Primitive Central Idempotents in Rational Group Algebras

Gurmeet K. Baksḥi

Panjab University
Chandigarh, India
email:gkbakshi@pu.ac.in

G : finite group

$\mathbb{Q}[G]$: the rational group algebra of G.

The problem of determining the Wedderburn decomposition of $\mathbb{Q}[G]$ naturally leads to the computation of the primitive central idempotents of $\mathbb{Q}[G]$.

G : finite group

$\mathbb{Q}[G]$: the rational group algebra of G.

The problem of determining the Wedderburn decomposition of $\mathbb{Q}[G]$ naturally leads to the computation of the primitive central idempotents of $\mathbb{Q}[G]$.

Aim of the talk

In this talk, I give a survey of the known results so far on computation of the primitive central idempotents of the rational group algebra $\mathbb{Q}[G]$ of a finite group G and also some of my recent results (with Passi) in this area.

Classical method of determining primitive central idempotents of $\mathbb{Q}[G]$.

The classical method is in two steps

(i) First compute the primitive central idempotents of the complex group algebra $\mathbb{C}[G]$ using the character table of G.

(ii) Then compute the primitive central idempotents of $\mathbb{Q}[G]$ using a group action.

Classical method of determining primitive central idempotents of $\mathbb{Q}[G]$.

The classical method is in two steps

(i) First compute the primitive central idempotents of the complex group algebra $\mathbb{C}[G]$ using the character table of G.

(ii) Then compute the primitive central idempotents of $\mathbb{Q}[G]$ using a group action.

Primitive central idempotents of $\mathbb{C}[G]$

The set $Irr(G)$ of complex irreducible characters of G $\longleftrightarrow$ The set of primitive central idempotents of $\mathbb{C}[G]$

$$e(\chi) = \frac{1}{o(G)} \sum_{g \in G} \chi(g^{-1}) g$$

Given $\chi \in Irr(G)$, is a primitive central idempotent of $\mathbb{C}[G]$, called the primitive central idempotent associated with χ.

Primitive central idempotents of $\mathbb{C}[G]$

The set $Irr(G)$ of complex irreducible characters of G $\longleftrightarrow$ The set of primitive central idempotents of $\mathbb{C}[G]$

Given $\chi \in Irr(G)$,

$$e(\chi) = \frac{1}{o(G)} \sum_{g \in G} \chi(g^{-1}) g$$

is a primitive central idempotent of $\mathbb{C}[G]$, called the primitive central idempotent associated with χ.

Primitive central idempotents of $\mathbb{Q}[G]$, from those of $\mathbb{C}[G]$.

- $Aut(\mathbb{C})$ acts on $\mathbb{C}[G]$ by acting on the coefficients. i.e.

$$\sigma.\sum a_g g = \sum \sigma(a_g)g, \quad (\sigma \in Aut(\mathbb{C}), \sum a_g g \in \mathbb{C}[G]).$$

- $Aut(\mathbb{C})$ acts on $Irr(G)$ by

$$\sigma.\chi = \sigma o \chi, \quad (\sigma \in \mathbb{A}, \chi \in Irr(G)).$$

Orbit of both χ and $e(\chi)$ can be obtained by applying the elements of $Gal(\mathbb{Q}(\chi)/\mathbb{Q})$ to χ and $e(\chi)$ respectively.

Primitive central idempotents of $\mathbb{Q}[G]$, from those of $\mathbb{C}[G]$.

- $Aut(\mathbb{C})$ acts on $\mathbb{C}[G]$ by acting on the coefficients. i.e.

$$\sigma.\sum a_g g = \sum \sigma(a_g)g, \quad (\sigma \in Aut(\mathbb{C}), \sum a_g g \in \mathbb{C}[G]).$$

- $Aut(\mathbb{C})$ acts on $Irr(G)$ by

$$\sigma.\chi = \sigma o \chi, \quad (\sigma \in \mathbb{A}, \chi \in Irr(G)).$$

Orbit of both χ and $e(\chi)$ can be obtained by applying the elements of $Gal(\mathbb{Q}(\chi)/\mathbb{Q})$ to χ and $e(\chi)$ respectively.

Primitive central idempotents of $\mathbb{Q}[G]$, from those of $\mathbb{C}[G]$.

- $Aut(\mathbb{C})$ acts on $\mathbb{C}[G]$ by acting on the coefficients. i.e.

$$\sigma.\sum a_g g = \sum \sigma(a_g)g, \quad (\sigma \in Aut(\mathbb{C}), \sum a_g g \in \mathbb{C}[G]).$$

- $Aut(\mathbb{C})$ acts on $Irr(G)$ by

$$\sigma.\chi = \sigma o \chi, \quad (\sigma \in \mathbb{A}, \chi \in Irr(G)).$$

Orbit of both χ and $e(\chi)$ can be obtained by applying the elements of $Gal(\mathbb{Q}(\chi)/\mathbb{Q})$ to χ and $e(\chi)$ respectively.

Primitive central idempotents of $\mathbb{Q}[G]$, from those of $\mathbb{C}[G]$.

- $Aut(\mathbb{C})$ acts on $\mathbb{C}[G]$ by acting on the coefficients. i.e.

$$\sigma . \sum a_g g = \sum \sigma(a_g) g, \quad (\sigma \in Aut(\mathbb{C}), \sum a_g g \in \mathbb{C}[G]).$$

- $Aut(\mathbb{C})$ acts on $Irr(G)$ by

$$\sigma . \chi = \sigma o \chi, \quad (\sigma \in \mathbb{A}, \chi \in Irr(G)).$$

Orbit of both χ and $e(\chi)$ can be obtained by applying the elements of $Gal(\mathbb{Q}(\chi)/\mathbb{Q})$ to χ and $e(\chi)$ respectively.

The set of distinct orbits of $Irr(G)$ under the action of $Aut(\mathbb{C})$. $\longleftrightarrow$ The set of primitive central idempotents of $\mathbb{Q}[G]$

Corresponding to an orbit of χ

$$e_{\mathbb{Q}}(\chi) = \sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi)$$

is a primitive central idempotent of $\mathbb{Q}[G]$.

The set of distinct orbits of *Irr*(G) under the action of *Aut*$(\mathbb{C})$. $\longleftrightarrow$ The set of primitive central idempotents of $\mathbb{Q}[G]$

Corresponding to an orbit of χ

$$e_{\mathbb{Q}}(\chi) = \sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi)$$

is a primitive central idempotent of $\mathbb{Q}[G]$.

The following question is of concern to mathematicians for the last many years

Question

Does there exist any character free expressions (avoiding the computations on the extension of rationals) of primitive central idempotents of $\mathbb{Q}[G]$?

The following question is of concern to mathematicians for the last many years

Question

Does there exist any character free expressions (avoiding the computations on the extension of rationals) of primitive central idempotents of $\mathbb{Q}[G]$?

Idempotents from subgroups.

Let H be a subgroup of a finite group G. Then

$$\hat{H} = \frac{1}{o(H)} \sum_{h \in H} h$$

is an idempotent of $\mathbb{Q}[G]$, called the idempotent determined by H.

$\boxed{\hat{H} \text{ is central if and only if } H \text{ is normal in } G.}$

Note that $\hat{G}$ is always a primitive central idempotent of $\mathbb{Q}[G]$, which we call the trivial primitive central idempotent of $\mathbb{Q}[G]$

Idempotents from subgroups.

