Strongly perfect lattices

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Lattices

Definition: Let $E := (V, (,))$ be an euclidian vector space and $(b_1, \ldots, b_n)$ linear independent then

$$L := \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_n$$

is a lattice and

$$G(\mathcal{B}) := ((b_i, b_j))_{1 \leq i,j \leq n}$$

is its Gram matrix.

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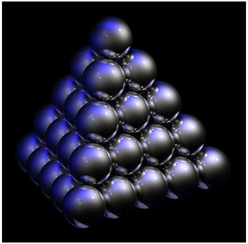
- $S(L) := \{\lambda \in L | (\lambda, \lambda) = \min(L)\}$.
- $|S(L)|$ is called the kissing number of L.

Sphere packings

History:

- Kepler's Conjecture (1611)



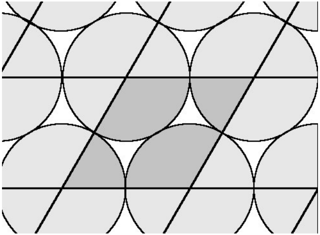
- Gauß proved Kepler's Conjecture for lattice sphere packings (1831)
- Kepler's Conjecture has been proven by Thomas Hales (1998)

Density of a lattice

Definition: Density of a lattice L is defined as

$$\Delta(L) := \frac{\mathrm{Vol}(S^{n-1})\left(\sqrt{\min(L)}/2\right)^n}{\mathrm{Vol(fundamental\ area)}} = \frac{\mathrm{Vol}(S^{n-1})}{2^n}\left(\frac{\min(L)^n}{\det(L)}\right)^{1/2}$$

where $S^{n-1} := \{x \in \mathbb{R}^n | (x,x) = 1\}$.



Hermite function

Definition:

- Hermite function

$$\gamma : \mathcal{L}_n \to \mathbb{R}_{>0} : \gamma(L) := \frac{\min(L)}{\det(L)^{1/n}} = \frac{2^n}{\mathrm{Vol}(S^{n-1})}\Delta(L)^{2/n}.$$

  where $\mathcal{L}_n$ is the set of n-dimensional lattices.

- Hermite constant $\gamma_n := \sup\{\gamma(L) | L \in \mathcal{L}_n\}$, upper boundaries are known up to $n = 36$.
- A lattice is called **extreme** if it is a local maximum of γ.

Extreme lattices

Definition:

- A lattice L is called perfect if

$$\langle \lambda^{tr}\lambda | \lambda \in S(L)\rangle_{\mathbb{R}} = \mathrm{Sym}_n(\mathbb{R}).$$

- L is eutactic if

$$G(L)^{-1} = \sum_{x \in S(L)} \rho_x x^{tr} x$$

with $\rho_x \in \mathbb{R}_{>0} \forall x \in S(L)$

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Theorem(Voronoi):

A lattice is extreme if and only if it is perfect and eutactic.

Classification of extreme lattices

Voronoi-algorithm: explicitly calculates all perfect lattices.
But does only work in dimensions smaller or equal to 8 because of complexity.

Classification of extreme lattices up to similarity:

dim	1	2	3	4	5	6	7	8	9
$\sharp\mathrm{Perf}_n$	1	1	1	2	3	7	33	10916	≥ 524289
$\sharp\mathrm{Ext}_n$	1	1	1	2	3	6	30	2408	≥ 12814

Strongly perfect lattices

Definition:

- A finite subset $X \subset S^{n-1}$ is called a spherical t-design if

$$\int_{S^{n-1}} f(x)dx = \frac{1}{|X|} \sum_{x \in X} f(x) \forall f \in \mathcal{F}_{n,m} m \leq t$$

  where $\mathcal{F}_{n,m}$ are all homogenous polynomials of degree m in $\mathbb{R}[X_1, \ldots, X_n]$.
- A lattice L is strongly perfect if $S(\frac{1}{\sqrt{\min(L)}} L)$ is a spherical 4-design.

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Theorem (Venkov):

Strongly perfect lattices are perfect and eutactic and hence extreme.

Strongly perfect lattices

Theorem: L is strongly perfect if and only if for all $\alpha \in \mathbb{R}^n$ holds:

$$\sum_{x \in S(L)} (x, \alpha)^2 = \frac{|S(L)| \min(L)}{n} (\alpha, \alpha)$$

$$\sum_{x \in S(L)} (x, \alpha)^4 = \frac{3|S(L)| \min(L)^2}{n(n+2)} (\alpha, \alpha)^2$$

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Theorem: Let L be a strongly perfect lattice in dimension n then

$$\frac{n+2}{3} \leq \min(L) \min(L^*) \leq \gamma_n^2$$

Classification of strongly perfect lattices

The classification is complete up to dimension 12 (Nebe, Venkov):

dim	1	2	4	6	7	8	10	12
	$\mathbb{Z}$	$\mathbb{A}_2$	$\mathbb{D}_4$	$\mathbb{E}_6, \mathbb{E}_6^*$	$\mathbb{E}_7, \mathbb{E}_7^*$	$\mathbb{E}_8$	$K'_{10}, (K'_{10})^*$	K_{12}, K_{12}^*

Remark: Further lattices are known in higher dimensions e.g.:

- Barnes-Wall lattice in dimension 16.
- Leech lattice in dimension 24.

Dual strongly perfect lattices

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Classification of dual strongly perfect lattices:

- $n = 13$: no dual strongly perfect lattice (Nebe, Venkov, N).
- $n = 14$: one lattice Q_{14} (Nebe, Venkov).
- $n = 15$: probably no dual strongly perfect lattice, classification almost complete.