(p-1)})/(D\, v_1\, T \oplus (D\,(0, w_2) + \ldots + D\,(0, w_q))) \;\cong\; D^{1\times(p-1)}/D^{1\times(q-1)}\, R',$$

where the presentation matrix $R' \in D^{(q-1)\times(p-1)}$ of M is formed by the vectors $w_2, \ldots, w_q$. Let $U \in \mathrm{GL}(q, D)$ be such that the rows of $U\, R\, T$ are $T\, v_1$, $(0, w_2)$, $\ldots$, $(0, w_q)$. Then we have $(U\, R\, T)\,(T^{-1}\, S\, U^{-1}) = I_q$, which shows that a right inverse of R' is obtained from $T^{-1}\, S\, U^{-1}$ by removing its first row and its first column. If $p-1$ is still greater than the stable rank of D, this reduction step can be applied again to R'.

The reduction technique discussed in the previous remark relies on Lemma 5.10 and Gaussian elimination over D. Iteration yields the following theorem.

Theorem 5.12 (cf. [QR07], Cor. 44)**.** *Let $M = D^{1\times p}/D^{1\times q}\, R$ be a stably free left D-module, where R admits a right inverse over D. If $p - q$ is greater than or equal to the stable rank of D, then M is free. If a constructive version of Lemma 5.10 is available over D, then a basis of M can be computed by applying Gaussian elimination over D to R.*

Proof. Iterating the reduction described in Remark 5.11 $q-1$ times we obtain a presentation matrix of M of shape $1\times(p-q+1)$ which admits a right inverse. By hypothesis, the reduction can be applied one more time to prove that we have $M \cong D^{1\times(p-q)}$. For computing a basis of M we keep the shape of the presentation matrices and refrain from subtracting suitable left D-multiples of $v_1 T$ from $v_2 T, \ldots, v_q T$. Then, the product of (extended versions of) the matrices $T \in \mathrm{GL}(p, D)$ provided by Lemma 5.10 is a matrix $Q \in \mathrm{GL}(p, D)$ such that $R\,Q$ is lower triangular with diagonal entries equal to 1. Therefore, the last $p-q$ columns of Q form an injective parametrization P of $R\,y = 0$ in $\mathcal{F}^{p\times 1}$ for any signal space $\mathcal{F}$, and the rows of a left inverse of P define a basis of M. □

Remark 5.13. An involution θ of D (cf. Remark 3.11) allows to apply similar reductions to the columns of $\theta(R)$ instead of the rows of R. Then we use a version of Lemma 5.10 for columns instead of rows, i.e., in Remark 5.9 we deal with *left* ideals of D instead of right ideals. Since the module M under consideration is a *left* D-module, we can then restrict our attention to the action of D from one side.

5.3 Linear systems with constant coefficients

A prominent class of linear functional systems is given by matrices of operators which do not involve multiplication by functions of the coordinates on which the unknown functions depend, e.g., time-invariant linear ordinary differential equations or time-invariant linear differential time-delay equations. The systems discussed in Example 4.2 are of this kind, but not, e.g., the system treated in Example 4.15. We call such a system a *linear system with constant coefficients*. In the algebraic framework described above, such a linear system can be dealt with by choosing D to be a commutative polynomial algebra.

Let $D = k[\partial_1, \ldots, \partial_n]$, where k is a field. *Serre's problem* (cf. [Lam06]) asks whether or not there exists a difference between finitely generated projective (or stably free) and free modules over D. Since for every finitely generated projective D-module M there exists a presentation matrix $R \in D^{q\times p}$ which admits a right inverse with entries in D, Serre's problem is equivalent to the question whether or not such a matrix can be augmented by $p-q$ rows with entries in D such that the resulting square matrix is invertible over D (cf. Remark 5.3). About twenty years after J.-P. Serre posed this problem, it was solved by D. Quillen and A. A. Suslin independently.

Theorem 5.14 (cf. [Qui76], [Sus76])**.** *Let $D = k[\partial_1, \ldots, \partial_n]$. For every matrix $R \in D^{q\times p}$ which admits a right inverse with entries in D there exists a matrix $B \in D^{(p-q)\times p}$ such that $(R^T \quad B^T)^T \in \mathrm{GL}(p, D)$. Every finitely generated projective module over D is free.*

Remark 5.15. Several authors have been working on constructive approaches to the Quillen-Suslin Theorem, cf., e.g., [LS92], [PW95], [LY05], [FQ07]. A recent implementation in Maple was developed in [Fab09]. For more details, we also refer to [FQ07].