Let H be a subgroup of a finite group G. Then

$$\hat{H} = \frac{1}{o(H)} \sum_{h \in H} h$$

is an idempotent of $\mathbb{Q}[G]$, called the idempotent determined by H.

$\boxed{\hat{H} \text{ is central if and only if } H \text{ is normal in } G.}$

Note that $\hat{G}$ is always a primitive central idempotent of $\mathbb{Q}[G]$, which we call the trivial primitive central idempotent of $\mathbb{Q}[G]$

A Question.

Is it possible to obtain primitive central idempotents of $\mathbb{Q}[G]$ from the subgroup structure of G?

We see progress in this direction

Sehgal(1978), Topics in Group Rings, Proposition 1.16

It was shown by Sehgal that if G is a finite abelian group, then the primitive of $\mathbb{Q}[G]$ can be written as a linear combination of the idempotents of the form $\hat{H}$, $H \leq G$.

Later Jespers Leal and Milies gave explicit description of the primitive (central) idempotents of $\mathbb{Q}[G]$, G a finite abelian group.

Primitive (central) idempotents of $\mathbb{Q}[G]$, when G is a cyclic group of prime power order (Jespers, Leal and Milies)

Let $G =< x >$ be a cyclic group of order p^m, $m \geq 1$, p a prime. Let

$$G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_m = 1$$

be the descending chain of all the subgroups of G; $G_i =< x^{p^i} >$, then the primitive (central) idempotents of the rational group algebra $\mathbb{Q}[G]$ are

$$e_0 = \hat{G} \quad and \quad e_i = \hat{G}_i - \widehat{G_{i-1}},\ 1 \leq i \leq m.$$

If G is any finite abelian group, then by decomposing G as a product of cyclic $p-$ groups, the primitive central idempotents of $\mathbb{Q}[G]$ can be obtained.

Primitive (central) idempotents of $\mathbb{Q}[G]$, when G is a cyclic group of prime power order (Jespers, Leal and Milies)

Let $G =< x >$ be a cyclic group of order p^m, $m \geq 1$, p a prime. Let

$$G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_m = 1$$

be the descending chain of all the subgroups of G; $G_i =< x^{p^i} >$, then the primitive (central) idempotents of the rational group algebra $\mathbb{Q}[G]$ are

$$e_0 = \hat{G} \quad \textit{and} \quad e_i = \hat{G}_i - \widehat{G_{i-1}},\ 1 \leq i \leq m.$$

If G is any finite abelian group, then by decomposing G as a product of cyclic $p-$ groups, the primitive central idempotents of $\mathbb{Q}[G]$ can be obtained.

Primitive (central) idempotents of $\mathbb{Q}[G]$, when G is a cyclic group of prime power order (Jespers, Leal and Milies)

Let $G =< x >$ be a cyclic group of order p^m, $m \geq 1$, p a prime. Let

$$G = G_0 \supseteq G_1 \supseteq ... \supseteq G_m = 1$$

be the descending chain of all the subgroups of G; $G_i =< x^{p^i} >$, then

the primitive (central) idempotents of the rational group algebra $\mathbb{Q}[G]$ are

$$e_0 = \hat{G} \quad and \quad e_i = \hat{G}_i - \hat{G_{i-1}},\ 1 \leq i \leq m.$$

If G is any finite abelian group, then by decomposing G as a product of cyclic $p-$ groups, the primitive central idempotents of $\mathbb{Q}[G]$ can be obtained.

Primitive (central) idempotents of $\mathbb{Q}[G]$, when G is an Abelian p- group

Let G is an Abelian p-group. If H is a subgroup of G such that G/H is cyclic and L/H is the minimal subgroup of G/H, then

$$e_H = \widehat{H} - \widehat{L}$$

is a primitive central idempotent of $\mathbb{Q}[G]$ with $\mathbb{Q}[G]e_H \cong \mathbb{Q}(\zeta_d)$, where $d = o(G/H)$ and ζ_d is primitive dth root of unity.

Moreover any non trivial primitive central idempotent $\mathbb{Q}[G]$ is of this type.

Primitive (central) idempotents of $\mathbb{Q}[G]$, when G is an Abelian p- group

Let G is an Abelian p-group. If H is a subgroup of G such that G/H is cyclic and L/H is the minimal subgroup of G/H, then

$$e_H = \widehat{H} - \widehat{L}$$

is a primitive central idempotent of $\mathbb{Q}[G]$ with $\mathbb{Q}[G]e_H \cong \mathbb{Q}(\zeta_d)$, where $d = o(G/H)$ and ζ_d is primitive dth root of unity.

Moreover any non trivial primitive central idempotent $\mathbb{Q}[G]$ is of this type.

Primitive (central) idempotents of $\mathbb{Q}[G]$, when G is an Abelian p- group

Let G is an Abelian p-group. If H is a subgroup of G such that G/H is cyclic and L/H is the minimal subgroup of G/H, then

$$e_H = \widehat{H} - \widehat{L}$$

is a primitive central idempotent of $\mathbb{Q}[G]$ with $\mathbb{Q}[G]e_H \cong \mathbb{Q}(\zeta_d)$, where $d = o(G/H)$ and ζ_d is primitive *dth* root of unity.

Moreover any non trivial primitive central idempotent $\mathbb{Q}[G]$ is of this type.

Some Notations as given by Jespers, Leal and Paques, 2003

- G :- a finite non trivial group.
- $\mathcal{M}(G)$:= the set of all minimal normal subgroups of G.
- $\varepsilon(G) := \prod_{M \in \mathcal{M}(G)} (1 - \hat{M})$
- For $N \trianglelefteq G$, $N \neq G$, put

$$\varepsilon(G, N) = \prod_{\overline{M} \in \mathcal{M}(G/N)} (\hat{N} - \hat{M})$$

  where M a subgroup of G containing N and $\overline{M} = M/N$..
- $\varepsilon(G, G) = \hat{G}$.

Some Notations as given by Jespers, Leal and Paques, 2003

- G :- a finite non trivial group.
- $\mathcal{M}(G)$:= the set of all minimal normal subgroups of G.
- $\varepsilon(G) := \prod_{M \in \mathcal{M}(G)} (1 - \hat{M})$
- For $N \trianglelefteq G$, $N \neq G$, put

$$\varepsilon(G, N) = \prod_{\overline{M} \in \mathcal{M}(G/N)} (\hat{N} - \hat{M})$$

where M a subgroup of G containing N and $\overline{M} = M/N$..

- $\varepsilon(G, G) = \hat{G}$.

Some Notations as given by Jespers, Leal and Paques, 2003

- G :- a finite non trivial group.
- $\mathcal{M}(G)$:= the set of all minimal normal subgroups of G.
- $\varepsilon(G) := \prod_{M \in \mathcal{M}(G)} (1 - \hat{M})$
- For $N \trianglelefteq G$, $N \neq G$, put

$$\varepsilon(G, N) = \prod_{\overline{M} \in \mathcal{M}(G/N)} (\hat{N} - \hat{M})$$

where M a subgroup of G containing N and $\overline{M} = M/N$..

- $\varepsilon(G, G) = \hat{G}$.

Some Notations as given by Jespers, Leal and Paques, 2003

- G :- a finite non trivial group.
- $\mathcal{M}(G) :=$ the set of all minimal normal subgroups of G.
- $\varepsilon(G) := \prod_{M \in \mathcal{M}(G)} (1 - \hat{M})$
- For $N \trianglelefteq G$, $N \neq G$, put

$$\varepsilon(G, N) = \prod_{\overline{M} \in \mathcal{M}(G/N)} (\hat{N} - \hat{M})$$

where M a subgroup of G containing N and $\overline{M} = M/N$..

