Torsion subgroups in $\mathrm{V}(\mathbb{Z}G)$

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Notations

G	finite group
$\mathbb{Z}G$	integral group ring of G
$\mathrm{U}(\mathbb{Z}G)$	group of units of $\mathbb{Z}G$
$\mathrm{V}(\mathbb{Z}G)$	group of normalized units of $\mathbb{Z}G$, i.e.

$$\mathrm{V}(\mathbb{Z}G) = \left\{ \sum_{g \in G} u_g g \in \mathrm{U}(\mathbb{Z}G) : \sum_{g \in G} u_g = 1 \right\}$$

As $\mathrm{U}(\mathbb{Z}G) = \pm\mathrm{V}(\mathbb{Z}G)$ we don't lose much, if we restrict our interest to $\mathrm{V}(\mathbb{Z}G)$.

First Zassenhaus conjecture

Question

How far does the structure of the group G determine the structure of the torsion units in $\mathbb{Z}G$?

G abelian $\Longrightarrow$ torsion elements of $\mathrm{V}(\mathbb{Z}G)$ are exactly the elements of G (Higman, 1939)

Hans Zassenhaus conjectured, that for an arbitrary group the normalized torsion units of $\mathbb{Z}G$ are not too "far away" from the group:

Conjecture (First Zassenhaus conjecture, 1970s)

(ZC1) *Every normalized torsion unit of the integral group ring $\mathbb{Z}G$ is rationally conjugate to an element of the group.*

[u and v rationally conjugate: $\exists\, x \in \mathrm{U}(\mathbb{Q}G)$ with $x^{-1}ux = v$]

(ZC1) has been verified for the following finite groups

- ✓ A_5 (Luthar, Passi, 1989)
- ✓ finite nilpotent groups (Weiss, 1991)
- ✓ groups of order at most 71 (Höfert, 2004)
- ✓ $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,13)$ (Hertweck, 2004)
- ✓ A_6 (Hertweck, 2008)
- ✓ finite metacyclic groups (Hertweck, 2008)

But especially for simple groups it seems to be hard to attack this conjecture and it seems plausible to study a weakened version first:

Prime graph question I

The prime graph (or Gruenberg-Kegel graph) of a group H is the undirected loop-free graph $\Pi(H)$ with

- Vertices: primes p, for which there exists an element of order p in H
- Edges: primes p and q joined iff there is an element of order pq in H

Example: $\Pi(\mathrm{A}_7)$ $\qquad |\mathrm{A}_7| = 2^3 \cdot 3^2 \cdot 5 \cdot 7$

Question (Prime graph question, Kimmerle, 2005)

(PQ) *Is it true, that* $\Pi(\mathrm{V}(\mathbb{Z}G)) = \Pi(G)$*?*

Clearly: **(ZC1)** $\Longrightarrow$ **(PQ)**.

Furthermore there is an affirmative answer to **(PQ)** for finite

- ✓ Frobenius groups (Kimmerle, 2006)
- ✓ soluble groups (Höfert, Kimmerle, 2006)
- ✓ 13 sporadic simple groups (Bovdi, Konovalov, et. al., 2005 - 2008)

A criterion for rational conjugacy

Let C be a conjugacy class of G, $u = \sum_{g \in G} u_g g \in \mathbb{Z}G$ then

$$\varepsilon_C(u) = \sum_{g \in C} u_g$$

is the partial augmentation of u with respect to C. For a representant $x \in C$ of the conjugacy class write $\varepsilon_x := \varepsilon_C$.

Theorem (Marciniak, Ritter, Sehgal, Weiss, 1987; Luthar, Passi, 1989)

Let $u \in \mathrm{V}(\mathbb{Z}G)$ be a torsion unit of order k. u is rationally conjugate to an element of $G \iff \forall\, d \,|\, k$ all partial augmentations of u^d but one vanish.

For confirming **(ZC1)**: Fix a possible order k of torsion units in $\mathrm{V}(\mathbb{Z}G)$ and

- if there is an element of this order in G, try to show that all units of this order in $\mathrm{V}(\mathbb{Z}G)$ satisfy the premise of the above theorem
- if there is no element of this order in G, try to show there is no element of this order in $\mathrm{V}(\mathbb{Z}G)$

Which orders of torsion units may appear?