Example 5.16. The following system of linear differential time-delay equations appears in a study of a flexible rod, cf. [Mou95, p. 120] or [MRPF95]:

$$\left\{ \begin{array}{rcl} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) & = & 0, \\ 2\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) & = & 0. \end{array} \right. \tag{22}$$

We define the commutative polynomial algebra $D = \mathbb{R}[\partial, \delta]$, where ∂ represents the differential operator $\frac{d}{dt}$ and δ the shift operator, and the matrix

$$R := \begin{pmatrix} \partial & -\partial\,\delta & -1 \\ 2\,\partial\,\delta & -\partial\,(1+\delta^2) & 0 \end{pmatrix} \in D^{2\times 3}.$$

Then (22) is expressed as $R\,(y_1, y_2, u)^T = 0$. Let, e.g., $\mathcal{F} = C^\infty(\mathbb{R})$, a choice which will be justified below. Finally, we define $M = D^{1\times 3}/D^{1\times 2}\,R$. Then a presentation of the torsion submodule $\mathrm{t}(M)$ of M is given by

$$\mathrm{t}(M) \;=\; (D^{1\times 3}\,S)/(D^{1\times 2}\,R) \;=\; (D\,(-2\,\delta \quad \delta^2+1 \quad 0))/(D^{1\times 2}\,R) \;\cong\; D/D\,\partial,$$

where

$$S := \begin{pmatrix} -2\,\delta & \delta^2+1 & 0 \\ \partial & -\partial\,\delta & -1 \\ \partial\,\delta & -\partial & \delta \end{pmatrix} \in D^{3\times 3}$$

is computed as explained in Section 4. Hence, the given system is not parametrizable, but the system module associated with $S\,(y_1, y_2, u)^T = 0$ is isomorphic to $M/\,\mathrm{t}(M)$ and hence torsion-free. Using the technique discussed in Remark 3.15, the free resolution

$$0 \longleftarrow M/\,\mathrm{t}(M) \longleftarrow D^{1\times 3} \xleftarrow{.S} D^{1\times 3} \xleftarrow{.(\partial \quad \delta \quad 1)} D \longleftarrow 0$$

can be reduced to the following one:

$$0 \longleftarrow M/\,\mathrm{t}(M) \longleftarrow D^{1\times 4} \xleftarrow{.\widetilde{S}} D^{1\times 3} \longleftarrow 0\,, \qquad \widetilde{S} := \begin{pmatrix} -2\,\delta & \delta^2+1 & 0 & 0 \\ \partial & -\partial\,\delta & -1 & 0 \\ \partial\,\delta & -\partial & \delta & 1 \end{pmatrix}.$$

Since $\widetilde{S}$ admits a right inverse, $M/\,\mathrm{t}(M)$ is stably free and therefore free by the Quillen-Suslin Theorem. We conclude that the linear system $S\,(y_1, y_2, u)^T = 0$ is flat. The Maple implementation `QuillenSuslin` of a constructive version of the Quillen-Suslin Theorem developed in [Fab09] completes the matrix $\widetilde{S}$ with a fourth row $(1 \quad -\delta/2 \quad 0 \quad 0)$ to a matrix in $\mathrm{GL}(4, D)$. Inversion of this matrix yields an injective parametrization

$$P := \begin{pmatrix} \delta^2+1 \\ 2\,\delta \\ -\partial\,(\delta^2-1) \end{pmatrix} \in D^{3\times 1}$$

of $S\,(y_1, y_2, u)^T = 0$ in $\mathcal{F}^{3\times 1}$, and the residue class of the above fourth row in $M/\,\mathrm{t}(M)$ is a flat output. In order to demonstrate the use of a flat output, we note first that every solution of (22) in $\mathcal{F}^{3\times 1}$ is of the form

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ u(t) \end{pmatrix} = P\,\xi(t) \tag{23}$$

for some $\xi(t) \in \mathcal{F}$ and that $\xi(t)$ is uniquely determined by

$$\xi(t) = y_1(t) - \frac{1}{2} y_2(t-1). \tag{24}$$

Substituting desired trajectories (in $\mathcal{F}$) for $y_1(t)$, $y_2(t)$, satisfying (22) for some $u(t) \in \mathcal{F}$, into (24), yields $\xi(t)$ and $u(t)$ via (23). Hence, we obtain an open-loop control law realizing the given trajectories.

The same reasoning applies to any D-module $\mathcal{F}$ (cf. Remark 5.6). For more details, we also refer to [FQ07, Ex. 5.3 and Ex. 5.5].

The construction of a matrix B (or B^{-1}) as in Theorem 5.14 usually eliminates variables from the system matrix R in an inductive fashion. An application of this strategy is the following.

Corollary 5.17 (cf. [FQ06] or [FQ07]). *Let $D = k[\partial_1, \ldots, \partial_n]$, $R(\partial_1, \ldots, \partial_n) \in D^{q \times p}$ a matrix admitting a right inverse with entries in D, and $\mathcal{F}$ an injective cogenerator for the category of D-modules (cf. Example 2.9 a)). The flat multidimensional linear system $R(\partial_1, \ldots, \partial_n)\, y = 0$ is equivalent (in the sense of Remark 2.10) to the controllable (i.e., parametrizable) one-dimensional linear system $R(\partial_1, 0, \ldots, 0)\, y = 0$.*

The next remark shows that a controlled enlargement of the ring D (e.g., admitting the inverses of shift operators in the context of differential time-delay systems) can turn a merely controllable linear system into a flat one.