- $\varepsilon(G, G) = \hat{G}$.

Some Notations as given by Jespers, Leal and Paques, 2003

- G :- a finite non trivial group.
- $\mathcal{M}(G)$:= the set of all minimal normal subgroups of G.
- $\varepsilon(G) := \prod_{M \in \mathcal{M}(G)} (1 - \hat{M})$
- For $N \trianglelefteq G$, $N \neq G$, put

$$\varepsilon(G, N) = \prod_{\overline{M} \in \mathcal{M}(G/N)} (\hat{N} - \hat{M})$$

where M a subgroup of G containing N and $\overline{M} = M/N$..

- $\varepsilon(G, G) = \hat{G}$.

Another description of the Primitive (central) idempotents of $\mathbb{Q}G$, when G is a finite Abelian Group given by Jespers, Leal and Paques (2003)

Jespers, Leal and Paques [2003]

If G is a finite abelian group, then the primitive (central) idempotents of $\mathbb{Q}[G]$ are precisely all the elements of the form $\epsilon(G, N)$, with N a subgroup of G so that G/N is cyclic.

Another description of the Primitive (central) idempotents of $\mathbb{Q}G$, when G is a finite Abelian Group given by Jespers, Leal and Paques (2003)

Jespers, Leal and Paques [2003]

If G is a finite abelian group, then the primitive (central) idempotents of $\mathbb{Q}[G]$ are precisely all the elements of the form $\epsilon(G, N)$, with N a subgroup of G so that G/N is cyclic.

Primitive Central Idempotents of $\mathbb{Q}G$, when G is a finite Nilpotent Group

Jespers, Leal and Paques [2003]

Let G be a finite nilpotent group. Then primitive idempotents of $\mathbb{Q}G$ are precisely all the elements of the form

$$\sum_{g \in G} (\varepsilon(G_m, H_m))^g$$

(the sum over all G - conjugates of $\varepsilon(G_m, H_m)$, where H_m and G_m are subgroups of G satisfying:

Primitive Central Idempotents of $\mathbb{Q}G$, when G is a finite Nilpotent Group

Jespers, Leal and Paques [2003]

Let G be a finite nilpotent group. Then primitive idempotents of $\mathbb{Q}G$ are precisely all the elements of the form

$$\sum_{g \in G} (\varepsilon(G_m, H_m))^g$$

(the sum over all G - conjugates of $\varepsilon(G_m, H_m)$) , where H_m and G_m are subgroups of G satisfying:

- $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m \subseteq G_m \subseteq G_{m-1} \subseteq \cdots \subseteq G_0 = G$,
- $H_i \trianglelefteq G_i$, $Z(G_i/H_i)$ is cyclic, $0 \leq i \leq m$.
- G_i/H_i is not abelian for $0 \leq i \leq m-1$ and G_m/H_m is abelian.
- $G_{i+1}/H_i = C_{G_i/H_i}(Z_2(G_i/H_i))$.
- $\displaystyle\bigcap_{x \in G_{i-1}/H_{i-1}} H_i^x = H_{i-1}, 1 \leq i \leq m$.

- $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m \subseteq G_m \subseteq G_{m-1} \subseteq \cdots \subseteq G_0 = G$,
- $H_i \trianglelefteq G_i$, $Z(G_i/H_i)$ is cyclic, $0 \leq i \leq m$.
- G_i/H_i is not abelian for $0 \leq i \leq m-1$ and G_m/H_m is abelian.
- $G_{i+1}/H_i = C_{G_i/H_i}(Z_2(G_i/H_i))$.
- $\bigcap_{x \in G_{i-1}/H_{i-1}} H_i^x = H_{i-1}, 1 \leq i \leq m$.

- $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m \subseteq G_m \subseteq G_{m-1} \subseteq \cdots \subseteq G_0 = G$,
- $H_i \trianglelefteq G_i$, $Z(G_i/H_i)$ is cyclic, $0 \leq i \leq m$.
- G_i/H_i is not abelian for $0 \leq i \leq m-1$ and G_m/H_m is abelian.
- $G_{i+1}/H_i = C_{G_i/H_i}(Z_2(G_i/H_i))$.
- $\bigcap_{x \in G_{i-1}/H_{i-1}} H_i^x = H_{i-1}, 1 \leq i \leq m.$

- $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m \subseteq G_m \subseteq G_{m-1} \subseteq \cdots \subseteq G_0 = G$,
- $H_i \trianglelefteq G_i$, $Z(G_i/H_i)$ is cyclic, $0 \leq i \leq m$.
- G_i/H_i is not abelian for $0 \leq i \leq m-1$ and G_m/H_m is abelian.
- $G_{i+1}/H_i = C_{G_i/H_i}(Z_2(G_i/H_i))$.
- $\displaystyle\bigcap_{x \in G_{i-1}/H_{i-1}} H_i^x = H_{i-1}, 1 \leq i \leq m.$

- $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m \subseteq G_m \subseteq G_{m-1} \subseteq \cdots \subseteq G_0 = G$,
- $H_i \trianglelefteq G_i$, $Z(G_i/H_i)$ is cyclic, $0 \leq i \leq m$.
- G_i/H_i is not abelian for $0 \leq i \leq m-1$ and G_m/H_m is abelian.
- $G_{i+1}/H_i = C_{G_i/H_i}(Z_2(G_i/H_i))$.
- $\bigcap_{x \in G_{i-1}/H_{i-1}} H_i^x = H_{i-1}, 1 \leq i \leq m.$

Primitive Central Idempotents of $\mathbb{Q}G$ associated with monomial irreducible characters.

Olivieri, Rio, Simon [2004]

Let G be a finite group,

$K \leq G$, χ be a linear complex character of K, and

χ^G the induced character on G.

If χ^G is irreducible, then the primitive central idempotent of $\mathbb{Q}G$

associated with χ^G is

$$e_{\mathbb{Q}}(\chi^G) = \frac{[Cen_G(\varepsilon(K,H)) : K]}{[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]} \sum_g \varepsilon(K,H)^g,$$

(where $H = ker\chi$ and the sum is over all distinct $G-$ conjugates of

$\varepsilon(K,H)$)

Primitive Central Idempotents of $\mathbb{Q}G$ associated with monomial irreducible characters.

Olivieri, Rio, Simon [2004]

Let G be a finite group,

$K \leq G$, χ be a linear complex character of K, and

χ^G the induced character on G.

If χ^G is irreducible, then the primitive central idempotent of $\mathbb{Q}G$

associated with χ^G is

$$e_{\mathbb{Q}}(\chi^G) = \frac{[Cen_G(\varepsilon(K,H)) : K]}{[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]} \sum_g \varepsilon(K,H)^g,$$

(where $H = ker\chi$ and the sum is over all distinct $G-$ conjugates of

$\varepsilon(K,H))$

Primitive Central Idempotents of $\mathbb{Q}G$ associated with monomial irreducible characters.

Olivieri, Rio, Simon [2004]

Let G be a finite group,

$K \leq G$, χ be a linear complex character of K, and

χ^G the induced character on G.

