$\mathbb{Q}$-forms and the theory of central simple G-algebras

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Consider $G = Q_8$ and choose $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = 2$.

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Observations:

- Ring of polynomial invariants has a generating set of rational polynomials although the representation is not rational
- For all $g \in G$ applying the GALOIS automorphism $I \mapsto -I$ to $\Delta(g)$ is afforded by conjugation with $\Delta(b) \rightsquigarrow \mathrm{Gal}(\mathbb{Q}[i]/\mathbb{Q})$ acts on $\Delta(G)$

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- $\mathbb{Q}[\zeta_7][x]^G$ is generated by: $x_1{x_2}^3 + x_2{x_3}^3 + x_3{x_1}^3, 270\, {x_3}^2{x_1}^2{x_2}^2 - 54\, {x_3}^5x_1 - 54\, {x_2}^5x_3 - 54\, {x_1}^5x_2, f_3, f_4$ where $f_3, f_4 \in \mathbb{Q}[x] \rightsquigarrow$ Generating set of rational invariants!

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Use this observation to give a precise formulation of this phenomena

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- Let G be a finite group and $\chi \in \mathrm{Irr}(G)$ faithful

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Definition of a $K/\mathbb{Q}$-form:

Define Δ to be a $K/\mathbb{Q}$-form if there exists $U \leq \mathrm{Aut}(G)$ and an isomorphism $^{-} : U \to \Gamma$ such that

$$\Delta(u(g)) = \overline{u}(\Delta(g)) \text{ for every } g \in G$$

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- Trivial Example: A representation $\Delta : G \to \mathrm{GL}_n(\mathbb{Q})$ is a $\mathbb{Q}/\mathbb{Q}$-form with $U = \langle 1 \rangle$.

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Comparison of $K/\mathbb{Q}$-forms:

Let Δ and Θ be $K/\mathbb{Q}$-forms then: $\Delta \sim \Theta$ if and only if there exists $Y \in \mathrm{GL}_n(\mathbb{Q})$ such that $Y\Delta(g)Y^{-1} = \Theta(g)$ for all $g \in G$

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Connection with invariant theory:

Theorem:

$\Delta : G \to \mathrm{GL}_n(K)$ is a $K/\mathbb{Q}$-form if and only if $K[x]^G$ is generated by polynomials with rational coefficients.

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Main question:

Given $\chi \in \mathrm{Irr}(G)$ and $U \leq \mathrm{Aut}(G)$ is there a GALOIS extension K of $\mathbb{Q}$ such that there exists a $K/\mathbb{Q}$-form Δ with this U?

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- U acts transitively on $\{\chi^\sigma \mid \sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})\}$ and $\overline{u} \circ \chi = \chi \circ u$ for all $u \in U$.

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We get

Theorem:

There exists a $K/\mathbb{Q}$-form if and only if $\lambda \sim 1 \in H^2(\Gamma, K^*)$.

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Let $e = \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma \circ e_\chi \in \mathbb{Q}G$ where e_χ is the central primitive idempotent of $\mathbb{C}[G]$ corresponding to χ. Consider $A := e\mathbb{Q}Ge$ as a simple $\mathbb{Q}$-algebra then:

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- U acts on $\mathbb{Q}G$ as automorphisms by the $\mathbb{Q}$-linear extension of ${}^u g := u(g)$
- U fixes e and so U acts on A as automorphisms
- For any representation $\Delta : G \to \mathrm{GL}_n(K)$ affording χ the action on A induces an action of U on $\Delta(\mathbb{Q}G)$

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Idea:

- Considering A as an element of the BRAUER group gives information about rationality questions of χ
- Hope: Find a generalization of the BRAUER group which takes a group action into account and maybe answers the ”rationality” question we are interested in.

G-algebras

The following definitions and theorems are due to TURULL

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- A G**-algebra** A is a finite dimensional associative k-algebra where G acts as automorphisms

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- We call two G-algebras A and B **equivalent** if there exists trivial G-algebras E_1 and E_2 such that: $A \otimes_k E_1 \cong_G B \otimes_k E_2$. We simply write $A \sim_G B$

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Theorem: [Turull 09]

Let F be a G-field with $F^G = k$. We define $\mathrm{BrCliff}(G, F)$ as set of all equivalence classes of central simple G-algebras such that the centres are G-isomorphic to F. Then $\mathrm{BrCliff}(G, F)$ is an abelian group with the following group structure:

$$\mathrm{BrCliff}(G, F) \times \mathrm{BrCliff}(G, F) \to \mathrm{BrCliff}(G, F)$$

$$([A], [B]) \mapsto [A \otimes_F B]$$

where G acts diagonally on the tensor product

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Let G be a finite group, $\chi \in \mathrm{Irr}(G)$ faithful with SCHUR index one over $\mathbb{Q}$. Given $U \leq \mathrm{Aut}(G)$ and a GALOIS field K such that the previous conditions are fulfilled, then there exists a $K/\mathbb{Q}$-form if and only if $[A] = [e\mathbb{Q}Ge] = [1] \in \mathrm{BrCliff}(U, \mathbb{Q}(\chi))$.

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From this we directly get the following corollary.

Corollary:

With the assumptions of the last theorem: There exists a $K/\mathbb{Q}$-form if and only if the irreducible $\mathbb{Q}G$ module M corresponding to the character $\sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} \sigma \circ \chi$ extends to an irreducible $\mathbb{Q}G \rtimes U$-module.

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- Find and implement algorithms in MAGMA or GAP