Burnside rings and arithmetical properties of finite simple groups

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Notations

G	finite group
$\Omega(G)$	Burnside ring of G
$M(G)$	table of marks of G
$\pi_e(G)$	the set of element orders of G called the spectrum of G
$\mathbb{C}G$	complex group algebra of G

Throughout we use freely the classification of the finite simple groups.

Arithmetical Properties of G

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It is a natural and old question of group theory to ask which properties of G are reflected by such an arithmetical property.

On the other hand, it is a fundamental question of representation theory to ask which properties of a group G are determined by its representations.

Questions

A typical question is which properties of a finite group are reflected by its Burnside ring $\Omega(G)$? This talk deals with the following two special cases.

- **Question 1.** Is a finite simple group G determined up to isomorphism by its Burnside ring $\Omega(G)$?

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A typical question is which properties of a finite group are reflected by its Burnside ring $\Omega(G)$? This talk deals with the following two special cases.

- **Question 1.** Is a finite simple group G determined up to isomorphism by its Burnside ring $\Omega(G)$?
- **Question 2.** More generally, are the composition factors of a finite group G determined by $\Omega(G)$?

Definition of the table of marks $M(G)$

Denote by $(G/V_i)^{V_j}$ the number of fixpoints of V_j on G/V_i.

The $m \times m$ -matrix $M(G) = (m_{ij})$ with $m_{ij} = (G/V_i)^{V_j}$ is called the table of marks.

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Note

$$m_{ij} = \frac{(\# V_j^g \leq V_i) \cdot |N_G(V_j)|}{|V_i|}$$

Example of a table of marks

$M(D_4)$

	D_4	$V_{4,1}$	C_4	$V_{4,2}$	$C_{2,1}$	Z	$C_{2,2}$	1
D_4/D_4	1	1	1	1	1	1	1	1
$D_4/V_{4,1}$	.	2	.	.	2	2	.	2
D_4/C_4	.	.	2	.	2	2	.	2
$D_4/V_{4,2}$	.	.	.	2	.	2	2	2
$D_4/C_{2,1}$	.	.	.	.	2	.	.	4
D_4/Z	.	.	.	.	.	4	.	4
$D_4/C_{2,2}$	.	.	.	.	.	.	2	4
$D_4/1$	.	.	.	.	.		.	8

Definition of $\Omega(G)$

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$\Omega(G)$ is a $\mathbb{Z}$ - order in $\mathbb{Q}^m$.

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- Hamiltonian Hall subgroups are isomorphic. (K 1989)

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Consequently

$$M(G) = M(H) \implies \Omega(G) \cong \Omega(H)$$

The converse is unknown.
It is known that the table of marks is not invariant under normalized automorphisms.

Structure of Burnside rings

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Block description of Burnside rings

$$\Omega(G) \;=\; \Omega(G)^s \times \Omega(N_G(P_2)/P_2)^s \times \ldots \times \Omega(N_G(P_k)/P_k)^s$$

where $P_1, P_2, \ldots, P_k$ are the representatives of the conjugacy classes of perfect subgroups of G with $P_1 = 1$.

Let X be a finite group. Then $\Omega(X)^s$ denotes the ideal generated by all X/H with H soluble.
$\Omega(X)^s$ is called the principal block of the Burnside ring $\Omega(X)$.

Reflected Properties by Burnside rings I

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- A hamiltonian group G is determined by $\Omega(G)$. (Raggi, Valero 2005) $\Omega(G)$.
- G is soluble if, and only if, $\Omega(G)$ has no non-trivial idempotents.
- The lattice of normal subgroups is determined provided G is soluble.

Immediate results on Questions 1 and 2

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- Question 2 has a positive answer provided G is a minimal simple group
- or more generally provided G is an automorphism group of a minimal simple group S containing $S = InnS$.

The answer to Question 2

Theorem

Let G and H be finite groups. Assume that G is simple. Then

$$\Omega(G) \cong \Omega(H) \Longrightarrow G \cong H$$

The object of the 2nd part of the talk is to present two approaches - using different arithmetical properties - for a proof of this result.

The answer to Question 2

Theorem

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The first approach relies on joint work with G.Raggi and F.Luca.