If χ^G is irreducible, then the primitive central idempotent of $\mathbb{Q}G$ associated with χ^G is

$$e_{\mathbb{Q}}(\chi^G) = \frac{[Cen_G(\varepsilon(K,H)) : K]}{[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]} \sum_g \varepsilon(K,H)^g,$$

(where $H = ker\chi$ and the sum is over all distinct $G-$ conjugates of $\varepsilon(K,H)$)

Primitive Central Idempotents of $\mathbb{Q}G$ associated with monomial irreducible characters.

Olivieri, Rio, Simon [2004]

Let G be a finite group,

$K \leq G$, χ be a linear complex character of K, and

χ^G the induced character on G.

If χ^G is irreducible, then the primitive central idempotent of $\mathbb{Q}G$ associated with χ^G is

$$e_{\mathbb{Q}}(\chi^G) = \frac{[Cen_G(\varepsilon(K,H)) : K]}{[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]} \sum_g \varepsilon(K,H)^g,$$

(where $H = ker\chi$ and the sum is over all distinct $G-$ conjugates of $\varepsilon(K,H)$)

Primitive Central Idempotents of $\mathbb{Q}G$ associated with monomial irreducible characters.

Olivieri, Rio, Simon [2004]

Let G be a finite group,

$K \leq G$, χ be a linear complex character of K, and

χ^G the induced character on G.

If χ^G is irreducible, then the primitive central idempotent of $\mathbb{Q}G$ associated with χ^G is

$$e_{\mathbb{Q}}(\chi^G) = \frac{[Cen_G(\varepsilon(K,H)) : K]}{[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]} \sum_g \varepsilon(K,H)^g,$$

(where $H = ker\chi$ and the sum is over all distinct $G-$ conjugates of $\varepsilon(K,H)$)

Primitive Central Idempotents of $\mathbb{Q}G$ associated with monomial irreducible characters.

Olivieri, Rio, Simon [2004]

Let G be a finite group,

$K \leq G$, χ be a linear complex character of K, and

χ^G the induced character on G.

If χ^G is irreducible, then the primitive central idempotent of $\mathbb{Q}G$ associated with χ^G is

$$e_{\mathbb{Q}}(\chi^G) = \frac{[Cen_G(\varepsilon(K,H)) : K]}{[\mathbb{Q}(\chi) : \mathbb{Q}(\chi^G)]} \sum_g \varepsilon(K,H)^g,$$

(where $H = ker\chi$ and the sum is over all distinct $G-$ conjugates of $\varepsilon(K,H)$)

My Recent Joint Work with I.B.S.Passi

Definition

A complex irreducible character χ of a finite group G, with an affording representation ρ, is defined to have the property $\mathcal{P}$ if for all $g \in G$, either $\chi(g) = 0$ or all the eigen-values of $\rho(g)$ have the same order.

We have derived explicit expression for the primitive central idempotent of $\mathbb{Q}[G]$ associated with a complex irreducible character having the property $\mathcal{P}$. Several consequences of this result are then obtained.

My Recent Joint Work with I.B.S.Passi

Definition

A complex irreducible character χ of a finite group G, with an affording representation ρ, is defined to have the property $\mathcal{P}$ if for all $g \in G$, either $\chi(g) = 0$ or all the eigen-values of $\rho(g)$ have the same order.

We have derived explicit expression for the primitive central idempotent of $\mathbb{Q}[G]$ associated with a complex irreducible character having the property $\mathcal{P}$. Several consequences of this result are then obtained.

My Recent Joint Work with I.B.S.Passi

Definition

A complex irreducible character χ of a finite group G, with an affording representation ρ, is defined to have the property $\mathcal{P}$ if for all $g \in G$, either $\chi(g) = 0$ or all the eigen-values of $\rho(g)$ have the same order.

We have derived explicit expression for the primitive central idempotent of $\mathbb{Q}[G]$ associated with a complex irreducible character having the property $\mathcal{P}$. Several consequences of this result are then obtained.

Some equivalent conditions for $\chi \in \mathrm{Irr}(G)$ to have the property $\mathcal{P}$.

Let $\chi \in \mathrm{Irr}(G)$ and ρ a representation of G affording the character χ. Let $\overline{\rho}$ denote the corresponding induced representation of $G/\ker(\chi)$ and $\overline{\chi}$ the character of $\overline{\rho}$. For $g \in G$, the following statements are equivalent:

(i) *All the eigen-values of* $\rho(g)$ *are of the same order.*

(ii) $\overline{\rho}$ *maps all primitive central idempotents of the rational group algebra* $\mathbb{Q}[\langle \ker(\chi)g \rangle]$ *to zero, except the idempotent* $\epsilon(\langle \ker(\chi)g \rangle, 1)$, *which gets mapped to the identity matrix .*

(ii) $\overline{\chi}|_{\langle \ker(\chi)g \rangle}$ *is a sum of faithful irreducible characters of* $\langle \ker(\chi)g \rangle$.

Some equivalent conditions for $\chi \in \mathrm{Irr}(G)$ to have the property $\mathcal{P}$.

Let $\chi \in \mathrm{Irr}(G)$ and ρ a representation of G affording the character χ. Let $\overline{\rho}$ denote the corresponding induced representation of $G/\ker(\chi)$ and $\overline{\chi}$ the character of $\overline{\rho}$. For $g \in G$, the following statements are equivalent:

(i) All the eigen-values of $\rho(g)$ are of the same order.

(ii) $\overline{\rho}$ maps all primitive central idempotents of the rational group algebra $\mathbb{Q}[\langle \ker(\chi)g \rangle]$ to zero, except the idempotent $\epsilon(\langle \ker(\chi)g \rangle, 1)$, which gets mapped to the identity matrix .

(ii) $\overline{\chi}|_{\langle \ker(\chi)g \rangle}$ is a sum of faithful irreducible characters of $\langle \ker(\chi)g \rangle$.

Some equivalent conditions for $\chi \in \mathrm{Irr}(G)$ to have the property $\mathcal{P}$.

Let $\chi \in \mathrm{Irr}(G)$ and ρ a representation of G affording the character χ. Let $\overline{\rho}$ denote the corresponding induced representation of $G/\ker(\chi)$ and $\overline{\chi}$ the character of $\overline{\rho}$. For $g \in G$, the following statements are equivalent:

(i) *All the eigen-values of $\rho(g)$ are of the same order.*

(ii) *$\overline{\rho}$ maps all primitive central idempotents of the rational group algebra $\mathbb{Q}[\langle \ker(\chi)g \rangle]$ to zero, except the idempotent $\epsilon(\langle \ker(\chi)g \rangle, 1)$, which gets mapped to the identity matrix .*

(ii) *$\overline{\chi}|_{\langle \ker(\chi)g \rangle}$ is a sum of faithful irreducible characters of $\langle \ker(\chi)g \rangle$.*

Some equivalent conditions for $\chi \in \mathrm{Irr}(G)$ to have the property $\mathcal{P}$.

Let $\chi \in \mathrm{Irr}(G)$ and ρ a representation of G affording the character χ. Let $\overline{\rho}$ denote the corresponding induced representation of $G/\ker(\chi)$ and $\overline{\chi}$ the character of $\overline{\rho}$. For $g \in G$, the following statements are equivalent:

(i) *All the eigen-values of* $\rho(g)$ *are of the same order.*

(ii) $\overline{\rho}$ *maps all primitive central idempotents of the rational group algebra* $\mathbb{Q}[\langle \ker(\chi) g \rangle]$ *to zero, except the idempotent* $\epsilon(\langle \ker(\chi) g \rangle, 1)$, *which gets mapped to the identity matrix .*

(ii) $\overline{\chi}|_{\langle \ker(\chi) g \rangle}$ *is a sum of faithful irreducible characters of* $\langle \ker(\chi) g \rangle$.