Remark 5.18. Let $D = k[\partial_1, \ldots, \partial_n]$ be as above and $S \subset D$ a multiplicatively closed subset, i.e., we have $0 \notin S$ and the product of each two elements of S is in S. Then the localization $S^{-1}D$ is the commutative ring whose elements are represented as d/s, where $d \in D$ and $s \in S$, and addition and multiplication are defined as usual for fractions. For any D-module M, the localization $S^{-1}M$ is an $S^{-1}D$-module. Then, for finitely generated D-modules M_1 and M_2 and every $n \in \mathbb{Z}_{\geq 0}$, there is an isomorphism of $S^{-1}D$-modules (cf., e.g., [Rot09, Thm. 7.39])

$$S^{-1}\, \mathrm{ext}^n_D(M_1, M_2) \cong \mathrm{ext}^n_{S^{-1}D}(S^{-1}M_1, S^{-1}M_2).$$

Using the fact that an element $m \in M$ satisfying $d\, m = 0$ for some non-zero $d \in D$ is zero in $S^{-1}M$ whenever $d \in S$, localization can arrange for vanishing of extension groups. Hence, by Theorem 4.17, obstructions to torsion-freeness, reflexiveness, etc., of M are encoded in the annihilators of the extension groups $\mathrm{ext}^n_D(N, D)$, and with an appropriate localization, $S^{-1}M$ is torsion-free, reflexive, etc. For more details about encoding these obstructions in a polynomial, called π-polynomial, we refer to, e.g., [Mou95], [CQR05].

5.4 Linear differential systems with polynomial coefficients

In this subsection we consider systems of linear partial differential equations which involve multiplication of the unknown functions or their derivatives by polynomials or rational functions in the independent variables. If at least one of these polynomials or rational functions is not constant, a representation of the system as $R\, y = 0$ is defined by a matrix R with entries in a Weyl algebra or a ring of differential operators with rational function coefficients (cf. Example 3.8 b)). For systems of this kind whose system module is free of rank at least 2 we describe an algorithm to compute a basis, i.e., a flat output (cf. [QR06a], [QR07], [GV03]).

The following theorem is of central importance for the techniques discussed in this subsection. For more details about Weyl algebras, we refer to [Bjö79].

Theorem 5.19 (Stafford, cf. [Sta78], Thm. 3.1)**.** *Let k be a field of characteristic zero, $n \in \mathbb{N}$. Then every left ideal of the Weyl algebra $D = A_n(k)$ is generated by two elements. More generally, for all a, b, $c \in D$ and $d \in D \setminus \{0\}$ there exist u_1, $u_2 \in D$ such that*

$$D\,a + D\,b + D\,c = D\,(a + d\,u_1\,c) + D\,(b + d\,u_2\,c). \tag{25}$$

Analogous statements hold for all right ideals of $A_n(k)$.

Example 5.20. The left ideal of $A_3(\mathbb{Q}) = \mathbb{Q}[x_1, x_2, x_3]\langle\partial_1, \partial_2, \partial_3\rangle$ which is generated by $a := \partial_1$, $b := \partial_2$, $c := \partial_3$ is also generated by $a' := a$ and $b' := b + x_1\,c$, because we have

$$c = (-x_1\,\partial_3 - \partial_2)\,a' + \partial_1\,b', \tag{26}$$

and then b is also a left $A_3(\mathbb{Q})$-linear combination of a' and b'. Here d was chosen to be 1.

For instance, if we choose $d = x_2$, then we have

$$A_3(\mathbb{Q})\,a + A_3(\mathbb{Q})\,b + A_3(\mathbb{Q})\,c = A_3(\mathbb{Q})\,a'' + A_3(\mathbb{Q})\,b''$$

with $a'' := a$ and $b'' := b + d\,x_1\,c$, because we have

$$c = \partial_3 = -(\partial_2^2 + x_1^2\,x_2^2\,\partial_3^2 + 2\,x_1\,x_2\,\partial_2\,\partial_3 + x_1\,\partial_3)\,a'' + (\partial_1\,\partial_2 + x_1\,x_2\,\partial_1\,\partial_3 - x_2\,\partial_3)\,b'',$$

and thus, b can be expressed as a left $A_3(\mathbb{Q})$-linear combination of a'' and b'' as well.

