The Ring of Monomial Representations

Burkhard Külshammer

Mathematical Institute
Friedrich Schiller University Jena, Germany

Arithmetic of Group Rings and Related Objects

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References

1. L. Barker, Fibred permutation sets and the idempotents and units of monomial Burnside rings, *J. Algebra* **281** (2004), 535-566
2. R. Boltje, A canonical Brauer induction formula, *Astérisque* **181-182** (1990), 31-59
3. A. Dress, The ring of monomial representations I. Structure theory, *J. Algebra* **18** (1971), 137-157
4. B. Fotsing and B. Külshammer, Modular species and prime ideals for the ring of monomial representations of a finite group, *Comm. Algebra* **33** (2005), 3667-3677
5. M. Müller, Zum Isomorphieproblem von Darstellungsringen, Ph. D. thesis, Jena 2008
6. V. P. Snaith, Explicit Brauer induction. With applications to algebra and number theory, Cambridge University Press, Cambridge 1994

The monomial category

G finite group

$\mathrm{Irr}(G)$ set of irreducible (complex) characters of G

$\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ character group of G

${}_{\mathbb{C}G}\textbf{mod}$ category of finitely generated $\mathbb{C}G$-modules

${}_{\mathbb{C}G}\textbf{mon}$ monomial category

Objects:
pairs $(V, \mathcal{L})$ where $V \in {}_{\mathbb{C}G}\textbf{mod}$ and $\mathcal{L}$ is a set of 1-dimensional subspaces (lines) of V such that $V = \bigoplus_{L \in \mathcal{L}} L$ and $gL \in \mathcal{L}$ for all $g \in G$, $L \in \mathcal{L}$.

Morphisms:
a morphism $f : (V, \mathcal{L}) \longrightarrow (W, \mathcal{M})$ is a homomorphism of $\mathbb{C}G$-modules $f : V \longrightarrow W$ such that, for $L \in \mathcal{L}$, there exists $M \in \mathcal{M}$ with $f(L) \subseteq M$.

Remarks:
(i) One should think of objects in ${}_{\mathbb{C}G}$**mon** as $\mathbb{C}G$-modules with additional structure.
(ii) Sometimes it is better to work with more general types of morphisms; however, this will not be important here.
(iii) The monomial category is not abelian (not even additive).

Example:
Every $\phi \in \widehat{G}$ gives rise to an object $(\mathbb{C}_\phi, \{\mathbb{C}_\phi\})$ in ${}_{\mathbb{C}G}$**mon** where $\mathbb{C}_\phi := \mathbb{C}$ and $gz := \phi(g)z$ for all $g \in G$, $z \in \mathbb{C}$.

Operations in the monomial category

Direct sum:

$$(V,\mathcal{L}) \oplus (W,\mathcal{M}) := (V \oplus W, \mathcal{L} \cup \mathcal{M})$$

An object in the monomial category is called indecomposable if it is non-zero and not isomorphic to an object of the form $(V,\mathcal{L}) \oplus (W,\mathcal{M})$ where $V \neq 0 \neq W$.

Tensor product:

$$(V,\mathcal{L}) \otimes (W,\mathcal{M}) := (V \otimes_{\mathbb{C}} W, \{L \otimes_{\mathbb{C}} M : L \in \mathcal{L},\ M \in \mathcal{M}\})$$

Some functors

${}_G\mathbf{set}$ category of finite G-sets

There is functor

$$ {}_G\mathbf{set} \longrightarrow {}_{\mathbb{C}G}\mathbf{mon} $$

sending each finite G-set Ω to the pair $(\mathbb{C}\Omega, \{\mathbb{C}\omega : \omega \in \Omega\})$. There is also a forgetful functor

$$ {}_{\mathbb{C}G}\mathbf{mon} \longrightarrow {}_{\mathbb{C}G}\mathbf{mod} $$

forgetting about the additional structure.

