Defect two blocks of symmetric groups over the p-adic integers

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Notation

- Σ_n: The symmetric group on n points
- $p \in \mathbb{Z}$ a prime
- $\mathbb{Z}_p$: The p-adic integers $\mathbb{Q}_p$: p-adic completion of $\mathbb{Q}$

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We are interested in defect two blocks of $\mathbb{Z}_p\Sigma_n$.
Examples of defect two blocks:
The principal blocks of $\mathbb{Z}_p\Sigma_n$ for $2 \cdot p \leqslant n \leqslant 3 \cdot p - 1$.

The problem

Basic orders/algebras

For a $\mathbb{Z}_p$-order Λ define its basic order Λ_0 as

$$\Lambda_0 := \mathrm{End}_\Lambda \left(\bigoplus_{S \text{ simple } \Lambda\text{-module}} \mathcal{P}(S) \right)$$

where $\mathcal{P}(S)$ denotes the projective cover of S.

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Blocks of defect one of $\mathbb{Z}_p\Sigma_n$

For each p there is (up to Morita-equivalence) only one block of defect one, and its basic order looks like this:

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as a $\mathbb{Z}_p$-order in the $\mathbb{Q}_p$-algebra $\mathbb{Q}_p \oplus \mathbb{Q}_p^{2\times 2} \oplus \ldots \oplus \mathbb{Q}_p^{2\times 2} \oplus \mathbb{Q}_p$.

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The question: What do the basic algebras for defect two blocks of $\mathbb{Z}_p\Sigma_n$ look like?

Properties of basic orders

Let Λ a $\mathbb{Z}_p$-order in a semisimple $\mathbb{Q}_p$-algebra $A := \mathbb{Q}_p \cdot \Lambda$, and Λ_0 its basic order.

- $A_0 := \mathbb{Q}_p \cdot \Lambda_0$ is semisimple as well and there is a canonical isomorphism

$$Z(A) \xrightarrow{\sim} Z(A_0)$$

  which restricts to an isomorphism $Z(\Lambda) \xrightarrow{\sim} Z(\Lambda_0)$. We will henceforth identify the centers.

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- If $\Lambda = \Lambda^{\sharp,u} := \{a \in A \mid \operatorname{Tr}(u \cdot a \cdot \Lambda) \subseteq \mathbb{Z}_p\}$ for some $u \in Z(A)$, then $\Lambda_0 = \Lambda_0^{\sharp,u}$ (for the same u).

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The latter justifies that we determine all $\varepsilon\Lambda_0$ (for all c.p.i.'s ε) first, and in a second step Λ_0 as a suborder of $\bigoplus_\varepsilon \varepsilon\Lambda_0$.

Some representation theory of Σ_n

- $P(n) := \{\lambda = (\lambda_1, \dots, \lambda_k) \mid \lambda_1 + \dots + \lambda_k = n\}$: partitions of n.
- $\lambda \in P(n)$ is called p-regular, if no p parts of λ are equal.
- $P(n)_{p\text{-reg}} := \{\lambda \in P(n) \mid \lambda \ p\text{-regular}\}$

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For each $\lambda \in P(n)$ there is a $\mathbb{Z}\Sigma_n$-lattice $S^\lambda_\mathbb{Z}$ called Specht module. For any ring R we define $S^\lambda_R := S^\lambda_\mathbb{Z} \otimes_\mathbb{Z} R \in \mathbf{mod}_{R\Sigma_n}$.

$$P(n) \leftrightarrow \{\text{ Abs. irr. } \mathbb{Q}\Sigma_n\text{-modules }\}: \ \lambda \mapsto S^\lambda_\mathbb{Q}$$

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Blocks of Σ_n

The p-blocks are parametrized by a so-called p-core (a partition) and a p-weight (a number). Defect two corresponds to weight 2 and $p > 2$.

The Jantzen-Schaper filtration

- There is a filtration
$$S^\lambda_{\mathbb{F}_p} = S^\lambda_0 \geqslant S^\lambda_1 \geqslant S^\lambda_2 \geqslant S^\lambda_3 = \{0\}$$
called the Jantzen-Schaper filtration.
- The subsets J_i of $P(n)_{p\text{-reg}}$ labeling the simple constituents of $S^\lambda_i / S^\lambda_{i+1}$ can be computed.
- The 0th layer is the top of $S^\lambda_{\mathbb{F}_p}$ (and simple) if λ is p-regular, otherwise it is zero.
- The 2nd layer is the socle of $S^\lambda_{\mathbb{F}_p}$ (and simple) if $\lambda^\top$ is p-regular, otherwise it is zero.