Some equivalent conditions for $\chi \in \mathrm{Irr}(G)$ to have the property $\mathcal{P}$.

Let $\chi \in \mathrm{Irr}(G)$ and ρ a representation of G affording the character χ. Let $\overline{\rho}$ denote the corresponding induced representation of $G/\ker(\chi)$ and $\overline{\chi}$ the character of $\overline{\rho}$. For $g \in G$, the following statements are equivalent:

(i) *All the eigen-values of* $\rho(g)$ *are of the same order.*

(ii) $\overline{\rho}$ *maps all primitive central idempotents of the rational group algebra* $\mathbb{Q}[\langle \ker(\chi)g \rangle]$ *to zero, except the idempotent* $\epsilon(\langle \ker(\chi)g \rangle, 1)$, *which gets mapped to the identity matrix .*

(ii) $\overline{\chi}|_{\langle \ker(\chi)g \rangle}$ *is a sum of faithful irreducible characters of* $\langle \ker(\chi)g \rangle$.

Some equivalent conditions for $\chi \in \mathrm{Irr}(G)$ to have the property $\mathcal{P}$.

Let $\chi \in \mathrm{Irr}(G)$ and ρ a representation of G affording the character χ. Let $\overline{\rho}$ denote the corresponding induced representation of $G/\ker(\chi)$ and $\overline{\chi}$ the character of $\overline{\rho}$. For $g \in G$, the following statements are equivalent:

(i) *All the eigen-values of $\rho(g)$ are of the same order.*

(ii) *$\overline{\rho}$ maps all primitive central idempotents of the rational group algebra $\mathbb{Q}[\langle \ker(\chi)g \rangle]$ to zero, except the idempotent $\epsilon(\langle \ker(\chi)g \rangle, 1)$, which gets mapped to the identity matrix .*

(ii) *$\overline{\chi}|_{\langle \ker(\chi)g \rangle}$ is a sum of faithful irreducible characters of $\langle \ker(\chi)g \rangle$.*

Theorem 1 (Bakshi-Passi, 2010)

Let G be a finite group and χ a complex irreducible character of a group G satisfying the property $\mathcal{P}$. Then the primitive central idempotent $e_{\mathbb{Q}}(\chi)$ of $\mathbb{Q}[G]$ associated with χ is given by

$$e_{\mathbb{Q}}(\chi) = \frac{1}{\sum_{g\in G,\chi(g)\neq 0}\left(\frac{\mu(d(g))}{\phi(d(g))}\right)^2} \sum_{g\in G,\, \chi(g)\neq 0} \frac{\mu(d(g))}{\phi(d(g))} g,$$

where, for $g \in G$, $d(g)$ denotes the order of g modulo $\ker(\chi)$, μ and ϕ denote respectively the Möbius mu and Euler phi function.

Theorem 1 (Bakshi-Passi, 2010)

Let G be a finite group and χ a complex irreducible character of a group G satisfying the property $\mathcal{P}$. Then the primitive central idempotent $e_{\mathbb{Q}}(\chi)$ of $\mathbb{Q}[G]$ associated with χ is given by

$$e_{\mathbb{Q}}(\chi) = \frac{1}{\sum_{g\in G, \chi(g)\neq 0} \left(\frac{\mu(d(g))}{\phi(d(g))}\right)^2} \sum_{g\in G,\, \chi(g)\neq 0} \frac{\mu(d(g))}{\phi(d(g))} g,$$

where, for $g \in G$, $d(g)$ denotes the order of g modulo $\ker(\chi)$, μ and ϕ denote respectively the Möbius mu and Euler phi function.

key Lemmas

Lemma 1

Let ζ be a primitive n-th root of unity, $n \geq 1$. Then

$$\sum_{1 \leq i \leq n,\, (i,\, n)=1} \zeta^i = \mu(n).$$

Lemma 2

Let G be a group of order n and ζ be a primitive n-th root of unity. If $\chi \in \mathrm{Irr}(G)$ and $g \in G$ are such that all the eigen-values of $\rho(g)$ have the same order, then

$$\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\chi(g)) = \mu(d)\chi(1)\frac{\phi(n)}{\phi(d)},$$

key Lemmas

Lemma 1

Let ζ be a primitive n-th root of unity, $n \geq 1$. Then

$$\sum_{1 \leq i \leq n,\, (i,\, n)=1} \zeta^i = \mu(n).$$

Lemma 2

Let G be a group of order n and ζ be a primitive n-th root of unity. If $\chi \in \mathrm{Irr}(G)$ and $g \in G$ are such that all the eigen-values of $\rho(g)$ have the same order, then

$$\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\chi(g)) = \mu(d)\chi(1)\frac{\phi(n)}{\phi(d)},$$

Proof of the Theorem

Let ζ be a primitive o(G)-th root of unity. From the classical method used to compute the primitive central idempotents of $\mathbb{Q}[G]$ associated with χ, we have

$$
\begin{aligned}
e_{\mathbb{Q}}(\chi) &= \sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi) \\
&= \frac{\chi(1)}{o(G)} \sum_{g \in G} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1} \\
&= \frac{\chi(1)}{o(G)} \sum_{g \in G, \chi(g) \neq 0} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1}
\end{aligned}
$$

Proof of the Theorem

Let ζ be a primitive o(G)-th root of unity. From the classical method used to compute the primitive central idempotents of $\mathbb{Q}[G]$ associated with χ, we have

$$
\begin{aligned}
e_{\mathbb{Q}}(\chi) &= \sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi) \\
&= \frac{\chi(1)}{o(G)} \sum_{g \in G} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1} \\
&= \frac{\chi(1)}{o(G)} \sum_{g \in G, \chi(g) \neq 0} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1}
\end{aligned}
$$

Proof of the Theorem

Let ζ be a primitive o(G)-th root of unity. From the classical method used to compute the primitive central idempotents of $\mathbb{Q}[G]$ associated with χ, we have

$$
\begin{aligned}
e_{\mathbb{Q}}(\chi) &= \sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi) \\
&= \frac{\chi(1)}{o(G)} \sum_{g \in G} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1} \\
&= \frac{\chi(1)}{o(G)} \sum_{g \in G, \chi(g) \neq 0} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1}
\end{aligned}
$$

Proof of the Theorem

Let ζ be a primitive o(G)-th root of unity. From the classical method used to compute the primitive central idempotents of $\mathbb{Q}[G]$ associated with χ, we have

$$\begin{aligned} e_{\mathbb{Q}}(\chi) &= \sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi) \\ &= \frac{\chi(1)}{o(G)} \sum_{g \in G} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1} \\ &= \frac{\chi(1)}{o(G)} \sum_{g \in G, \chi(g) \neq 0} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1} \end{aligned}$$

$$= \frac{\chi(1)}{o(G)} \sum_{g\in G,\, \chi(g)\neq 0} \frac{1}{[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \left(\sum_{\sigma\in Gal(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1},$$

$$= \frac{\chi(1)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g\in G,\, \chi(g)\neq 0} \mu(d(g))\chi(1)\frac{\phi(n)}{\phi(d(g))} g^{-1}, \ \ using\ \ Lemma\, 2$$

$$= \frac{(\chi(1))^2\phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g\in G,\, \chi(g)\neq 0} \frac{\mu(d(g))}{\phi(d(g))} g^{-1}$$

$$= \frac{(\chi(1))^2\phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g\in G,\, \chi(g)\neq 0} \frac{\mu(d(g))}{\phi(d(g))} g$$