H subgroup of G

Then there is a restriction functor

$$\mathrm{Res}_H^G : {}_{\mathbb{C}G}\mathbf{mon} \longrightarrow {}_{\mathbb{C}H}\mathbf{mon}$$

defined in the obvious way. On the other hand, there is an induction functor

$$\mathrm{Ind}_H^G : {}_{\mathbb{C}H}\mathbf{mon} \longrightarrow {}_{\mathbb{C}G}\mathbf{mon}$$

sending an object $(M, \mathcal{M}) \in {}_{\mathbb{C}H}\mathbf{mon}$ to $(\mathbb{C}G \otimes_{\mathbb{C}H} W, \{g \otimes M : g \in G,\ M \in \mathcal{M}\})$.

In particular, every $\phi \in \widehat{H}$ gives rise to an object $\mathrm{Ind}_H^G(\mathbb{C}_\phi, \{\mathbb{C}_\phi\}) \in {}_{\mathbb{C}G}\mathbf{mon}$. These objects will be very important later.

Example:

Let $G = Q_8$ be a quaternion group of order 8. Then G has 3 maximal subgroups H_1, H_2, H_3, all cyclic of order 4. For $i = 1, 2, 3$, let $\phi_i \in \widehat{H_i}$ be a monomorphism. Then $(\mathbb{C}_{\phi_i}, \{\mathbb{C}_{\phi_i}\}) \in {}_{\mathbb{C}H_i}\mathbf{mon}$, and $\mathrm{Ind}_{H_i}^{G}(\mathbb{C}_{\phi_i}, \{\mathbb{C}_{\phi_i}\}) \in {}_{\mathbb{C}G}\mathbf{mon}$. These three objects in ${}_{\mathbb{C}G}\mathbf{mon}$ are pairwise non-isomorphic, although the underlying modules $\mathrm{Ind}_{H_i}^{G}(\mathbb{C}_{\phi_i})$ are all isomorphic; in fact, they are isomorphic to the unique irreducible $\mathbb{C}G$-module of dimension 2.

Classification of indecomposables

Theorem

(i) For $H \leq G$ and $\phi \in \widehat{H}$,

$$\mathrm{Ind}_H^G(\mathbb{C}_\phi, \{\mathbb{C}_\phi\})$$

is an indecomposable object in ${}_{\mathbb{C}G}\mathbf{mon}$.
(ii) Every indecomposable object in ${}_{\mathbb{C}G}\mathbf{mon}$ arises in this way, up to isomorphism.
(iii) For $H, K \leq G$ and $\phi \in \widehat{H}$, $\psi \in \widehat{K}$, we have

$$\mathrm{Ind}_H^G(\mathbb{C}_\phi, \{\mathbb{C}_\phi\}) \cong \mathrm{Ind}_K^G(\mathbb{C}_\psi, \{\mathbb{C}_\psi\})$$

iff the pairs (H, ϕ) and (K, ψ) are conjugate in G.

The poset $\mathcal{M}(G)$

In the theorem above, the conjugation action on the set

$$\mathcal{M}(G) := \{(H, \phi) : H \leq G,\ \phi \in \widehat{H}\}$$

of monomial pairs is defined by

$${}^{g}(H, \phi) := ({}^{g}H, {}^{g}\phi)$$

where ${}^{g}H := gHg^{-1}$ and $({}^{g}\phi)({}^{g}h) := \phi(h)$ for all $h \in H$. We denote the stabilizer of (H, ϕ) under the conjugation action by $N_G(H, \phi)$, a subgroup of $N_G(H)$.

$\mathcal{M}(G)$ becomes a poset (partially ordered set) where

$$(K, \psi) \leq (H, \phi) :\Longleftrightarrow K \leq H \quad \text{and} \quad \phi|K = \psi.$$

Note that the conjugation action is compatible with the partial order, so that $\mathcal{M}(G)$ becomes a G-poset. We denote by $\mathcal{M}(G)/G$ the set of G-orbits $[H, \phi]_G$.

Quaternion and dihedral group

$G = Q_8$:

G : gives 4 elements in $\mathcal{M}(G)/G$
3 maximal subgroups: each gives 3 elements in $\mathcal{M}(G)/G$
1 subgroup of order 2: gives 2 elements in $\mathcal{M}(G)/G$
1 subgroup of order 1: gives 1 element in $\mathcal{M}(G)/G$

Thus ${}_{\mathbb{C}G}$**mon** has 16 indecomposable objects, up to isomorphism.

