Strongly perfect lattices

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Lattices

Definition: Let $E := (V, (,))$ be an euclidian vector space and $(b_1, \dots, b_n)$ linear independent then

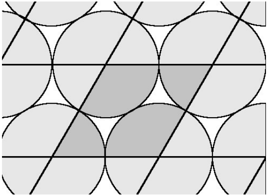
- $L := \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_n$ is a lattice.
- $G(\mathcal{B}) := ((b_i, b_j))_{1 \leq i,j \leq n}$ is its Gram matrix.
- $\det(L) := \det(G(\mathcal{B}))$ is independent of the choice of $\mathcal{B}$.
- $L^* := \{v \in V | (v, \lambda) \in \mathbb{Z} \forall \lambda \in L\}$ is the dual lattice of L.
- $\min(L) := \min\{(\lambda, \lambda) | 0 \neq \lambda \in L\}$.
- $S(L) := \{\lambda \in L | (\lambda, \lambda) = \min(L)\}$.
- $|S(L)|$ is called the kissing number of L.

Density of a lattice

Definition: Density of a lattice L is defined as

$$\Delta(L) := \frac{\mathrm{Vol}(S^{n-1})\left(\sqrt{\min(L)}/2\right)^n}{\mathrm{Vol}(\text{fundamental area})} = \frac{\mathrm{Vol}(S^{n-1})}{2^n}\left(\frac{\min(L)^n}{\det(L)}\right)^{1/2}$$

where $S^{n-1} := \{x \in \mathbb{R}^n | (x,x) = 1\}$.



Definition:

Lattices that are local maxima of Δ are called extreme.

Strongly perfect lattices

Definition:

- A finite subset $X \subset S^{n-1}$ is called a spherical t-design if

$$\int_{S^{n-1}} f(x)dx = \frac{1}{|X|} \sum_{x \in X} f(x) \qquad \forall f \in \mathcal{F}_{n,m} m \leq t$$

  where $\mathcal{F}_{n,m}$ are all homogenous polynomials of degree m in $\mathbb{R}[X_1, \ldots, X_n]$.
- A lattice L is strongly perfect if $S(\frac{1}{\sqrt{\min(L)}} L)$ is a spherical 4-design.

Theorem (Venkov):

Strongly perfect lattices are extreme.

Strongly perfect lattices

Theorem: L is strongly perfect if and only if for all $\alpha \in \mathbb{R}^n$ holds:

$$\sum_{x \in S(L)} (x,\alpha)^2 = \frac{|S(L)| \min(L)}{n} (\alpha,\alpha)$$

$$\sum_{x \in S(L)} (x,\alpha)^4 = \frac{3|S(L)| \min(L)^2}{n(n+2)} (\alpha,\alpha)^2$$

Methods for classification

- Theorem above applied for $\alpha \in L^*$.
- Known boundaries for $|S(L)|$ and $\min(L)\min(L^*)$.
- θ-series for lattices.
- Maximal even superlattices.

Classification of strongly perfect lattices

The classification is complete up to dimension 12 (Nebe, Venkov):

dim	1	2	4	6	7	8	10	12
	$\mathbb{Z}$	$\mathbb{A}_2$	$\mathbb{D}_4$	$\mathbb{E}_6, \mathbb{E}_6^*$	$\mathbb{E}_7, \mathbb{E}_7^*$	$\mathbb{E}_8$	$K'_{10}, (K'_{10})^*$	K_{12}, K_{12}^*

Classification of dual strongly perfect lattices:

- $n = 13$: no dual strongly perfect lattice (Nebe, Venkov, N).
- $n = 14$: one lattice Q_{14} (Nebe, Venkov).
- $n = 15$ and 17: probably no dual strongly perfect lattice, classification almost complete.

Remark: Further lattices are known in higher dimensions e.g.: Barnes-Wall lattice in dimension 16, Leech lattice in dimension 24.