Remark 5.21. Effective versions of Theorem 5.19 were developed in [HS01] and [Ley04]. The strategy can be outlined as follows. Two elements u_1, $u_2 \in A_n(k)$ satisfy (25) if c is a left $A_n(k)$-linear combination of $a' := a + d\,u_1\,c$ and $b' := b + d\,u_2\,c$, as exemplified in (26). We may assume that a, b, c, $d \in A_n(k)$ are all non-zero. A computation of u_1, u_2 starts with a representation

$$q\,c = h_1\,a + h_2\,b, \tag{27}$$

where q, h_1, $h_2 \in A_n(k)$, $q \neq 0$. Since a common left multiple of each pair of elements of a Weyl algebra can be computed, we obtain such a representation, e.g., by considering the pair (a, c). By repeatedly adding terms to a and b in a clever way such that in an updated representation (27) the indeterminates $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ are eliminated from the factor q, one finally arrives at a representation (27), where q is a non-zero element of k and hence invertible. In order to achieve this goal, quite a few Gröbner or Janet basis computations are necessary in general, which indicates the complexity of such an algorithm and the size of its output. Implementations of such a procedure are available in the Macaulay2 package `Dmodules` (cf. [LT]) and in the Maple package **`Stafford`** (cf. [QR07]).

We are going to demonstrate that an effective version of Theorem 5.19 allows to compute bases of free left modules over $A_n(k)$ or $B_n(k)$ of rank at least 2. Since we are in position to use an involution of $A_n(k)$ or $B_n(k)$ (cf. Example 3.12), we apply the reduction process for presentation matrices described in Subsection 5.2 to columns instead of rows (cf. Remark 5.13).

Remark 5.22. Let $D = A_n(k)$ be the Weyl algebra, where k is a field of characteristic zero. Let $(v_1, v_2, \ldots, v_m)^T \in D^{m\times 1}$ be a unimodular vector, where $m \geq 3$. By Theorem 5.19, there exist u_1, $u_2 \in D$ such that $D\,v_1 + D\,v_2 + D\,v_m = D\,(v_1 + u_1\,v_m) + D\,(v_2 + u_2\,v_m)$. Hence,

$$D\,(v_1 + u_1\,v_m) + D\,(v_2 + u_2\,v_m) + D\,v_3 + \ldots + D\,v_{m-1} = D\,v_1 + D\,v_2 + \ldots + D\,v_m = D,$$

which shows that $(v_1+u_1\,v_m, v_2+u_2\,v_m, v_3, \ldots, v_{m-1})^T$ is unimodular. We conclude that the stable rank of D, considered as a right D-module, is at most 2, and the same is true if we consider D as a left D-module.

In fact, the following theorem is a corollary of Theorem 5.19.

Theorem 5.23 (cf. [Sta78], Cor. 3.2 (a))**.** *Let k be a field of characteristic zero, $n \in \mathbb{N}$. Then the stable ranks of the Weyl algebra $A_n(k)$ and the ring $B_n(k)$ of differential operators with rational function coefficients are equal to* 2*.*

Example 5.24. Let us consider the following linear partial differential equation for three unknown functions y_1, y_2, y_3 of the independent variables x_1, x_2.

$$\frac{\partial y_1}{\partial x_1} - y_1 + \frac{\partial y_2}{\partial x_2} + x_2\,\frac{\partial y_3}{\partial x_1} = 0.$$

Defining $D := A_2(\mathbb{R})$ and the matrix

$$R = \begin{pmatrix} \partial_1 - 1 & \partial_2 & x_2\,\partial_1 \end{pmatrix} \in D^{1\times 3},$$

the equation can be written as $R\,y = 0$, where $y = (y_1, y_2, y_3)^T$. The matrix R admits the right inverse $(x_2\,\partial_2, x_2, -\partial_2)^T$. Hence, the left D-module $M := D^{1\times 3}/D\,R$ is stably free, and by Theorems 5.12 and 5.23, M is free. Therefore, the linear system under consideration is flat. We apply the combination of Remarks 5.11 and 5.22 in order to compute a basis of M, i.e., we perform row operations on the adjoint $\widetilde{R} = (-\partial_1 - 1, -\partial_2, -x_2\,\partial_1)^T$ of R (cf. Example 3.12) as follows. For the entries $a := -\partial_1 - 1$, $b := -\partial_2$, $c := -x_2\,\partial_1$ of $\widetilde{R}$ we have

$$D\,a+D\,b+D\,c = D\,(a+u_1\,c)+D\,(b+u_2\,c) = D\,(a+c)+D\,(b+c), \quad u_1 := 1, \quad u_2 := 1, \tag{28}$$

because we obtain $(a, b, c)^T$ as

$$\begin{pmatrix} -\partial_1 - 1 \\ -\partial_2 \\ -x_2\,\partial_1 \end{pmatrix} \begin{pmatrix} (x_2+1)\,(x_2\,\partial_1 + \partial_2) - 1 & (-x_2 - 1)\,((x_2+1)\,\partial_1 + 1) \end{pmatrix} \begin{pmatrix} a+c \\ b+c \end{pmatrix}.$$

Since $\widetilde{R}$ admits a left inverse over D, the left ideal of D which is generated by a, b, and c is equal to D. Therefore, (28) implies that $(a+c, b+c)^T$ admits a left inverse $(v_1, v_2) \in D^{1\times 2}$. Now, multiplying the following matrix product from the left to $\widetilde{R}$ yields $(1, 0, 0)^T$:

$$\begin{pmatrix} 1 & 0 & 0 \\ -(b+c) & 1 & 0 \\ -(a-1+c) & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (a-1)\,v_1 & (a-1)\,v_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}$$

The adjoint U of this matrix satisfies $R\,U = (1, 0, 0)$. Hence, by Proposition 5.4, the residue classes in M of the last two rows of $T := U^{-1}$ form a basis of M.