$\varepsilon^\lambda \Lambda_0$ as a graduated order

From now on: Λ a defect two block of $\mathbb{Z}_p\Sigma_n$, $A := \mathbb{Q}_p \otimes \Lambda$, $\varepsilon^\lambda \in Z(A)$ the c.p.i. corresponding to $S^\lambda_{\mathbb{Q}_p}$.

- Decomposition numbers: $d_{\lambda,\mu} := \left[\mathbb{Q}_p \otimes \mathcal{P}(D^\mu) : S^\lambda_{\mathbb{Q}_p}\right] \leqslant 1$ (according to Scopes). The decomposition numbers can be calculated.
- $c_\mu := \{\lambda \in P(n) \mid d_{\lambda,\mu} = 1\}$. The Cartan numbes: $|c_\mu \cap c_\nu| \leqslant 2$ for $\mu \neq \nu$ (also according to Scopes).
- $r_\lambda := \{\mu \in P(n)_{p\text{-reg}} \mid d_{\lambda,\mu} = 1\}$.

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$$\begin{tikzcd}
\varepsilon^\lambda \operatorname{End}_A\left(\bigoplus_\mu \mathbb{Q}_p \otimes \mathcal{P}(D^\mu)\right) \arrow[r, "\sim"] & \operatorname{End}_A\left(\bigoplus_{\mu \in r_\lambda} S^\lambda_{\mathbb{Q}_p}\right) \arrow[d, "\wr"] \\
\varepsilon^\lambda \operatorname{End}_\Lambda\left(\bigoplus_\mu \mathcal{P}(D^\mu)\right) \arrow[u, hook] \arrow[r, hook] & \mathbb{Q}_p^{r_\lambda \times r_\lambda}
\end{tikzcd}$$

$$\varepsilon^\lambda \pi_{\mathcal{P}(D^\mu)} \longmapsto e_{\mu\mu} := (\delta_{\mu\nu}\delta_{\mu\eta})_{\nu\eta}$$

We therefore identify $\varepsilon^\lambda \Lambda_0 \subset \mathbb{Q}_p^{r_\lambda \times r_\lambda}$. $\varepsilon^\lambda \Lambda_0$ contains all diagonal idempotents.

Exponent matrices

$$\varepsilon^{\lambda}\Lambda_0 = (p^{m_{\mu\nu}}\mathbb{Z}_p)_{\mu\nu} \subset \mathbb{Q}_p^{r_\lambda \times r_\lambda} \quad \text{for some matrix } m \in \mathbb{Z}^{r_\lambda \times r_\lambda} \text{ (exponent matrix)}$$

For example: $\varepsilon^{(4,2,1)}\mathbb{Z}_3\Sigma_7 \cong \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ (p) & \mathbb{Z}_p & (p) & \mathbb{Z}_p \\ (p) & (p) & \mathbb{Z}_p & \mathbb{Z}_p \\ (p^2) & (p) & (p) & \mathbb{Z}_p \end{pmatrix} \quad m = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{pmatrix}$

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- Now let λ be p-singular. Two cases are to be considered:
  - $\lambda^\top$ **p-regular:** The functor $- \otimes_{\mathcal{O}}$ sgn induces an equivalence between $\mathbf{mod}_{\varepsilon^\lambda\Lambda}$ and $\mathbf{mod}_{\varepsilon^{\lambda^\top}\Lambda}$.

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    ($\Longrightarrow$ reduction to λ p-regular).
  - $\lambda^\top$ **p-singular:** In this case r_λ has just one element. Hence $\varepsilon^{\lambda}\Lambda_0 \cong \mathbb{Z}_p^{1\times 1}$.

So all $\epsilon^{\lambda}\Lambda_0$ can be determined.

The embedding $\Lambda_0 \hookrightarrow \bigoplus_\lambda \varepsilon^\lambda \Lambda_0$

- Identify $\mathrm{End}_\Lambda(\bigoplus_\mu \mathcal{P}(D^\mu)) = \Lambda_0 \subset \bigoplus_\lambda \varepsilon^\lambda \Lambda_0 \subset \bigoplus_\lambda \mathbb{Q}_p^{r_\lambda \times r_\lambda}$.
- Define $e^\eta_{\mu\nu} \in \bigoplus_\lambda \mathbb{Q}_p^{r_\lambda \times r_\lambda}$ to be the element that has a 1 in the η-component at position (μ, ν) (and 0's everywhere else).

We have

$$\pi_{\mathcal{P}(D^\mu)} = \sum_{\lambda \in c_\mu} e^\lambda_{\mu\mu}$$

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Remark

The elements in $\Lambda_{\mu\mu}$ will not be needed to generate the order Λ_0
($Ext^1_{\mathbb{F}_p \Sigma_n}(D^\mu, D^\mu) = \{0\}$ all D^μ in a defect two block of Σ_n)
That is, we only have to determine $\Lambda_{\mu\nu}$ with $\mu \neq \nu$.

