Nilpotency indices of symmetric elements in group algebras

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PI and GI properties

Denote by $F[x_1, x_2, \cdots, x_n]$ the polynomial ring over the field F with the non-commuting indeterminates $x_1, x_2, \cdots, x_n$.
Let A be an algebra over the field F and $S \subseteq A$ a subset.

Definition (Polynomial identities)

We say that S satisfies a polynomial identity (PI) if there exists a nonzero polynomial $f(x_1, x_2, \cdots, x_m) \in F[x_1, x_2, \cdots, x_n]$ such that $f(s_1, s_2, \cdots, s_n) = 0$ for all $s_i \in S$.

Denote by $U(S)$ the set of units in the subset S of A.

Definition (Group identities)

$U(S)$ is said to satisfy a group identity (GI) if there exists a nontrivial word $w(x_1, x_2, \cdots, x_n)$ in the free group generated by $x_1, x_2, \cdots, x_n$ such that $w(u_1, u_2, \cdots, u_n) = 1$ for all $u_1, u_2, \cdots, u_n \in U(S)$.

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symmetric elements and PI properties

Let $*$ be an involution on A. An element $x \in A$ is called symmetric (skew symmetric) with respect to $*$, if $x^* = x$ ($x^* = -x$). Denote A^+ and A^- the set of symmetric and skew symmetric elements of A, respectively.

Theorem (Amitsur, 1968)

Let A be an algebra with an involution $*$. A is PI if and only if A^+ (A^-) is PI.

Of course, the polynomial identity which is satisfied by the algebra is not necessarily the same as the one which is satisfied by the symmetric elements.

symmetric and skew-symmetric elements

It is well-known that group algebra FG is an algebra with involution.

Definition

The canonical involution of FG is defined by $x = \sum_{g \in G} \alpha_g g \to x^* = \sum_{g \in G} \alpha_g g^{-1}$.

Denote by $G_* = \{g \in G \mid g = g^*\}$ the symmetric elements of G. Then FG^+ is generated as an F-module by the set

$$\{g + g^* \mid g \in G, g \neq G_*\} \cup G_*$$

and FG^- is generated as an F-module by the set

$$\{g - g^* \mid g \in G\}.$$

Then FG^+ is a Jordan algebra and FG^- is a Lie algebra.

Nilpotency, Lie nilpotency

Definition (Lie nilpotency)

The subset $S \subseteq FG$ is Lie nilpotent, if for some $n \geq 2$, $[x_1, x_2, \cdots, x_n] = 0$ for all $x_i \in S$, where $[x_1, x_2] = x_1 x_2 - x_2 x_1$, and $[x_1, x_2, \cdots, x_n] = [[x_1, x_2, \cdots, x_{n-1}], x_n]$.
The smallest such n is called the Lie nilpotency index of S and is denoted by $t(S)$.

The corresponding group identity is the nilpotency.

Definition (nilpotency)

The subset $S \subseteq U(FG)$ is nilpotent, if for some $n \geq 2$, $(x_1, x_2, \cdots, x_n) = 1$ for all $x_i \in S$, where $(x_1, x_2) = x_1^{-1} x_2^{-1} x_1 x_2$, and $(x_1, x_2, \cdots, x_n) = ((x_1, x_2, \cdots, x_{n-1}), x_n)$.

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Commutativity of symmetric and skew-symmetric elements

Theorem (Broche, 2003)

Let G be a nonabelian group and let F be a commutative ring of characteristic different from 2. Then, FG^+ is a commutative ring if and only if G is a Hamiltonian 2-group.

Theorem (Broche, Polcino Milies, 2007)

Let R be a commutative ring with unity, char $R \neq 2, 4$ *and let G be a group. Then RG^- is commutative if and only if one of the following conditions holds:*

- *G is abelian;*
- *$A = \langle\, g \in G \,|\, |g| \neq 2 \,\rangle$ is a normal abelian subgroup of G;*
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Lie nilpotency

Theorem (Giambruno, Sehgal, 1993)

Let G be a group with no 2-elements and F a field with $\operatorname{char} F \neq 2$. Then FG^+ (FG^-) is Lie nilpotent if and only if FG is Lie nilpotent.