An implementation of this technique to compute bases of free left D-modules of rank at least 2 is available in the Maple package `Stafford` (cf. [QR07]), which is based on `OreModules`.

In the context of control theory we obtain the following corollary.

Corollary 5.25 (cf. [QR06a], Cor. 2)**.** *Every controllable linear ordinary differential system with polynomial or rational function coefficients and with at least two inputs is flat.*

5.5 Linear differential systems with coefficients of a more general kind

The results of the previous subsection were generalized to systems of linear ordinary differential equations with formal or convergent power series coefficients in one variable in [QR10]. The case of partial differential equations is also settled if formal or convergent Laurent series coefficients are allowed (cf. the second paragraph in Remark 5.26 below), but partial differential equations with formal or convergent power series coefficients need still to be investigated in this respect. For related module-theoretic results for Dedekind prime rings and certain simple Dedekind domains, we refer to [MR00, Thm. 5.7.8 and Cor. 7.11.6].

An important source of systems of linear partial differential equations with non-constant coefficients are systems of non-linear partial differential equations. A linearization of such a non-linear system is given by the formal Fréchet derivatives of the left hand sides of the equations (cf. [Rob06], and, for a more geometric point of view, [Pom01]). The linearized equations can also be expressed algebraically in terms of Kähler differentials. In general, the coefficients of the resulting linear system depend on the unknown functions of the original system. A treatment of the linearization along the lines of the previous sections then also needs to take into account the equations of the non-linear system, as they imply relations for the coefficients of the linearization.

First we consider systems of linear ordinary differential equations with formal or convergent power series coefficients.

Remark 5.26. Let D be either $k[[t]]\langle\partial\rangle$ or $k\{t\}\langle\partial\rangle$, i.e., the skew polynomial ring generated by $k[[t]]$ or $k\{t\}$ and ∂ with the commutation rules that are implied by the product rule for $\partial = \frac{d}{dt}$. Here $k[[t]]$ and $k\{t\}$ denote the rings of formal and convergent power series in t, respectively, and we assume that k is either $\mathbb{R}$ or $\mathbb{C}$.

Using a result of Coutinho and Holland (cf. [CH88]), one can prove (cf. [QR10, Corollaries 2 and 3]) that the ring D satisfies the statements of Stafford's Theorem 5.19. By a theorem of Caro and Levcovitz (cf. [CL10]), the same is true for $D = A\langle\partial_1, \ldots, \partial_n\rangle$, the ring of differential operators with coefficients in the field of fractions A of $k[[t]]$ or $k\{t\}$ (cf. also [QR]).

Using a constructive version of Lemma 5.10 for a ring D as above, Remark 5.11 (and, if necessary, also Remark 5.22) becomes effective, and bases of free left D-modules of rank at least 2 can be computed. A Maple package `StaffordAnalytic` is under development which implements these routines for the case in which the coefficients are given in finite terms, e.g., by rational functions in exp, sin, cos, etc.

Example 5.27. Let us restrict the linear system (15) in Example 4.15 to the first two equations, i.e., let $D = k\{t\}\langle\partial\rangle$, where ∂ represents the differential operator $\frac{d}{dt}$, $k \in \{\mathbb{R}, \mathbb{C}\}$, and define

$$R := \begin{pmatrix} \partial - 1 & \sin(2t) & -\cos(2t) & -\sin(t) \\ 0 & \partial & 2 & -\cos(t) \end{pmatrix} \in D^{2\times 4}.$$

The system module corresponding to the linear system $R\,y = 0$, where $y = (x_1, x_2, x_3, u)^T$, is $M := D^{1\times 4}/D^{1\times 2}\,R$. The matrix R admits the right inverse

$$S := \frac{1}{d}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\cos(t) & \sin(t) \\ -2 & -\cos(2t) \end{pmatrix} \in D^{4\times 2}, \qquad \text{where} \quad d := \cos(t)\cos(2t) + 2\sin(t).$$

In particular, the rows of R are D-linearly independent. By applying Gaussian elimination (e.g., as described in Remark 5.9, modified according to Remark 5.13) to the rows of

$$\theta(R) = \begin{pmatrix} -\partial - 1 & 0 \\ \sin(2t) & -\partial \\ -\cos(2t) & 2 \\ -\sin(t) & -\cos(t) \end{pmatrix},$$

computing the product of the elementary matrices which realize these operations, and applying the involution, we obtain the matrix

$$Q := \frac{1}{d} \begin{pmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \\ -\cos(t) & \sin(t) & \cos(t)\,\partial - \cos(t) & -\sin(t)\,\partial + \sin(2t)\cos(t) \\ -2 & -\cos(2t) & 2(\partial + 1) & \cos(2t)\,\partial + 2\sin(2t) \end{pmatrix} \in \mathrm{GL}(4, D),$$

which satisfies

$$R\,Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In fact, Q is the inverse of the square matrix which is obtained by appending the standard basis vectors e_1 and e_2 of $D^{1\times 4}$ to R, and the first two columns of Q coincide with those of S. Hence, by Remark 5.3, a basis for the free left D-module M is given by $(e_1 + D^{1\times q}\,R, e_2 + D^{1\times q}\,R)$.