Using selfduality

Λ_0 being selfdual implies the following:

- In the case $m^{\lambda}_{\mu\nu} + m^{\lambda}_{\nu\mu} = 2$ for all $\lambda \in c_\mu \cap c_\nu$:

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- In the case $m_{\mu\nu}^{\lambda} + m_{\nu\mu}^{\lambda} = 1$ für alle $\lambda \in c_\mu \cap c_\nu$:

$$\Lambda_{\mu\nu} = \left\langle \begin{pmatrix} \alpha_{\mu\nu}^{\eta} \cdot p^{m_{\mu\nu}^{\eta}} & p^{m_{\mu\nu}^{\lambda}} \\ 0 & p^{m_{\mu\nu}^{\lambda}+1} \end{pmatrix} \cdot \begin{pmatrix} e_{\mu\nu}^{\eta} \\ e_{\mu\nu}^{\lambda} \end{pmatrix} \right\rangle_{\mathbb{Z}_p} \quad \text{wo } c_\mu \cap c_\nu = \{\eta, \lambda\}$$

for certain parmeters $\alpha_{\mu\nu}^{\eta} \in \mathbb{Z}_p^{\times}$.

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Remark

The selfduality of Λ_0 also implies

$$\alpha^\eta_{\nu\mu} = -(\alpha^\eta_{\mu\nu})^{-1} \cdot \frac{\dim S^\lambda}{\dim S^\eta}$$

Now these parameters have to be eliminated by conjugation!.

The Ext-quiver (Part I)

Definition

For a $\mathbb{Z}_p$-order Γ its Ext-quiver is (in our case) defined as the following undirected graph:

- vertices $\leftrightarrow$ simple Γ-modules
- $\#$ edges $S - T = \dim_{\mathbb{F}_p} \mathrm{Ext}^1_{\mathbb{F}_p \otimes \Gamma}(S, T)$ $(= \dim_{\mathbb{F}_p} \mathrm{Ext}^1_{\mathbb{F}_p \otimes \Gamma}(T, S))$

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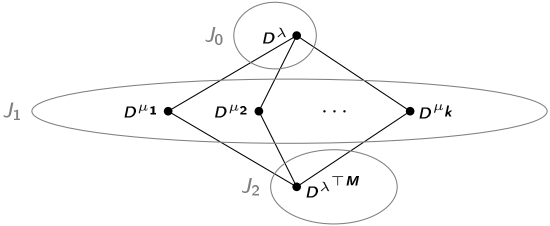
We look at the Ext-quiver of $\varepsilon^\lambda \Lambda_0$ *(λ and $\lambda^\top$ p-regular)*:



The Ext-quiver of $\varepsilon^\lambda \Lambda_0$ is a maximally bipartite graph.

The Ext-quiver (Part II)

- The Ext-quiver of any $\varepsilon^\lambda\Lambda_0$ is a maximally bipartite graph.
- **Known:** The Ext-quiver of Λ_0 is a bipartite graph.
- The epimorphisms $\Lambda_0 \twoheadrightarrow \varepsilon^\lambda\Lambda_0$ yield that the Ext-quivers of $\varepsilon^\lambda\Lambda_0$ are subquivers of the Ext-quiver of Λ_0.

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For μ, ν with $[S^\lambda : D^\mu] \neq 0$ and $[S^\lambda : D^\nu] \neq 0$ the following holds:

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The Ext-quiver (Part II)

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Remark

Only those $\Lambda_{\mu\nu}$ with $\mathrm{Ext}^1_{\mathbb{F}_p \otimes \Lambda_0}(D^\mu, D^\nu) \neq 0$ are needed to generate Λ_0 as a $\mathbb{Z}_p$-algebra (i. e. only the corresponding $\alpha^\lambda_{\mu\nu}$ need to be determined).

Elimination of parameters

Theorem

Let $\mu > \nu \in \mathcal{P}(n)_{p\text{-reg}}$ be partitions in a defect two block. Then

$$\mathrm{Ext}^1_{\mathbb{F}_p \otimes \Lambda_0}(D^\mu, D^\nu) \neq 0 \wedge |c_\mu \cap c_\nu| = 2 \Longrightarrow \nu \in c_\mu \cap c_\nu$$

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Corollary

W.l.o.g. we only have parameters $\alpha^\nu_{\mu\nu}$ with $\mu > \nu$. By successive conjugation these can all be eliminated (i. e., set to be $= 1$).

Together with what we have already seen before, this Corollary completely determines the basic orders of defect two blocks of $\mathbb{Z}_p \Sigma_n$.