Theorem (Lee, 1999)

Suppose $Q_8 \not\subseteq G$ and $\operatorname{char} F = p \neq 2$. Then the following are equivalent:

- *FG is Lie nilpotent;*
- *FG^+ is Lie nilpotent;*
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Let F be a field $\operatorname{char} F \neq 2$, and let G be a group. Then FG^- is Lie nilpotent if and only if either

- *G has a nilpotent p-abelian normal subgroup H with $(G \setminus H)^2 = 1$;*
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Lie nilpotency indices

Theorem (Bovdi, Spinelli, 2004, Shalev, 1993 $p > 3$)

Let FG be Lie nilpotent. Then $t^L(FG) \leq |G'| + 1$ equality holds if and only if either G' is cyclic, or $p = 2$, G' is a noncentral elementary abelian group of order 4 and $\gamma_3(G) \neq 1$. Moreover if $t^L(FG) = |G'| + 1$ then $t(FG) = t^L(FG)$.

Theorem (Balogh, Juhász, 2010)

Let FG be a Lie nilpotent group algebra of odd characteristic. Then $t((FG)^+) = |G'| + 1$ if and only if G' is cyclic.

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Theorem (Balogh, Juhász, 2010)

Let FG be a group algebra of characteristic $p > 2$ such that $(FG)^+$ is Lie nilpotent but FG is not, and assume that the Sylow p-subgroup P of G is of order p^n with $n \geq 1$. Then

- $1 + n(p-1) \leq t((FG)^+) \leq t_N(P)$;
- *if P is a powerful group, then $t((FG)^+) = t_N(P)$;*
- *if P is abelian, then for all $k \geq 2$ the subspace $\gamma^k((FG)^+)$ is spanned by all elements of the form $(h_1 - h_1^{-1}) \cdots (h_k - h_k^{-1})(1 - a^2)a$, where $h_i \in P$ and a is a noncentral 2-element of G.*

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Commutativity under involutions of the first kind

Definition

Let φ be an involution of G. Then the F-linear extension of φ

$$x = \sum \alpha_g g \mapsto x^\varphi = \sum \alpha_g \varphi(g)$$

is an involution of FG.

φ is an involution of the first kind.

Theorem (Jespers, Ruiz Marin, 2005)

Let R be a commutative ring with $\operatorname{char} R \neq 2, 3, 4$. Suppose G is a non-abelian group and φ is an involution on G. Then RG_φ^- is commutative if and only if one of the following conditions holds:

- *$K = \langle\, g \in G | g \notin G_\varphi \,\rangle$ is an abelian subgroup of index 2 in G;*
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LC group

Definition (LC group)

Let G be a group and $Z(G)$ its center. G is said to be LC group (lack of commutativity) if any pair of elements $g, h \in G$, it is the case that $gh = hg$ if and only if $g \in Z(G)$ or $h \in Z(G)$ or $gh \in Z(G)$.

Theorem

Let G be a group and $Z(G)$ its center. G is LC group if and only if it is a finite 2-group such that $G/Z(G) \cong C_2 \times C_2$ and the derived subgroup $G' = \langle\, s \mid s^2 = 1 \,\rangle$.

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LC-involution

For an LC-group G we have an involution $\odot : G \to G$ defined by

$$g^{\odot} = \begin{cases} g, & \text{if } g \in Z(G); \\ gs, & \text{otherwise.} \end{cases}$$

Then $\odot$ is an anti-automorphism of order two. Thus the linear extension of this anti-automorphism into FG

$$x = \sum \alpha_g g \mapsto x^{\odot} = \sum \alpha_g g^{\odot}$$

is an involution.