$G = D_8$:

G : gives 4 elements in $\mathcal{M}(G)/G$
3 maximal subgroups: each gives 3 elements in $\mathcal{M}(G)/G$
3 conjugacy classes of subgroups of order 2: each gives 2 elements in $\mathcal{M}(G)/G$
1 subgroup of order 1: gives 1 element in $\mathcal{M}(G)/G$

Thus ${}_{\mathbb{C}G}$**mon** has 20 indecomposable objects, up to isomorphism.

The monomial ring

The monomial ring $D(G)$ is the Grothendieck ring of the category ${}_{\mathbb{C}G}\mathbf{mon}$. Addition comes from direct sums, and multiplication comes from tensor products. Then $D(G)$ is a free $\mathbb{Z}$-module; a basis is given by the elements

$$\mathrm{Ind}_H^G(\mathbb{C}_\phi, \{\mathbb{C}_\phi\})$$

where (H, ϕ) ranges over a set of representatives for $\mathcal{M}(G)/G$.

The identity element of $D(G)$ is $(\mathbb{C}, \{\mathbb{C}\})$ where $\mathbb{C}$ denotes the trivial $\mathbb{C}G$-module. Moreover, we have

$$\mathrm{Ind}_H^G(\mathbb{C}_\phi, \{\mathbb{C}_\phi\}) \cdot \mathrm{Ind}_K^G(\mathbb{C}_\psi, \{\mathbb{C}_\psi\})$$
$$= \sum_{HgK \in H\backslash G/K} \mathrm{Ind}_{H\cap gKg^{-1}}^G(\mathbb{C}_{\phi\cdot {}^g\psi}, \{\mathbb{C}_{\phi\cdot {}^g\psi}\})$$

Connection with other representation rings

The functors

$$ {}_G\mathbf{set} \longrightarrow {}_{\mathbb{C}G}\mathbf{mon} \longrightarrow {}_{\mathbb{C}G}\mathbf{mod} $$

induce ring homomorphisms between the relevant Grothendieck rings:

$$ B(G) \longrightarrow D(G) \longrightarrow R(G). $$

Here $B(G)$ denotes the Burnside ring of G, and $R(G)$ denotes the character ring of G.

By Brauer's Induction Theorem, the ring homomorphism $b_G : D(G) \longrightarrow R(G)$ is surjective.

Restriction and induction

Let $H \leq G$. Then the restriction functor

$$\mathrm{Res}_H^G : {}_{\mathbb{C}G}\mathbf{mon} \longrightarrow {}_{\mathbb{C}H}\mathbf{mon}$$

induces a ring homomorphism

$$\mathrm{res}_H^G : D(G) \longrightarrow D(H).$$

On the other hand, the induction functor

$$\mathrm{Ind}_H^G : {}_{\mathbb{C}H}\mathbf{mon} \longrightarrow {}_{\mathbb{C}G}\mathbf{mon}$$

induces a homomorphism of groups

$$\mathrm{ind}_H^G : D(H) \longrightarrow D(G)$$

whose image is an ideal in $D(G)$. Usually, this is not a ring homomorphism.

Theorem. (Boltje 1989)

There are unique group homomorphisms $a_G : R(G) \longrightarrow D(G)$ such that

- $a_G(\chi) = (\mathbb{C}_\chi, \{\mathbb{C}_\chi\})$ for $\chi \in \widehat{G}$,
- $a_G(\chi)$ does not involve any $(\mathbb{C}_\phi, \{\mathbb{C}_\phi\})$, for $\chi, \phi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1 = \phi(1)$, and
- $\mathrm{res}^G_H \circ a_G = a_H \circ \mathrm{res}^G_H : R(G) \longrightarrow D(H)$ for all subgroups $H \leq G$.

Moreover, we have $b_G \circ a_G = \mathrm{id}_{R(G)}$ where $b_G : D(G) \longrightarrow R(G)$ is canonical.

This means that $a_G(\chi)$ gives a canonical way to write χ as a sum of monomial characters.

