Lie properties of a crossed product

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In 1929 Emmy Noether introduced the notion of the classic crossed product and 40s Jacobson generalized it, but this concept did not play a significant role until 1960. At that time I generalized it further, introducing the notion of the crossed product of a semigroup and an arbitrary ring, and began to investigate it from the viewpoint of group algebras. The crossed products have been the subject of intensive research, many deep results have been achieved concerning their ring-theoretical properties.

Let $U(R)$ be the group of units, let $Aut(R)$ be the group of automorphisms of the ring R.
Assume that the mappings

$$\sigma : G \to Aut(R), \quad \lambda : G \times G \to U(R)$$

are such that

$$\lambda(a, bc)\lambda(b, c) = \lambda(ab, c)\lambda(a, b)^{\sigma(c)}, \tag{1}$$

and

$$\alpha^{\sigma(a)\cdot\sigma(b)} = \lambda(a, b)^{-1}\alpha^{\sigma(ab)}\lambda(a, b) \tag{2}$$

for any $a, b, c \in G$ and $\alpha \in R$, where $\alpha^{\sigma(a)}$ is the action of the automorphism $\sigma(a)$ on the α of R.
The λ is called the **twisted fuction** or the **factor system** of the group G over the ring R relative to σ.

Assign to every $g \in G$ a symbol $\widetilde{g}$, and let

$$R_\sigma^\lambda[G] = \{\sum_{g\in G} \widetilde{g}\alpha_g \mid \alpha_g \in R\}$$

be the set of all formal sums with only finitely many nonzero coefficients α_g.
We remark that the elements of $R_\sigma^\lambda[G]$

$$x = \sum_{g\in G} \widetilde{g}\alpha_g \quad \text{and} \quad y = \sum_{g\in G} \widetilde{g}\beta_g$$

are equal if and only if $\alpha_g = \beta_g$ for any $g \in G$.

On $R_\sigma^\lambda[G]$ addition and multiplication are defined as follows:

(i) $\sum_{g\in G} \widetilde{g}\alpha_g + \sum_{g\in G} \widetilde{g}\beta_g = \sum_{g\in G} \widetilde{g}(\alpha_g + \beta_g)$;

(ii) $\widetilde{g}\widetilde{h} = \widetilde{gh}\lambda(g,h)$, where λ is a twisted function of G over R;

(iii) $\alpha\widetilde{g} = \widetilde{g}\alpha^{\sigma(g)}$

and the product of arbitrary elements x and y is determined according to distributivity.
The ring $R_\sigma^\lambda[G]$ is called a **crossed product** of the group G over the ring R.

In the theory of ordinary group rings the Lie properties play an important role. Group algebras with the many "good" Lie properties were described during the 70's using the theory on polynomial identities. Later these results were applied to the study of the group of units. In most cases the Lie structure reflects very well the characteristics of the group of units, there is a close relationship between their properties of them.
We describe the structure of those crossed products which are upper (lower) Lie nilpotent and Lie (n, m)-Engel. We generalize results of Passi, Passman and Sehgal which were obtained for the group algebras.

Let $F^\lambda[G]$ be a twisted group algebra with normalized twisting function λ.
If the twisted group algebra $F^\lambda[G]$ has an F-basis

$$\widetilde{G} = \{\widetilde{g} \mid g \in G\}$$

such that for each g of G there exists an element $d_g \in F$ such that the elements of the set $\{d_g\widetilde{g} \mid g \in G\}$ form a group basis for $F^\lambda[G]$, then $F^\lambda[G]$ is called **untwisted.**
In this situation $F^\lambda[G]$ is isomorphic to FG via this diagonal change of the basis.
$F^\lambda[G]$ is called **stably untwisted** if there exists an extension K of the field F such that $K^\lambda[G]$ is untwisted.

Let $F^\lambda[G]$ be a twisted group algebra. For each $h \in G$ of order k we have

$$\widetilde{h}^k = \prod_{i=1}^{k-1} \lambda(h^i, h) \cdot \widetilde{1}. \tag{3}$$

It is convenient to say that

$$\mu(h) = \prod_{i=1}^{k-1} \lambda(h^i, h)$$

is the **twist** of $\widetilde{h}$ which plays an important role in the study of $F^\lambda[G]$.

An p-element u of G is called an **untwisted p-element** if the order of u coincides with the order of $\widetilde{u}\gamma$ for some $\gamma \in F$.