Similarly to Corollary 5.25 we obtain a corresponding statement about linear control systems which give rise to left D-modules as considered above.

Corollary 5.28 (cf. [QR10], Cor. 4)**.** *Every controllable linear ordinary differential system with convergent power series coefficients and with at least two inputs is flat.*

The last class of linear systems we discuss here arises from linearizing systems of non-linear (ordinary or partial) differential equations. Although we will not elaborate on the notion of Kähler differentials, we understand the linearization of a differential equation in this sense, i.e., the linearized equation is obtained by applying the universal derivation d to the non-linear equation, and we write Y, $\dot{Y}$, $\ddot{Y}$, etc. for dy, $d\dot{y}$, $d\ddot{y}$, respectively, when linearizing an ordinary differential equation for an unknown function y, and similarly for partial differential equations. The linearization can also be interpreted as a formal Fréchet derivative. For more details, we refer to [Pom01, Ch. VI] and [Rob06, Sect. 3.2].

Remark 5.29. We define a system described by (not necessarily linear) partial differential equations to be controllable if every observable is free (cf. Remark 4.3). Since linearization of an autonomous equation shows that some observable of the linearized system is autonomous, controllability of the linearized system implies controllability of the given system (cf. [Pom01, p. 809]). The linearized system is described by linear partial differential equations whose coefficients are subject to the given (not necessarily linear) equations. In order to be able to apply the techniques of the previous sections, the arithmetic of the coefficient field needs to be implemented (cf. [Rob06] for more details). In favorable situations the given equations can be solved for the terms involving the highest derivatives. Then these equations can be used as rewriting rules for the coefficients of the linearized system. However, as already mentioned in Remark 3.1, deciding whether or not a given non-linear expression is the left hand side of a consequence of the given system requires a particularly suited generating set. In general, such a preprocessing is necessary to obtain a confluent and terminating rewriting system for the coefficients. For systems of polynomially non-linear partial differential equations, the problem is solved by a decomposition of the radical differential ideal generated by the equations into prime differential ideals, or similar decompositions (cf., e.g., [Dio92], [BLOP09], [Gri89], [Wan01], [Rob12], [LHR13], and the references therein). Since we confine ourselves to linear systems in this article, we only mention here work in progress by T. Cluzeau, A. Quadrat, and the author of this paper investigating certain classes of quasi-linear differential systems using this approach.

We treat an example taken from [OM02, 5.2.3] (cf. also the references therein). For more details about the techniques that are applied in this example, we refer to [Rob06] and [Pom01].

Example 5.30. Let us consider the following system of non-linear ordinary differential equations for unknown functions x_1, x_2, and u:

$$\left\{ \begin{array}{rcl} \dot{x}_1 + (u + \frac{1}{2}u^2)\, x_1 & = & 0, \\ \dot{x}_2 - u\, x_1 & = & 0. \end{array} \right. \tag{29}$$

The linearization of this system is given by

$$\left\{ \begin{array}{rcl} \dot{X}_1 + (u + \frac{1}{2}u^2) X_1 + (1+u)\, x_1\, U & = & 0, \\ \dot{X}_2 - u\, X_1 - x_1\, U & = & 0. \end{array} \right.$$

Let $\mathbb{Q}\{x_1, x_2, u\}$ be the differential polynomial ring over $\mathbb{Q}$ in the differential indeterminates x_1, x_2, u with one derivation, i.e., the polynomial ring in infinitely many indeterminates $(x_1)_i$, $(x_2)_j$, u_k, where $i, j, k \in \mathbb{Z}_{\geq 0}$. These indeterminates represent the derivatives of the unknown functions x_1, x_2, and u. The derivation is trivial on $\mathbb{Q}$ and maps $(x_1)_i$, $(x_2)_j$, u_k to $(x_1)_{i+1}$, $(x_2)_{j+1}$, u_{k+1}, respectively, where $(x_1)_0 = x_1$, $(x_2)_0 = x_2$, $u_0 = u$.