Explicit formula

The map $a_G : R(G) \longrightarrow D(G)$ satisfies, for $\chi \in \mathrm{Irr}(G)$:

$$a_G(\chi) = \frac{1}{|G|} \sum_{(K,\psi)\leq(H,\phi) \text{ in } \mathcal{M}(G)} |K|\mu_{(K,\psi),(H,\phi)}(\phi, \chi|H)\mathrm{Ind}_K^G(\mathbb{C}_\psi, \{\mathbb{C}_\psi\}).$$

Here μ denotes the Möbius function of the poset $\mathcal{M}(G)$.

For a finite poset $(P, \leq)$, the Möbius function is defined by $\mu_{xx} = 1$ and $\sum_{x \leq z \leq y} \mu_{xz} = 0$ for different $x, y \in P$.

It is not obvious that the coefficients in the formula are integers; but this can be proved.

Example:

Let $G = Q_8$ and $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 2$. Then

$$\begin{aligned} a_G(\chi) &= \mathrm{Ind}_{H_1}^{G}(\mathbb{C}_{\phi_1}, \{\mathbb{C}_{\phi_1}\}) + \mathrm{Ind}_{H_2}^{G}(\mathbb{C}_{\phi_2}, \{\mathbb{C}_{\phi_2}\}) \\ &\quad + \mathrm{Ind}_{H_3}^{G}(\mathbb{C}_{\phi_3}, \{\mathbb{C}_{\phi_3}\}) - \mathrm{Ind}_{Z(G)}^{G}(\mathbb{C}_{\phi}, \{\mathbb{C}_{\phi}\}) \end{aligned}$$

where H_1, H_2, H_3 are the maximal subgroups of G and ϕ_1, ϕ_2, ϕ_3, ϕ are all injective.

Species

Recall that $D(G)$ is a free $\mathbb{Z}$-module of rank $|\mathcal{M}(G)/G|$ and a ring, i. e. a $\mathbb{Z}$-order. Thus $\mathbb{C}D(G) := \mathbb{C} \otimes_{\mathbb{Z}} D(G)$ is a commutative $\mathbb{C}$-algebra of dimension $|\mathcal{M}(G)/G|$.

A $\mathbb{C}$-algebra homomorphism $s : \mathbb{C}D(G) \longrightarrow \mathbb{C}$ is called a species of $\mathbb{C}D(G)$.

Let us determine all species of $\mathbb{C}D(G)$.

Example:

For $H \leq G$ and $h \in H$, we get a species

$$s_{(H,hH')} : D(G) \longrightarrow D(H) \longrightarrow R(H/H') \longrightarrow \mathbb{C}$$

where the map $D(G) \longrightarrow D(H)$ is given by restriction, the map $\pi_H : D(H) \longrightarrow R(H/H')$ is linear with $\pi_H(\mathrm{Ind}_K^H(\mathbb{C}_\psi, \{\mathbb{C}_\psi\})) = \psi$ whenever $K = H$, and $\pi_H(\mathrm{Ind}_K^H(\mathbb{C}_\psi, \{\mathbb{C}_\psi\})) = 0$ if $K < H$. Finally, $t_g : R(G) \longrightarrow \mathbb{C}$ is defined by $t_g(\chi) := \chi(g)$ for $\chi \in \mathrm{Irr}(G)$.

Theorem.

(i) Every species of $\mathbb{C}D(G)$ arises in the way described above.
(ii) For $H, K \leq G$ and $h \in H$, $k \in K$, we have $s_{(H,hH')} = s_{(K,kK')}$ iff (H, hH') and (K, kK') are G-conjugate.

Here the conjugation action of G on the set

$$\mathcal{D}(G) := \{(H, hH') : H \leq G,\ h \in H\}$$

is defined by ${}^{g}(H, hH') := ({}^{g}H, {}^{g}h{}^{g}H')$ for $g \in G$ and $(H, hH') \in \mathcal{D}(G)$. We denote the set of orbits $[H, hH']_G$ by $\mathcal{D}(G)/G$.