$(as\ \chi(g) \neq 0 \Leftrightarrow \chi(g^{-1}) \neq 0\ and\ d(g) = d(g^{-1})).$

$$= \frac{\chi(1)}{o(G)} \sum_{g\in G,\, \chi(g)\neq 0} \frac{1}{[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \left(\sum_{\sigma\in Gal(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1},$$

$$= \frac{\chi(1)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g\in G,\, \chi(g)\neq 0} \mu(d(g))\chi(1)\frac{\phi(n)}{\phi(d(g))} g^{-1}, \ \textit{using Lemma}\, 2$$

$$= \frac{(\chi(1))^2\phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g\in G,\, \chi(g)\neq 0} \frac{\mu(d(g))}{\phi(d(g))} g^{-1}$$

$$= \frac{(\chi(1))^2\phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g\in G,\, \chi(g)\neq 0} \frac{\mu(d(g))}{\phi(d(g))} g$$

$(as\ \chi(g) \neq 0 \Leftrightarrow \chi(g^{-1}) \neq 0\ and\ d(g) = d(g^{-1})).$

$$= \frac{\chi(1)}{o(G)} \sum_{g \in G,\, \chi(g) \neq 0} \frac{1}{[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \left(\sum_{\sigma \in Gal(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1},$$

$$= \frac{\chi(1)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g \in G,\, \chi(g) \neq 0} \mu(d(g))\chi(1) \frac{\phi(n)}{\phi(d(g))} g^{-1}, \; using \; Lemma\, 2$$

$$= \frac{(\chi(1))^2 \phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g \in G,\, \chi(g) \neq 0} \frac{\mu(d(g))}{\phi(d(g))} g^{-1}$$

$$= \frac{(\chi(1))^2 \phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g \in G,\, \chi(g) \neq 0} \frac{\mu(d(g))}{\phi(d(g))} g$$

$(as\ \chi(g) \neq 0 \Leftrightarrow \chi(g^{-1}) \neq 0\ and\ d(g) = d(g^{-1})).$

$$
\begin{aligned}
&= \frac{\chi(1)}{o(G)} \sum_{g\in G,\, \chi(g)\neq 0} \frac{1}{[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \left(\sum_{\sigma\in Gal(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(\chi(g)) \right) g^{-1}, \\
&= \frac{\chi(1)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g\in G,\, \chi(g)\neq 0} \mu(d(g))\chi(1)\frac{\phi(n)}{\phi(d(g))} g^{-1}, \ \ using \ \ Lemma\, 2 \\
&= \frac{(\chi(1))^2\phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g\in G,\, \chi(g)\neq 0} \frac{\mu(d(g))}{\phi(d(g))} g^{-1} \\
&= \frac{(\chi(1))^2\phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} \sum_{g\in G,\, \chi(g)\neq 0} \frac{\mu(d(g))}{\phi(d(g))} g \\
&\quad (as \ \ \chi(g)\neq 0 \Leftrightarrow \chi(g^{-1})\neq 0 \ \ and \ \ d(g)=d(g^{-1})).
\end{aligned}
$$

Since $e_{\mathbb{Q}}(\chi)^2 = e_{\mathbb{Q}}(\chi)$, we obtain, by comparing the coefficient of 1 on both sides of this equation we get that

$$\frac{(\chi(1))^2\phi(n)}{o(G)[\mathbb{Q}(\zeta):\mathbb{Q}(\chi)]} = \frac{1}{\sum_{g\in G,\chi(g)\neq 0}\left(\frac{\mu(d(g))}{\phi(d(g))}\right)^2},$$

which completes the proof of our theorem. $\square$

Some Consequences

Notation.

Let $H \trianglelefteq G$, and $K/H = \langle Ha \rangle$ a cyclic subgroup of G/H.

- If $K \neq H$ and $o(K/H) = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, p_i's distinct primes, α_i's ≥ 1, we define

$$E_{H,\,K} := \frac{(p_1 - 1)(p_2 - 1) \cdots (p_n - 1)}{o(H) p_1 p_2 \ldots p_n} \sum_{g \in L} \frac{\mu(d(g))}{\phi(d(g))} g,$$

  where, for any $g \in L$, $d(g)$ is the order of g modulo H and L/H is the subgroup of K/H of order $p_1 p_2 \ldots p_n$.

- Set $E_{H,\,H} = \hat{H}$.

Notation.

Let $H \trianglelefteq G$, and $K/H = \langle Ha \rangle$ a cyclic subgroup of G/H.

- If $K \neq H$ and $o(K/H) = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, p_i's distinct primes, α_i's ≥ 1, we define

$$E_{H,\,K} := \frac{(p_1-1)(p_2-1)\cdots(p_n-1)}{o(H)p_1p_2\dots p_n} \sum_{g \in L} \frac{\mu(d(g))}{\phi(d(g))} g,$$

  where, for any $g \in L$, $d(g)$ is the order of g modulo H and L/H is the subgroup of K/H of order $p_1 p_2 \dots p_n$.

- Set $E_{H,\,H} = \hat{H}$.

Notation.

Let $H \trianglelefteq G$, and $K/H = \langle Ha \rangle$ a cyclic subgroup of G/H.

- If $K \neq H$ and $o(K/H) = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, p_i's distinct primes, α_i's ≥ 1, we define

$$E_{H,\,K} := \frac{(p_1-1)(p_2-1)\,\cdots\,(p_n-1)}{o(H)p_1p_2\,\dots\,p_n}\sum_{g\in L}\frac{\mu(d(g))}{\phi(d(g))}g,$$

  where, for any $g \in L$, $d(g)$ is the order of g modulo H and L/H is the subgroup of K/H of order $p_1p_2\dots p_n$.

- Set $E_{H,\,H} = \hat{H}$.

$\chi \in \mathrm{Irr}(G)$,

$Z(\chi)/\ker(\chi) := Z(G/\ker(\chi))$.

It is known that $Z(G/\ker(\chi))$ *is cyclic and* $\chi(1)^2 \leq [G : Z(\chi)]$.

We determine the primitive central idempotent of $\mathbb{Q}[G]$ associated with χ provided $\chi(1)^2 = [G : Z(\chi)]$.

$\chi \in \mathrm{Irr}(G)$,

$Z(\chi)/\ker(\chi) := Z(G/\ker(\chi))$.

It is known that $Z(G/\ker(\chi))$ *is cyclic and* $\chi(1)^2 \leq [G : Z(\chi)]$.

We determine the primitive central idempotent of $\mathbb{Q}[G]$ associated with χ provided $\chi(1)^2 = [G : Z(\chi)]$.

Corollary 1

Let χ be an irreducible character of a group G of degree $\sqrt{[G : Z(\chi)]}$.

Then

$$e_{\mathbb{Q}}(\chi) = E_{\ker(\chi),\, Z(\chi)}.$$

If, in addition, $o(Z(\chi)/\ker(\chi)) = p^k$ for some prime p and $k \geq 1$, then

$$e_{\mathbb{Q}}(\chi) = \hat{\ker(\chi)} - \hat{L},$$

where $L/\ker(\chi)$ is the unique subgroup of $Z(G/\ker(\chi))$ of order p.

Corollary 1

Let χ be an irreducible character of a group G of degree $\sqrt{[G : Z(\chi)]}$.