Normalized isomorphisms

1) (Raggi - Valero 2004 based on Nicolson 1978 and Ki.-Roggenkamp 1995)
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$$\sigma(G/1) = H/1.$$

2) If σ is a normalized isomorphism, then for each soluble subgroup U of G

$$\sigma(G/U) = H/U_* + \sum z_i H/V_i, \;\; z_i \in \mathbb{Z}$$

with $|V_i| < |U_*|$.

Quasiidempotents

$\Omega(G)$ has finite index in $\mathbb{Q}^m$. Therefore for each primitive idempotent e_U of $\mathbb{Q}^m$ there exists a smallest $m_u \in \mathbb{N}$ such that

$$\underbrace{m_u \cdot e_u}_{:=q_U} \in \Omega(G).$$

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It follows that $\Omega(G)$ determines $|G|$.

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- Therefore the Sylow numbers of G and H coincide.
- U cyclic $\Longrightarrow$ U_* cyclic
- It follows that G and H have the same spectrum, i.e. $\pi_e(G) = \pi_e(H)$

Shi's conjecture

(Conjecture Wujie Shi 1987, Kourovka Notebook 12.39)

Let G be a finite group and let S be a finite simple group. Assume that $\pi_e(S) = \pi_e(G)$ and that $|S| = |G|$. Then $G \cong S$.

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(iv) G is a simple group of Lie type not isomorphic to a symplectic group or an orthogonal group of order $\geq 10^8$.

(v) G is a simple group of Lie type of type ${}^2D_n(q) (n \geq 4)$ or $D_n(q)$ (n odd, $n \geq 5$).

Preliminary result

Corollary.

Question 1 has an affirmative answer for each simple group except possibly for some series of orthogonal and symplectic groups

Sylow numbers

Wolfgang Kimmerle

Introduction
Notations

Arithmetical Properties

Objects of the talk

Burnside rings
Reflected Properties by M(G)

Some results on $\Omega(G)$
The Result
Spectra
Ingredients for a proof

2nd approach
Prime graph
Back to spectra

Xianhua Li, Thesis Potsdam 2008

Let G and H be finite groups. Assume that G is simple and that G and H have the same Sylow numbers and the same order. Then

$$G \cong H \quad \text{or} \quad \{G, H\} = \{B_n(q), C_n(q)\}$$

Centralizers of Involutions

Now $\Omega(G) \cong \Omega(H) \Longrightarrow |G| = |H|$ and that the Sylow numbers coincide. Using Xianhua Li's result it remains to consider the case $B_n(q)$ versus $C_n(q)$. This is settled via centralizers of involutions.

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Because

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for each soluble element U of $V(G)$, it follows that the class lengths of involutions of H and G coincide. In particular the number of involutions of H and G coincides.

But $B_n(q)$ and $C_n(q)$ have different numbers of involutions.

This completes the second approach to the theorem establishing a complete proof.

Gruenberg - Kegel graph

A main tool for the proof of the conjecture of Wujie Shi for many simple groups is the Gruenberg - Kegel graph $\Gamma(G)$ of a finite group (also called the prime graph of G).

This is the graph whose vertices are the primes p dividing $|G|$. Two different vertices p and q are joined by an edge provided there is a group element of order pq.

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K. W. Gruenberg and O. Kegel have shown that groups with disconnected prime graph have a very restricted structure. Using the classification of the finite simple groups, J. S. Williams proved that G has a nilpotent $\pi-$ Hall subgroup provided the primes of π form a connecting component of $\Gamma(G)$ consisting of odd primes.

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Because groups with isomorphic Burnside rings have isomorphic sublattice of nilpotent groups this provides in contrast to the use of solely the spectrum an easier and more direct way for such groups to decide whether they are determined by their Burnside ring.

The connecting components of finite simple groups with disconnected Gruenberg - Kegel graph have been determined by J.S.Williams and A.S.Kondratiev.

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Note. A finite group has at most 6 prime graph components. A finite soluble group has at most 2 prime graph components.

C.Höfert, Thesis Stuttgart 2008

Let G and H be finite groups. Assume that the Gruenberg-Kegel graph $\Gamma(G)$ has at least three components. Then $\Omega(G) \cong \Omega(H)$ implies that G and H have the same composition factors.

Shi's Conjecture II

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Shi's Conjecture II

- Grichkoseeva, Mazurov, Vasiliev completed independently of Li the proof of Shi's conjecture (announced Groups St.Andrews 2009 in Bath)
- Xianhua Li gave 2008 a sketch of a unified proof of Shi' s conjecture
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Thank you for your attention .