It is important to observe that

$$|\mathcal{M}(G)/G| = |\mathcal{D}(G)/G|.$$

Thus the species give rise to an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}D(G) \cong \prod_{[H,hH']_G \in \mathcal{D}(G)/G} \mathbb{C}.$$

Hence $\mathbb{C}D(G)$ is a semisimple $\mathbb{C}$-algebra; in particular, 0 is the only nilpotent element in $\mathbb{C}D(G)$. Moreover, the species are essentially the projections on the various factors in the isomorphism above.

Since

$$\mathbb{C}D(G) \cong \prod_{[H,hH']_G \in \mathcal{D}(G)/G} \mathbb{C}$$

the primitive idempotents $e_{(H,hH')}$ of $\mathbb{C}D(G)$ are in bijection with the species of $\mathbb{C}D(G)$. This bijection can be characterized by

$$s_{(K,kK')}(e_{(H,hH')}) = 0$$

whenever $[K, kK']_G \neq [H, hH']_G$. The idempotents $e_{(H,hH')}$ are given by the formula

$$e_{(H,hH')} = \frac{|H'|}{|N_G(H, hH')| \cdot |H|} \sum_{K \leq H} |K| \mu_{KH} \sum_{\phi \in \widehat{H}} \phi(h^{-1}) \mathrm{Ind}_K^G(\mathbb{C}_\phi, \{\mathbb{C}_\phi\}).$$

Example.

Let $G = S_3$ and $H = A_3$. Then

$$e_{(H,1)} = \frac{1}{6}\mathrm{Ind}_H^G(\mathbb{C}, \{\mathbb{C}\}) + \frac{1}{3}\mathrm{Ind}_H^G(\mathbb{C}_\phi, \{\mathbb{C}_\phi\}) - \frac{1}{6}\mathrm{Ind}_1^G(\mathbb{C}, \{\mathbb{C}\})$$

where $1 \neq \phi \in \mathrm{Irr}(H)$.

Corollary (Barker 2004).

The idempotents of $D(G)$ are all contained in its subring $B(G)$. Thus the primitive idempotents of $D(G)$ are in bijection with the conjugacy classes of perfect subgroups of G.

The prime spectrum

For a commutative ring R, the spectrum $\mathrm{Spec}(R)$ of R is the set of all prime ideals of R.
Let ζ be a primitive $|G|$th root of unity in $\mathbb{C}$, and let

$$\mathbb{Z}[\zeta]D(G) := \mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} D(G).$$

For $(H, hH') \in \mathcal{D}(G)$ and $P \in \mathrm{Spec}(\mathbb{Z}[\zeta])$, we set

$$\mathcal{P}(H, hH', P) := \{x \in \mathbb{Z}[\zeta]D(G) : s_{(H,hH')}(x) \in P\}.$$

Theorem.

$\mathrm{Spec}(\mathbb{Z}[\zeta]D(G) =$
$\{\mathcal{P}(H, hH', P) : (H, hH') \in \mathcal{D}(G),\ P \in \mathrm{Spec}(\mathbb{Z}[\zeta])\}.$

When is $\mathcal{P}(H, hH', P) = \mathcal{P}(K, kK', Q)$, for $(H, hH'), (K, kK') \in \mathcal{D}(G)$ and $P, Q \in \mathrm{Spec}(\mathbb{Z}[\zeta])$?

The prime spectrum

For a commutative ring R, we set

$$\mathrm{Spec}_0(R) := \{P \in \mathrm{Spec}(R) : \mathrm{char}(R/P) = 0\}$$

and

$$\mathrm{Spec}_p(R) := \{P \in \mathrm{Spec}(R) : \mathrm{char}(R/P) = p\}$$

$(p \in \mathbb{P})$ where $\mathbb{P}$ denotes the set of all prime numbers.

Theorem.

Then

$$\mathrm{Spec}_0(\mathbb{Z}[\zeta]D(G)) = \{\mathcal{P}(H, hH', 0) : (H, hH') \in \mathcal{D}(G)\}.$$

Moreover, for $(H, hH'), (K, kK') \in \mathcal{D}(G)$, we have $\mathcal{P}(H, hH', 0) = \mathcal{P}(K, kK', 0)$ iff (H, hH') and (K, kK') are conjugate in G.