Theorem (Cohn, Livingstone, 1965)

Let $u \in \mathbb{Z}G$ be a normalized torsion unit of order k. Then k divides the exponent of G.

Which conjugacy classes have to be taken into account?

Theorem (Berman, 1955; Higman, 1939)

Let $u = \sum_{g \in G} u_g g \in \mathbb{Z}G$ *a normalized torsion unit,* $u \neq 1$. *Then* $u_1 = 0$.

Theorem (Hertweck, 2004)

Let $u \in \mathbb{Z}G$ *be a normalized torsion unit and* C *a conjugacy class of* G. *If the order of the elements of* C *does not divide the order of* u, *then* $\varepsilon_C(u) = 0$.

In many cases this is not sufficient. A method developed by Luthar and Passi, and improved by Hertweck, may provide help.

Theorem (Luthar, Passi, 1989; Hertweck, 2004)

Let $u \in \mathbb{Z}G$ be a torsion unit of order k and let

- $p = 0$ *and* χ *an ordinary character of* G *or*
- p *a prime not dividing* k *and* χ *be a* p*-modular Brauer character of* G *(as function with values in* $\mathbb{C}$*).*

If $\zeta \in \mathbb{C}$ *is a primitive* k*-th root of unity, then for every integer* ℓ *the number*

$$\mu_\ell(u, \chi, p) = \frac{1}{k} \sum_{d|k} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\zeta^{-d\ell})$$

is a non-negativ integer.

$$\begin{aligned}\mu_\ell(u,\chi,p) &= \frac{1}{k}\sum_{d|k}\mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\zeta^{-d\ell}) \\ &= \underbrace{\frac{1}{k}\sum_{\substack{d|k\\ d\neq 1}}\mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\zeta^{-d\ell})}_{=:a_{\chi,\ell}} + \frac{1}{k}\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(u)\zeta^{-\ell})\end{aligned}$$

As $\chi(u) = \sum_C \varepsilon_C(u)\chi(C)$, where $\chi(C)$ is the value of χ on the class C, we obtain linear equations

$$t_{C_1}\varepsilon_{C_1}(u) + t_{C_2}\varepsilon_{C_2}(u) + \ldots + t_{C_h}\varepsilon_{C_h}(u) + a_{\chi,\ell} = \mu_\ell(u,\chi,p).$$

for the partial augmentations $\varepsilon_{C_i}(u)$, with “known” coefficients $t_{C_j}, a_{\chi,\ell}$ and $\mu_\ell(u,\chi,p)$.

Example: The smallest Suzuki group $\mathrm{Sz}(8)$ ...

- ... is a simple group
- ... of order $|\mathrm{Sz}(8)| = 2^6 \cdot 5 \cdot 7 \cdot 13 = 29\,120$
- ... has conjugacy classes

  `1a 2a 4a 4b 5a 7a 7b 7c 13a 13b 13c`

- ... and prime graph

Example: torsion units of order 13 in $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$

Let $u \in \mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$ be a torsion unit of order 13, then u has possibly non-trivial partial augmentation on the following conjugacy classes

1a 2a 4a 4b 5a 7a 7b 7c 13a 13b 13c

We are not done yet. But using the Luthar-Passi method with φ_2, one of the irreducible degree 4 Brauer characters modulo 2 we get the inequalities

Example: torsion units of order 13 in $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$

$$\begin{array}{rrrrrrrrrr}
13\,\mu_1(u,\varphi_2,2) & = & & 9\,\varepsilon_{\mathtt{13a}}(u) & - & 4\,\varepsilon_{\mathtt{13b}}(u) & - & 4\,\varepsilon_{\mathtt{13c}}(u) & + & 4 \geq 0 \\
13\,\mu_2(u,\varphi_2,2) & = & - & 4\,\varepsilon_{\mathtt{13a}}(u) & + & 9\,\varepsilon_{\mathtt{13b}}(u) & - & 4\,\varepsilon_{\mathtt{13c}}(u) & + & 4 \geq 0 \\
 & & - & 5\,\varepsilon_{\mathtt{13a}}(u) & - & 5\,\varepsilon_{\mathtt{13b}}(u) & - & 5\,\varepsilon_{\mathtt{13c}}(u) & + & 5 \geq 0 \\
\hline
 & & & & & & - & 13\,\varepsilon_{\mathtt{13c}}(u) & + & 13 \geq 0
\end{array}$$