The left hand sides of (29) generate a differential ideal I of $\mathbb{Q}\{x_1, x_2, u\}$ which is prime because the residue class ring $\mathbb{Q}\{x_1, x_2, u\}/I$ is an integral domain. Hence, the field of fractions $\mathrm{Quot}(\mathbb{Q}\{x_1, x_2, u\}/I)$ of this residue class ring exists, and we define the skew polynomial ring $D = \mathrm{Quot}(\mathbb{Q}\{x_1, x_2, u\}/I)\langle \partial \rangle$ whose commutation rules are given by the product rule of differentiation (cf. Example 3.8 b)). By abusing notation we also write x_1, x_2, u for the residue classes of x_1, x_2, u, respectively, in $\mathrm{Quot}(\mathbb{Q}\{x_1, x_2, u\}/I)$.

The rewriting rules $\dot{x}_1 \mapsto -(u + \frac{1}{2}u^2)\, x_1$, $\dot{x}_2 \mapsto u\, x_1$ define a unique normal form for each coefficient of any skew polynomial in D. By applying these rewriting rules, the module-theoretic constructions discussed in Section 3 become effective for the ring D.

The system module M is defined by

$$0 \longleftarrow M \longleftarrow D^{1\times 3} \xleftarrow{.R} D^{1\times 2}, \qquad R := \begin{pmatrix} \partial + (u + \frac{1}{2}u^2) & 0 & (1+u)\, x_1 \\ -u & \partial & -x_1 \end{pmatrix}.$$

In order to determine structural properties of the linearization, we compute $\mathrm{ext}^1_D(N, D)$, where N is the Auslander transpose of $M = D^{1\times 3}/D^{1\times 2}\, R$. Similarly to Example 4.15, using the standard involution θ of D, we actually compute $\mathrm{ext}^1_D(\widetilde{N}, D)$, where $\widetilde{N} := D^{1\times 2}/D^{1\times 3}\, R_1$, $R_1 := \theta(R)$. We obtain the exact sequence of left D-modules

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 3} \xrightarrow{.R_1} D^{1\times 2} \longrightarrow \widetilde{N} \longrightarrow 0 \tag{30}$$

where

$$\begin{aligned} R_2 \;=\; & \big(\, 2(1+u)\,(2\dot{u}-u^2-u^3)\,x_1\,\partial - (4(1+u)\,\ddot{u} - 8\dot{u}^2 - 2u\,(1+u)\,(2+u)\,\dot{u})\,x_1, \\ & -2(2\dot{u}-u^2-u^3)\,x_1\,\partial + (4\ddot{u} - 4u\,(1+2u)\,\dot{u} + u^4\,(1+u))\,x_1, \\ & 2(2\dot{u}-u^2-u^3)\,\partial^2 - (4\ddot{u} - 8u\,(1+u)\,\dot{u} + u^3\,(1+u)\,(2+u))\,\partial \\ & +u\,(2u\ddot{u} + u^2\,(2+u)\,\dot{u} - 8\dot{u}^2)\,\big) \in D^{1\times 3}. \end{aligned} \tag{31}$$

We apply the functor $\hom_D(-, D)$ to (30) and transform the right D-modules in the resulting complex into left D-modules by using the involution θ. Another syzygy computation yields a matrix S such that the bended complex in

$$\begin{array}{ccccc} D & \xleftarrow{.\widetilde{R}_2} & D^{1\times 3} & \xleftarrow{.R} & D^{1\times 2} \\ & & & \nwarrow^{.S} & \\ & & & & D^{1\times 2} \end{array}$$

is exact, where $\widetilde{R}_2 := \theta(R_2)$. In fact, the choice $S = R$ is possible, which implies that we have $\mathrm{ext}^1_D(N, D) = \{0\}$. Hence, the linearized system is parametrizable, and therefore, the non-linear system (29) is controllable. The parametrization $\widetilde{R}_2 \in D^{3\times 1}$ is actually injective because a left inverse of $\widetilde{R}_2$ is given by

$$\left(\begin{array}{ccc} \dfrac{2}{(2\dot{u}-u^2-u^3)^2\,x_1} & \dfrac{2(1+u)}{(2\dot{u}-u^2-u^3)^2\,x_1} & 0 \end{array} \right) \in D^{1\times 3}. \tag{32}$$

Clearly, a specialization of the above reasoning to particular functions x_1, x_2, and u is only legitimate if the denominators (e.g., in (32)) and leading coefficients (e.g., in (31)) arising in these computations do not vanish. In fact, if u is chosen to satisfy $2\dot{u} - u^2 - u^3 = 0$, the corresponding system module is not torsion-free.

6 Software packages

In this short last section some software packages providing implementations of the methods described in the previous sections are listed. All of these are freely available.

The **C**ontrol **L**ibrary **I**n **P**lural & **S**ingular (or `Singular:Control.lib`, cf. [LZ05]) is a library developed by V. Levandovskyy and E. Zerz using the computer algebra system Singular. Module-theoretic computations can be performed over commutative polynomial algebras and, more generally, over G-algebras using Gröbner bases. The package provides tools to determine various autonomy and controllability degrees of linear systems. An additional package realizing a purity filtration (cf., e.g, [Qua13]) is also under development.