Commutativity of RG_φ^+

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Let φ be an involution on a non-abelian group G and let R be a commutative ring with $\operatorname{char} R \neq 2$. *The following are equivalent:*

- *RG_φ^+ is commutative;*
- *The group G has the LC property, a unique nontrivial commutator s and the involution $\varphi = \odot$.*
- *$G/Z(G) \cong C_2 \times C_2$, $\varphi(g) = g$ if $g \in Z(G)$ and otherwise $\varphi(g) = h^{-1}gh$ for all $h \in G$ with $(g,h) \neq 1$.*

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Let φ be an involution on a non-abelian group G and let R be a commutative ring with $\operatorname{char} R \neq 2$. The following are equivalent:

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Theorem (Giambruno, Polcino Milies, Sehgal, 2009)

Let G be a group with no 2-elements and F a field of characteristic $p \neq 2$. Then, $(FG)^+$ is Lie nilpotent if and only if FG is Lie nilpotent.

Theorem (Lee, Sehgal, Spinelli, 2009)

*Let F be a field of characteristic $p > 2$, and let G be a group with involution *. Suppose that FG is not Lie nilpotent. Then FG^+ is Lie nilpotent if and only if G is nilpotent, and G has a finite normal *-invariant p-subgroup N such that G/N is an SLC-group.*

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Involutions of the second kind

Definition (oriented involution)

Define the map $\circledast : FG \to FG$ *by the following way. Let* $\sigma : G \to \{1, -1\}$ *a group homomorphism. Set*

$$x = \sum \alpha_g g \mapsto x^{\circledast} = \sum \alpha_g \sigma(g) g^{-1}.$$

Commutativity under oriented involutions

Theorem (Broche, Polcino Milies, 2004)

Let R be a commutative ring with unity and let G be a non-abelian group with involution φ and non-trivial orientation homomorphism σ with kernel N. Then RG^+ is a commutative ring if and only if one of the following conditions holds:

- *N is an abelian group and $G \setminus N \subset G_\varphi$;*
- *G and N have the LC property, and there exists a unique nontrivial commutator s such that the involution φ is given by*

$$\varphi(g) = \begin{cases} g & \text{if} \quad g \in N \cap Z(G) \text{ or } g \in (G \setminus N) \setminus Z(G). \\ sg & \text{otherwise.} \end{cases}$$

- char $R = 4$, *G has the LC property, and there exists a unique nontrivial commutator s such that the involution $\varphi = \odot$.*

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Let * be the canonical involution.

Theorem (Bovdi, Kovacs, Sehgal, 2003)

Let G be a locally finite non-abelian p-group and let R be a commutative ring of characteristic p. Then $U(FG^+)$ forms a multiplicative group if and only if $p = 2$, G is a direct product of an elementary abelian 2-group and a group H satisfying one of the following conditions:

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Nilpotent symmetric units

Theorem (Giambruno, Sehgal, Valenti, 1998)

Let F be a field of characteristic $p \neq 2$ and G a torsion group. Then $U(FG^+)$ is nilpotent if and only if FG^+ is Lie nilpotent.

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Let F be a field of characteristic $p \neq 2$ and G a torsion group. Suppose $Q_8 \not\subseteq G$. Then the following are equivalent:

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- *$p > 2$ and $G \cong Q_8 \times E \times P$ where E is an elementary abelian 2-group and P is a finite p-group;*
- *$p = 0$ and $G \cong Q_8 \times E$, where E is an elementary abelian 2-group.*

Nilpotent symmetric units

Theorem (Lee, 2003)

Let F be a field of characteristic $p \neq 2$ and G a torsion group. Suppose $Q_8 \not\subseteq G$. Then the following are equivalent:

- *$U(FG^+)$ is nilpotent;*
- *$U(FG)$ is nilpotent;*
- *G is nilpotent and p-abelian.*

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Nilpotency class of symmetric units

Theorem (Balogh, Juhász, 2010)

Let FG be a Lie nilpotent group algebra of odd characteristic. If G is a torsion group, then $\mathrm{cl}(U(FG^+)) = |G'|$ *if and only if G' is cyclic.*

Theorem (Balogh, Juhász, 2010)

Let FG be a group algebra of characteristic $p > 2$ such that FG^+ is Lie nilpotent but FG is not, and assume that the Sylow p-subgroup P of G is of order p^n with $n \geq 1$. If $t(FG^+) = t_N(P)$, then $\mathrm{cl}(U(FG^+)) = t(FG^+) - 1$.

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