We obtain the third inequality since u is normalizd. By adding the three upper inequalites, we obtain the fourth.
Hence $\varepsilon_{\mathtt{13c}}(u) \leq 1$. Similarly

$$0 \;\leq\; \varepsilon_{\mathtt{13a}}(u), \varepsilon_{\mathtt{13b}}(u), \varepsilon_{\mathtt{13c}}(u) \;\leq\; 1$$

and as $\varepsilon_{\mathtt{13a}}(u) + \varepsilon_{\mathtt{13b}}(u) + \varepsilon_{\mathtt{13c}}(u) = 1$ there must be exactly one non-vanishing partial augmentation.

Theorem (Marciniak, Ritter, Sehgal, Weiss, 1987; Luthar, Passi, 1989)

Let $u \in \mathrm{V}(\mathbb{Z}G)$ be a torsion unit of order k. u is rationally conjugate to an element of $G \iff \forall\, d \mid k$ all partial augmentations of u^d but one vanish.

Proposition

Let $u \in \mathbb{Z}\mathrm{Sz}(8)$ be a normalized torsion unit, then

1. *the order of u coincides with the order of an element of $\mathrm{Sz}(8)$,*
2. *if u is of order $2, 5$ or 13, then u is rationally conjugate to an element of $\mathrm{Sz}(q)$,*
3. *if u is of order 4, there are at most 12 possible tupels of partial augmentations for u,*
4. *if u is of order 7, there are at most 6 possible tupels of partial augmentations for u.*

This proposition gives a postive answer to **(PQ)** for $\mathrm{Sz}(8)$:

$$\Pi(\mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))) = \Pi(\mathrm{Sz}(8))$$

$$\begin{array}{cccc} 2 & 5 & 7 & 13 \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

We even obtain that every cyclic subgroup of $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(8))$ is isomorphic to one of $\mathrm{Sz}(8)$.

Prime graph question II

The answer to **(PQ)** is yes for

- ✓ $\mathrm{Sz}(8)$
- ✓ $\mathrm{PSL}(3,3)$
- ✓ $\mathrm{PSL}(2,p^f)$

  (Hertweck, 2004)
- ✓ $\mathrm{SL}(2,p^f)$

p an odd prime

$\mathrm{SL}(2,p^f) \twoheadrightarrow \mathrm{PSL}(2,p^f)$ induces
$\mathrm{V}(\mathbb{Z}\mathrm{SL}(2,p^f)) \longrightarrow \mathrm{V}(\mathbb{Z}\mathrm{PSL}(2,p^f))$
torsion-part of the kernel: $\{1,t\}$,
t the central involution of $\mathrm{SL}(2,p^f)$

Example: The groups $\mathrm{SL}(2, q)$

Proposition

Let p be an odd prime and $G = \mathrm{SL}(2, p^f)$. Further let $u \in \mathbb{Z}G$ be a normalized torsion unit of prime order r, then

- *if $r = p$ and $f \leq 2$ then u is rationally conjugate to an element of G,*
- *if $r \neq p$ then u is rationally conjugate to an element of G.*

Example: The groups $\mathrm{SL}(2, q)$

Elements $u \in \mathrm{V}(\mathbb{Z}\mathrm{SL}(2, p^f))$, of prime order $r \neq p$ are rationally conjugate to elements of $\mathrm{SL}\,(2, q)$.