The Maple package `OreModules` (cf. [CQR07]), developed by F. Chyzak, A. Quadrat, and the author of this article, performs module-theoretic constructions over Ore algebras as described in the previous sections using Gröbner bases or Janet bases. It allows to compute extension groups, parametrizations, flat outputs, and π-polynomials and provides tools to solve, e.g., linear quadratic optimal control problems. A library of examples with origin in control theory and mathematical physics illustrates the use of `OreModules`.

The package `OreModules` is complemented by a couple of additional packages. The Maple package `QuillenSuslin` (cf. [Fab09]), developed by A. Fabiańska, realizes a constructive version of the Quillen-Suslin Theorem. Methods to compute bases of free left modules of rank at least 2 over Weyl algebras are implemented in the Maple package `Stafford` (cf. [QR07]), developed by A. Quadrat and the author of this paper. It is based on a constructive version of Stafford's Theorem (cf. Theorem 5.19). An extension of this package to (a certain class of) power series coefficients is under development by the same authors and is called `StaffordAnalytic`. It is based on an extension of the `Janet` package, which is mentioned next.

The packages listed in the previous two paragraphs profit, in particular, from Maple packages (and C++ extensions) `Involutive`, `Janet`, and `JanetOre`, developed by the author of this paper, implementing the involutive basis algorithm for the computation of Janet bases (cf. [BCG+03], [Ger05], [Rob06]).

Another Maple package which builds on `OreModules` is called `OreMorphisms` (cf. [CQ09]) and is developed by T. Cluzeau and A. Quadrat. It implements the computation of homomorphisms between finitely presented left modules over Ore algebras and provides various tools to study equivalences, factorization, and simplification of linear functional systems.

The computation of purity filtrations (cf. [Qua13]) for linear systems over Ore algebras is possible using the Maple package `PurityFiltration` developed by A. Quadrat, building on `OreModules`.

T. Cluzeau, A. Quadrat and the author of this article develop `AlgebraicAnalysis`, a Maple package for the study of linearized systems of partial differential equations (cf. Remark 5.29). It is based on the package `Janet` (cf. above) which in turn uses some procedures of the Maple package `jets` (cf. [Bar01]), developed by M. Barakat, implementing jet calculus.

Finally, a GAP package `AbelianSystems` is under development by M. Barakat and A. Quadrat which implements various methods discussed above using the package `homalg`, which realizes methods of homological algebra in GAP (cf. `http://homalg.math.rwth-aachen.de`; cf. also [BR08] for the predecessor of `homalg` in Maple).

7 Conclusion

The module associated with a system of linear functional equations reflects structural properties of the solution set in a signal space which is an injective cogenerator for the module category under consideration. The purpose of this paper is to give an overview on some recent progress in developing effective methods in this context, i.e., algorithms deciding to what extent a given behavior is parametrizable or whether or not it is autonomous (e.g., determining to which class in the hierarchy of modules the given system module belongs, cf. Proposition 2.12 and Theorem 4.17) and algorithms computing parametrizations of a certain kind (e.g., computing bases of finitely generated free modules, cf. Section 5). The effectiveness depends on the ring to be dealt with, and the interpretation of the module-theoretic constructions in terms of behaviors depends on the duality between equations and solutions. Investigations of relevant classes of rings and their module structures and of appropriate signal spaces are topics of current research.

For example, for most cases of systems of linear functional equations with non-constant coefficients no suitable concrete signal space is known (cf. also [BO12] for recent work on the partial differential-difference case with constant coefficients employing the frequency domain). Both theory and applications to engineering sciences motivate the study of largely unexplored classes of rings of operators, e.g., those arising for boundary value problems (cf. [QR13a], [GRR14] and the references therein). Moreover, restricting the domains of definition of functions to non-trivial varieties leads to further interesting classes of rings of operators (cf., e.g., [CH88]), which is also related to problems in the theory of D-modules, sheaves, etc.

For lack of space, this paper does not address more refined studies of autonomous behaviors, viz., the technique of purity or grade filtration (cf., e.g., [Bar10], [Qua13]), nor questions on how controllable and autonomous behaviors can be interconnected (cf., e.g., [ZL01], [QR05a]). Currently the practical impact of computations of grade filtrations on symbolic solving of differential systems is studied.

Finally, generalizations to non-linear systems are investigated. In the case of differential systems, besides applying genuinely non-linear approaches using differential geometry or differential algebra, linearization techniques may reduce (some aspects of) the non-linear case to the context discussed in this paper (cf. Subsection 5.5). In general a decomposition of the radical differential ideal associated with the differential system into prime differential ideals or a related decomposition is necessary before linearizing (cf., e.g., [Dio92], [BLOP09], [Gri89], [Wan01], [Rob12], [LHR13]). Close relationships of (certain aspects of) non-linear systems and their linearizations are now explored (cf., e.g., [CCQ11]).

Some freely available software packages addressing the topics of this paper were listed in Section 6.

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