The prime spectrum

Now let $p \in \mathbb{P}$, and let

$$\mathcal{D}_p(G) := \{(H, hH') \in \mathcal{D}(G) : |\langle h \rangle| \not\equiv 0 \not\equiv |N_G(H, hH') : H| \pmod{p}\}.$$

Theorem.

Then

$$\mathrm{Spec}_p(\mathbb{Z}[\zeta]D(G)) =$$

$$\{\mathcal{P}(H, hH', P) : (H, hH') \in \mathcal{D}_p(G),\ P \in \mathrm{Spec}_p(\mathbb{Z}[\zeta])\}.$$

Moreover, for $(H, hH'), (K, kK') \in \mathcal{D}_p(G)$ and $P, Q \in \mathrm{Spec}_p(\mathbb{Z}[\zeta])$, we have $\mathcal{P}(H, hH', P) = \mathcal{P}(K, kK', Q)$ iff (H, hH') and (K, kK') are conjugate in G, and $P = Q$.

Remark.

One can use the results above, together with a little Galois theory, in order to determine $\mathrm{Spec}(D(G))$, but we do not give the details here.

Theorem.

For $(H, hH') \in \mathcal{D}(G)$, we have

$$|N_G(H, hH') : H'| = \min\{n \in \mathbb{N} : n \cdot e_{(H,hH')} \in \mathbb{Z}[\zeta]D(G)\}.$$

This shows that all the indices $|N_G(H, hH') : H'|$ are determined by the ring $D(G)$; in particular, the order $|G|$ is determined by $D(G)$.

Torsion units

For a commutative ring R, we denote by $U_t(R)$ the group of torsion units of R. We also denote by $\mathcal{N}(G)$ the set of normal subgroups of G. Moreover, for $N \in \mathcal{N}(G)$, we denote by $D(G)_N$ the subgroup of $D(G)$ generated by the elements $\mathrm{Ind}_H^G(\mathbb{C}_\phi, \{\mathbb{C}_\phi\})$ where $[H, \phi]_G$ ranges over the elements in $\mathcal{M}(G)/G$ such that

- $N \leq H$;
- $N \leq M \leq H \quad \text{and} \quad M \in \mathcal{N}(G) \Longrightarrow N = M$.

Then

$$D(G) = \bigoplus_{N \in \mathcal{N}(G)} D(G)_N$$

and

$$D(G)_M D(G)_N \subseteq D(G)_{M \cap N},$$

for $M, N \in \mathcal{N}(G)$.

Torsion units

Proposition.

$U_t(D(G))$ is finite, and $\exp(U_t(D(G)))$ divides $2|G|$.

Theorem.

Let $n \in \mathbb{N}$ such that $\exp(U_t(D(G))) \mid n$. Moreover, for $N \in \mathcal{N}(G)$, let

$$N^* := \{a \in D(G)_N : (1+a)^n = 1\}.$$

Then every $u \in U_t(D(G))$ can be written uniquely in the form

$$u = \pm(\mathbb{C}_\psi, \{\mathbb{C}_\psi\}) \prod_{G \neq N \in \mathcal{N}(G)} (1 + u_N)$$

where $\psi \in \widehat{G}$ and $u_N \in N^*$ for $N \in \mathcal{N}(G)$. Thus

$$|U_t(D(G))| = 2|\widehat{G}| \prod_{G \neq N \in \mathcal{N}(G)} |N^*|.$$

Example:
If $G = S_3$ then

$$U_t(D(G)) \cong \langle -1 \rangle \times \widehat{G} \times \{1\}^* \times A_3^*$$

where $\{1\}^*$ is elementary abelian of order 4 and A_3^* has order 2.

Example:
If G is abelian then

$$U_t(D(G)) \cong G \times C_2^{m+1}$$

where m denotes the number of subgroups of G of index 2.

Corollary.

Let G be abelian, and let H be a finite group such that $D(G) \cong D(H)$. Then $G \cong H$.

The isomorphism problem

Theorem.

Let G and H be finite groups. Suppose that all Sylow subgroups of G and H are cyclic (for all primes). Then $D(G) \cong D(H)$ iff $G \cong H$.

Problem:

Find non-isomorphic finite groups G and H such that

$$D(G) \cong D(H).$$