Proof. Conjugacy classes of elements of order r are represented by

$$\mathbf{j} := \begin{pmatrix} \xi^j & . \\ . & \xi^{-j} \end{pmatrix}, \qquad 1 \leq j \leq \left\lceil \frac{r-1}{2} \right\rceil,$$

with $\xi \in \overline{\mathbb{F}_q}^{\times}$, a primitive r-th root of unity. Let φ be the Brauer character afforded by the "natural" p-modular representation Θ. Then

$$\varphi(u) = \sum_{1 \leq j \leq \lceil \frac{r-1}{2} \rceil} \varepsilon_{\mathbf{j}}(u) \varphi(\mathbf{j}) = \sum_{1 \leq j \leq \lceil \frac{r-1}{2} \rceil} \varepsilon_{\mathbf{j}}(u) \left(\zeta^j + \zeta^{-j} \right).$$

On the other hand $\Theta(u)$ is diagonalizable, so $\varphi(u) = \zeta^k + \zeta^\ell$. Hence there is only one nonvanishing partial augmentation and u is rationally conjugate to a group element.

Subgroups

Question (Subgroup question)

$$H \le \mathrm{V}(\mathbb{Z}G) \text{ finite} \quad \Longrightarrow \quad H \text{ isomorphic to a subgroup of } G?$$

The answer is in general no. (Hertweck, 1998)

Proposition

For any prime p the elementary abelian p-subgroups of $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(q))$ are isomorphic to subgroups of $\mathrm{Sz}(q)$.

Corollary

If $p \in \{2, 5\}$ this isomorphism can be taken as conjugation with an unit of $\mathbb{Q}\mathrm{Sz}(q)$.

Proposition

For any prime p the elementary abelian p-subgroups of $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(q))$ are isomorphic to subgroups of $\mathrm{Sz}(q)$.

Proof. If $p \neq 2$: The Sylow p-subgroups of $\mathrm{Sz}(q)$ are cyclic and the result follows immediately from Hertweck's $\mathrm{C}_p \times \mathrm{C}_p$ theorem.

Proposition

For any prime p the elementary abelian p-subgroups of $\mathrm{V}(\mathbb{Z}\mathrm{Sz}(q))$ are isomorphic to subgroups of $\mathrm{Sz}(q)$.

Proof. If $p = 2$: Let $H \leq \mathrm{V}(\mathbb{Z}\mathrm{Sz}(q))$, $H \cong \mathrm{C}_2^k$.
Let $q = 2^{2m+1}$, $r = 2^m$ and $t \in \mathrm{Sz}(q)$ be an involution

	1	x^a	y^b	z^c	t	f	f^{-1}
δ_1	$r(q-1)$	.	1	-1	$-r$	ri	$-ri$

$(\delta_1)_H \colon H \to \mathbb{C}$. We have

$$\langle\, (\delta_1)_H \mid 1_H \,\rangle_H = \frac{1}{|H|} \sum_{h \in H} \delta_1(h) = \frac{1}{|H|} \Big(r(q-1) + (2^k - 1)(-r) \Big) \geq 0.$$

Hence

$$2^{2m+1} - 1 \geq 2^k - 1,$$

and $k \leq 2m+1$. On the other hand $\mathrm{Z}(F) \cong \mathrm{C}_2^{2m+1}$, where $F \in \mathrm{Syl}_2(\mathrm{Sz}(q))$.

A finite simple group with a partition is isomorphic to a group of the series

- $\mathrm{PSL}(2, q)$,
- $\mathrm{Sz}(q)$.

Combinig the last result, and

Theorem (Hertweck, Höfert, Kimmerle, 2008)

For any prime p the elementary abelian p-subgroups of $\mathrm{V}(\mathbb{Z}\mathrm{PSL}(2, q))$, q a prime power, are isomorphic to subgroups of $\mathrm{PSL}(2, q)$.

we obtain

Proposition

Let G be a finite simple group with a partition, p a prime number. Then any elementary abelian p-group $H \leq \mathrm{V}(\mathbb{Z}G)$ is isomorphic to a subgroup of G.

Proposition

Let $G = \mathrm{SL}(2, q)$ with q a power of the prime p and r a prime.

- *if $p = 2$ then all finite r-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are isomoprhic to subgroups of G.*
- *if $p \neq 2$ then all elementary-abelian r-subgroups of $\mathrm{V}(\mathbb{Z}G)$ are isomoprhic to subgroups of G if $r \neq p$.*