Let R be an associative ring. The lower Lie central series in R is defined inductively as follows:

$$\gamma_1(R) = R, \quad \gamma_2(R) = [\gamma_1(R), R], \ldots, \gamma_n(R) = [\gamma_{n-1}(R), R], \ldots.$$

The two-sided ideal $R^{[n]} = \gamma_n(R)R$ of R is called the **n-th lower Lie power** of R.
Let us define by induction a second set of ideals in R:

$$R^{(1)} = R, \quad R^{(2)} = [R^{(1)}, R]R, \ldots, R^{(n)} = [R^{(n-1)}, R]R, \ldots$$

The ideal $R^{(n)}$ is called **the n-th upper Lie power** of R.
R is called **upper Lie nilpotent** if $R^{(n)} = 0$ for some n.
The ring R with $R^{[m]} = 0$ for some m is called **lower Lie nilpotent**.

Theorem

Let $F^{\lambda}_{\sigma}[G]$ be a crossed product of a group G and the field F of characteristic 0 or p. Then

1. *Any upper (lower) Lie nilpotent crossed product $F^{\lambda}_{\sigma}[G]$ is a twisted group algebra.*
2. **The twisted group algebra** $F^{\lambda}[G]$ *is lower (upper) Lie nilpotent if and only if one of the following condition holds:*
   - (2.i) $F^{\lambda}[G]$ *is a commutative algebra (i.e G is abelian and the twisting function is symmetric).*
   - (2.ii) *$\mathrm{char}(F) = p$, G is a nilpotent group with commutator subgroup of p-power order and the untwisted p-elements of G form a subgroup. Moreover, for any $a, b \in G$ the group commutator (a, b) is an untwisted p-element and*

$$\left(\lambda(a,b)^{-1}\lambda(b,b^{-1}ab)\lambda(a,(a,b)\lambda((b,a),(a,b))^{-1}\right)^{-p^m} = \mu((a,b)),$$

where p^m is the order of (a, b).

Using the notation of untwisting, Theorem can be formulated as

Corollary

Let $F^{\lambda}[G]$ be a twisted group algebra of a group G and a field F of characteristic 0 or $p > 0$. The algebra $F^{\lambda}[G]$ is lower (upper) Lie nilpotent if and only if one of the following conditions holds:

(i) *$F^{\lambda}[G]$ is commutative;*

(ii) *$char(F) = p$, G is a nilpotent group such that G' is a finite p-group and $F^{\lambda}[G]$ is stably untwisted.*

Let R be an associative ring and let n, m be fixed positive integers. If

$$[a, \underbrace{b^m, b^m, \ldots, b^m}_{n}] = 0$$

for all elements $a, b \in R$, then R is called (n, m)**-Engel.** Clearly, an (n, m)-Engel ring satisfies the polynomial identity

$$[x, \underbrace{y^m, y^m, \ldots, y^m}_{n}].$$

Let p^t be the smallest positive integer such that $n \leq p^t$ and let $m = p^l r$ with $(p, r) = 1$. If R is an (n, m)-Engel ring of $char(R) = p > 0$, then it is (p^{l+t}, r)-Engel ring, too.

Theorem

Let $F_\sigma^\lambda[G]$ be a crossed product of G and a field F of characteristic $p > 0$.

(1) *Any (n, m)-Engel crossed product $F_\sigma^\lambda[G]$ is a twisted gr algebra.*

(2) *If $F^\lambda[G]$ is an (n, m)-Engel twisted group algebra, then either $F^\lambda[G]$ is commutative, or:*

  (2.i) *G has a normal subgroup B of a finite index such that commutator subgroup B' has p-power order, the p-Sylow subgroup P/B of G/B is a normal subgroup, G/P is a finite abelian group of an exponent that divides m and P is a nilpotent subgroup.*

  (2.ii) *the untwisted p-elements of G form a subgroup and for all $a, b \in G$ (a, b) is an untwisted p-element such that*

$$\left(\lambda(a,b)^{-1}\lambda(b,b^{-1}ab)\lambda(a,(a,b)\lambda((b,a),(a,b))^{-1}\right)^{-p^m} = \mu((a,b)),$$

  where p^m is the order of (a, b). Moreover, $F^\lambda[B]$ is stably untwisted and $|G : B||B'|$ is bounded by a fixed function of n and m.

Theorem

The twisted group algebra $F^\lambda[G]$ of positive characteristic p is n-Engel if and only if either $F^\lambda[G]$ is commutative, or the following conditions hold:

(i) *G is a nilpotent group with a normal subgroup B of a finite p-power index, B' is a finite p-group and $F^\lambda[B]$ is stably untwisted;*

(ii) *the untwisted p-elements of G form a subgroup, the commutator (a, b) is an untwisted p-element for all $a, b \in G$ and*

$$\left(\lambda(a,b)^{-1}\lambda(b,b^{-1}ab)\lambda(a,(a,b)\lambda((b,a),(a,b))^{-1}\right)^{-p^m} = \mu((a,b))$$

where p^m is the order of (a, b).