Then

$$e_{\mathbb{Q}}(\chi) = E_{\ker(\chi),\, Z(\chi)}.$$

If, in addition, $o(Z(\chi)/\ker(\chi)) = p^k$ for some prime p and $k \geq 1$, then

$$e_{\mathbb{Q}}(\chi) = \hat{\ker(\chi)} - \hat{L},$$

where $L/\ker(\chi)$ is the unique subgroup of $Z(G/\ker(\chi))$ of order p.

Corollary 2

If $\chi \in \mathrm{Irr}(\chi)$ with $\chi(1)^2 = [G : Z(G)]$, then $e_{\mathbb{Q}}(\chi) = E_{\mathrm{ker}(\chi), Z(G)}$.

Corollary 3

If $\chi \in \mathrm{Irr}(G)$ such that $G/Z(\chi)$ is abelian, then $E_{\ker(\chi),Z(\chi)}$ is the primitive central idempotent of $\mathbb{Q}[G]$ associated with χ.

Applications

We now apply these results to write quite simple expressions of the primitive central idempotents in the rational group algebra $\mathbb{Q}[G]$ of

- an extra special p-group,
- a CM_{p-1}-group,
- a nilpotent group of class ≤ 2 and
- those associated with normally monomial irreducible characters.

Applications

We now apply these results to write quite simple expressions of the primitive central idempotents in the rational group algebra $\mathbb{Q}[G]$ of

- an extra special p-group,
- a CM_{p-1}-group,
- a nilpotent group of class ≤ 2 and
- those associated with normally monomial irreducible characters.

Applications

We now apply these results to write quite simple expressions of the primitive central idempotents in the rational group algebra $\mathbb{Q}[G]$ of

- an extra special p-group,
- a CM_{p-1}-group,
- a nilpotent group of class ≤ 2 and
- those associated with normally monomial irreducible characters.

Applications

We now apply these results to write quite simple expressions of the primitive central idempotents in the rational group algebra $\mathbb{Q}[G]$ of

- an extra special p-group,
- a CM_{p-1}-group,
- a nilpotent group of class ≤ 2 and
- those associated with normally monomial irreducible characters.

Applications

We now apply these results to write quite simple expressions of the primitive central idempotents in the rational group algebra $\mathbb{Q}[G]$ of

- an extra special p-group,
- a CM_{p-1}-group,
- a nilpotent group of class ≤ 2 and
- those associated with normally monomial irreducible characters.

Applications

We now apply these results to write quite simple expressions of the primitive central idempotents in the rational group algebra $\mathbb{Q}[G]$ of

- an extra special p-group,
- a CM_{p-1}-group,
- a nilpotent group of class ≤ 2 and
- those associated with normally monomial irreducible characters.

Extra special p-groups

Recall that a finite p-group G is said to be extra special if

(i) $G' = Z(G)$,

(ii) $o(Z(G)) = p$, *and*

(iii) $G/Z(G)$ *is an elementary abelian group.*

Extra special p-groups

Recall that a finite p-group G is said to be extra special if

(i) $G' = Z(G)$,

(ii) $o(Z(G)) = p$, *and*

(iii) $G/Z(G)$ *is an elementary abelian group.*

It is known that every extra special p-group has order p^{2n+1} for some positive integer n and the irreducible complex representations of an extra special p-group G of order p^{2n+1} are given as follows:

(i) There are exactly p^{2n} irreducible representations of dimension 1; these representations are just the ones corresponding to the representations of the abelian group G/G'.

(ii) There are exactly $p-1$ faithful irreducible characters χ_i of degree p^n, which vanish outside $Z(G)$ and satisfy $\chi_i|_{Z(G)} = p^n\lambda_i$, λ_i a faithful linear character of $Z(G)$.

Note that all of these characters have the property $\mathcal{P}$.

It is known that every extra special p-group has order p^{2n+1} for some positive integer n and the irreducible complex representations of an extra special p-group G of order p^{2n+1} are given as follows:

(i) There are exactly p^{2n} irreducible representations of dimension 1; these representations are just the ones corresponding to the representations of the abelian group G/G'.

(ii) There are exactly $p-1$ faithful irreducible characters χ_i of degree p^n, which vanish outside $Z(G)$ and satisfy $\chi_i|_{Z(G)} = p^n\lambda_i$, λ_i a faithful linear character of $Z(G)$.

Note that all of these characters have the property $\mathcal{P}$.

It is known that every extra special p-group has order p^{2n+1} for some positive integer n and the irreducible complex representations of an extra special p-group G of order p^{2n+1} are given as follows:

(i) There are exactly p^{2n} irreducible representations of dimension 1; these representations are just the ones corresponding to the representations of the abelian group G/G'.

(ii) There are exactly $p-1$ faithful irreducible characters χ_i of degree p^n, which vanish outside $Z(G)$ and satisfy $\chi_i|_{Z(G)} = p^n\lambda_i$, λ_i a faithful linear character of $Z(G)$.

Note that all of these characters have the property $\mathcal{P}$.

It is known that every extra special p-group has order p^{2n+1} for some positive integer n and the irreducible complex representations of an extra special p-group G of order p^{2n+1} are given as follows:

(i) There are exactly p^{2n} irreducible representations of dimension 1; these representations are just the ones corresponding to the representations of the abelian group G/G'.

(ii) There are exactly $p-1$ faithful irreducible characters χ_i of degree p^n, which vanish outside $Z(G)$ and satisfy $\chi_i|_{Z(G)} = p^n\lambda_i$, λ_i a faithful linear character of $Z(G)$.

Note that all of these characters have the property $\mathcal{P}$.

If χ is one of the linear characters as listed in (i) above and $H = \ker(\chi)$, then, G/H is cyclic and by Theorem 1 and arguing as in Corollary 1, it follows that

$$e_{\mathbb{Q}}(\chi) = \begin{cases} \hat{G} & \textit{if } \ker(\chi) = G \\ \hat{H} - \hat{L} & \textit{if } \ker(\chi) \neq G, \end{cases}$$

where L/H is the subgroup of G/H of order p. Also we note that for any normal subgroup H of G with G/H cyclic, there is a linear character in the list (i) above with $\ker(\chi) = H$.

If χ is any of the $p-1$ faithful irreducible characters as listed in (ii) above, then by Theorem 1, it turns out that $e_{\mathbb{Q}}(\chi) = 1 - Z(\hat{G})$. Note that the primitive central idempotents $\hat{G}$, $1 - Z(\hat{G})$, $\hat{H} - \hat{L}$, where H runs through all proper subgroups of G such that G/H is cyclic and L/H is the subgroup of G/H of order p, are all distinct. Therefore, we have the following:

If χ is one of the linear characters as listed in (i) above and $H = \ker(\chi)$, then, G/H is cyclic and by Theorem 1 and arguing as in Corollary 1, it follows that

$$e_{\mathbb{Q}}(\chi) = \begin{cases} \hat{G} & \text{if } \ker(\chi) = G \\ \hat{H} - \hat{L} & \text{if } \ker(\chi) \neq G, \end{cases}$$

where L/H is the subgroup of G/H of order p. Also we note that for any normal subgroup H of G with G/H cyclic, there is a linear character in the list (i) above with $\ker(\chi) = H$.

If χ is any of the $p-1$ faithful irreducible characters as listed in (ii) above, then by Theorem 1, it turns out that $e_{\mathbb{Q}}(\chi) = 1 - \hat{Z(G)}$. Note that the primitive central idempotents $\hat{G}$, $1 - \hat{Z(G)}$, $\hat{H} - \hat{L}$, where H runs through all proper subgroups of G such that G/H is cyclic and L/H is the subgroup of G/H of order p, are all distinct. Therefore, we have the following:

If χ is one of the linear characters as listed in (i) above and $H = \ker(\chi)$, then, G/H is cyclic and by Theorem 1 and arguing as in Corollary 1, it follows that

$$e_{\mathbb{Q}}(\chi) = \begin{cases} \hat{G} & \text{if } \ker(\chi) = G \\ \hat{H} - \hat{L} & \text{if } \ker(\chi) \neq G, \end{cases}$$

where L/H is the subgroup of G/H of order p. Also we note that for any normal subgroup H of G with G/H cyclic, there is a linear character in the list (i) above with $\ker(\chi) = H$.

If χ is any of the $p-1$ faithful irreducible characters as listed in (ii) above, then by Theorem 1, it turns out that $e_{\mathbb{Q}}(\chi) = 1 - Z(\hat{G})$. Note that the primitive central idempotents $\hat{G}$, $1 - Z(\hat{G})$, $\hat{H} - \hat{L}$, where H runs through all proper subgroups of G such that G/H is cyclic and L/H is the subgroup of G/H of order p, are all distinct. Therefore, we have the following:

Theorem 2

If G is an extra special p-group, then all the primitive central idempotents of $\mathbb{Q}[G]$ are given by

$$\hat{G},\ 1 - Z(\hat{G})\ \textit{and}\ \hat{H} - \hat{L},$$

where H runs through all the proper normal subgroups H of G with G/H cyclic and L/H is the unique subgroup of G/H of order p.

CM_{p-1} groups

Recall that a p-group G is called a CM_{p-1}-group if every proper normal subgroup H of G with $Z(G/H)$ cyclic is the kernel of exactly $p-1$ irreducible characters of G.

It is known that every non-principal irreducible character of a CM_{p-1}-group G satisfies $\chi(1)^2 = [G : Z(\chi)]$.

CM_{p-1} groups

Recall that a p-group G is called a CM_{p-1}-group if every proper normal subgroup H of G with $Z(G/H)$ cyclic is the kernel of exactly $p-1$ irreducible characters of G.

It is known that every non-principal irreducible character of a CM_{p-1}-group G satisfies $\chi(1)^2 = [G : Z(\chi)]$.

Thus as an immediate consequence of Corollary 1, we have the following :

Theorem 3

If G is a CM_{p-1}-group, then all the primitive central idempotents of $\mathbb{Q}[G]$ are given by

$$\hat{G} \text{ and } \hat{H} - \hat{L},$$

where H runs over all proper normal subgroups of G with $Z(G/H)$ cyclic and L/H is the unique subgroup of $Z(G/H)$ of order p.

Nilpotent groups of class ≤ 2

For an abelian group G, it is known that there is 1-1 correspondence between the primitive (central) idempotents of $\mathbb{Q}[G]$ and the (normal) subgroups H of G such that G/H is cyclic; the following theorem extends this result to nilpotent groups of class ≤ 2.

Theorem 4 Let G be a nilpotent group of class ≤ 2. Then there is 1-1 correspondence between the primitive central idempotents of $\mathbb{Q}[G]$ and the normal subgroups H of G such that $Z(G/H)$ is cyclic. Moreover, for a normal subgroup H of G with $Z(G/H)$ cyclic, $E_{H,Z}$ is the primitive central idempotent of $\mathbb{Q}[G]$ which is associated with it, where $Z/H = Z(G/H)$.

Nilpotent groups of class ≤ 2

For an abelian group G, it is known that there is 1-1 correspondence between the primitive (central) idempotents of $\mathbb{Q}[G]$ and the (normal) subgroups H of G such that G/H is cyclic; the following theorem extends this result to nilpotent groups of class ≤ 2.

Theorem 4 Let G be a nilpotent group of class ≤ 2. Then there is 1-1 correspondence between the primitive central idempotents of $\mathbb{Q}[G]$ and the normal subgroups H of G such that $Z(G/H)$ is cyclic. Moreover, for a normal subgroup H of G with $Z(G/H)$ cyclic, $E_{H,Z}$ is the primitive central idempotent of $\mathbb{Q}[G]$ which is associated with it, where $Z/H = Z(G/H)$.

Monomial irreducible characters

K be a subgroup of a group G

χ_K : a linear character on K

χ_K^G : the induced character on G (such a character of G is called a monomial character)

If $\chi_K^G \in \mathrm{Irr}(G)$, Oliveri, Rio and Simon(2004) obtained an expression of the primitive central idempotent $e_{\mathbb{Q}}(\chi_K^G)$ of $\mathbb{Q}[G]$ associated with χ_K^G.

Monomial irreducible characters

K be a subgroup of a group G

χ_K : a linear character on K

χ_K^G : the induced character on G (such a character of G is called a monomial character)

If $\chi_K^G \in \mathrm{Irr}(G)$, Oliveri, Rio and Simon(2004) obtained an expression of the primitive central idempotent $e_{\mathbb{Q}}(\chi_K^G)$ of $\mathbb{Q}[G]$ associated with χ_K^G.

Here is a more simplified expression of $e_{\mathbb{Q}}(\chi_K^G)$, when K and $\ker(\chi_K)$ are both normal subgroups of G.

Theorem 5 Let χ_K^G be a monomial irreducible character of a group G. If K and $H = \ker(\chi_K)$ are both normal in G, then

$$e_{\mathbb{Q}}(\chi_K^G) = E_{H,K}$$

. If, in addition, $o(K/H) = p^k$ for some prime p and $k \geq 1$, then

$$e_{\mathbb{Q}}(\chi_K^G) = E_{H,K} = \hat{H} - \hat{L},$$

where L/H is the unique subgroup of K/H of order p.

Here is a more simplified expression of $e_{\mathbb{Q}}(\chi_K^G)$, when K and $\ker(\chi_K)$ are both normal subgroups of G.

Theorem 5 Let χ_K^G be a monomial irreducible character of a group G. If K and $H = \ker(\chi_K)$ are both normal in G, then

$$e_{\mathbb{Q}}(\chi_K^G) = E_{H,K}$$

. If, in addition, $o(K/H) = p^k$ for some prime p and $k \geq 1$, then

$$e_{\mathbb{Q}}(\chi_K^G) = E_{H,\,K} = \hat{H} - \hat{L},$$

where L/H is the unique subgroup of K/H of order p.

Shoda Pair, (Olivieri, Rio and Simon, 2004)

A pair (H, K) of subgroups of G is called a Shoda pair if it satisfies the following conditions:

(i) *$H \trianglelefteq K$,*

(ii) *K/H is cyclic, and*

(iii) *if $g \in G$ and $[K, g] \cap K \subseteq H$, then $g \in K$.*

By Shoda's Theorem (Shoda, 1933), if χ is a linear character of a subgroup K of G with kernel H, then the induced character χ^G is irreducible if and only if (H, K) is a Shoda pair (Olivieri, Rio and Simon, 2004).

As an immediate consequence of Theorem 5, we have the following:

Corollary

If (H, K) is a Shoda pair in G with $H, K \trianglelefteq G$, then $E_{H, K}$ is a primitive central idempotent of $\mathbb{Q}[G]$.

It may be noted that our expressions for primitive central idempotents in these results are quite simple and, as such, may possibly be of help in further studies.

THANK YOU