The Trivial Source Ring of a Finite Group

Burkhard Külshammer

Mathematical Institute
Friedrich Schiller University Jena, Germany

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Notation

F algebraically closed field of characteristic $p > 0$

G finite group

$G_{p'}$ set of p'-elements of G (i.e. $|\langle g\rangle| \not\equiv 0 \pmod p$ for $g \in G_{p'}$)

FG group algebra

${}_{FG}\mathbf{mod}$ category of finitely generated FG-modules

We write $M \mid N$ if M is isomorphic to a direct summand of N, for $M, N \in {}_{FG}\mathbf{mod}$.

We fix an isomorphism $\epsilon \longmapsto \widehat{\epsilon}$ between the $|G|_{p'}$th roots of unity in F and the $|G|_{p'}$th roots of unity in $\mathbb{C}$. Here $n_{p'}$ denotes the p'-part of a positive integer n and, similarly, n_p denotes the p-part of n. Thus $n = n_p n_{p'}$, n_p is the largest power of p dividing n, and $\gcd(n_p, n_{p'}) = 1$.

Brauer characters

Let $M \in {}_{FG}\mathbf{mod}$ and $g \in G_{p'}$. Moreover, let $\epsilon_1, \ldots, \epsilon_n$ be the eigenvalues of the F-linear map

$$M \longrightarrow M, \quad m \longmapsto gm,$$

counted with multiplicities (so that $n = \dim_F M$). Then $\epsilon_1, \ldots, \epsilon_n$ are $|G|_{p'}$th roots of unity. We set

$$\phi_M(g) := \widehat{\epsilon}_1 + \cdots + \widehat{\epsilon}_n.$$

The map $\phi_M : G_{p'} \longrightarrow \mathbb{C}$ defined in this way is then called the Brauer character of M.

Properties:

(i) $\phi_{M\oplus N} = \phi_M + \phi_N$ and $\phi_{M\otimes_F N} = \phi_M \phi_N$ for $M, N \in {}_{FG}\mathbf{mod}$;
(ii) $\phi_F(g) = 1$ for $g \in G_{p'}$ where $F = F_G$ denotes the trivial FG-module.

(iii) $\phi_M(g) = \phi_M(h)$ whenever $g, h \in G_{p'}$ are conjugate in G.

(iv) $\phi_M = \phi_N$ iff M, N have the same composition factors (counted with multiplicities).

Thus the Brauer characters of G form a commutative ring $R(FG)$. As an abelian group, $R(FG)$ is free of rank $\ell(G)$ where $\ell(G)$ denotes the number of conjugacy classes of p'-elements in G. Thus

$$\mathbb{C}R(FG) := \mathbb{C} \otimes_{\mathbb{Z}} R(FG)$$

is a commutative $\mathbb{C}$-algebra of dimension $\ell(G)$ which can also be viewed as the ring of all class functions $G_{p'} \longrightarrow \mathbb{C}$.

Vertices and sources

Let $M \in {}_{FG}\mathbf{mod}$ be indecomposable, and let $P \leq G$ be minimal (with respect to $\subseteq$) such that $M \mid \mathrm{Ind}_P^G(\mathrm{Res}_P^G(M))$. Then P is called a vertex of M.
In this case, P is a p-subgroup of G and unique up to conjugation in G.

Now let $M \in {}_{FG}\mathbf{mod}$ be indecomposable with vertex P. Then there is an indecomposable FP-module S such that $M \mid \mathrm{Ind}_P^G(S)$. Moreover, S is unique up to isomorphism and conjugation with elements in $N_G(P)$, and S is called a source of M in P.

Green correspondence

Let $P \leq G$ be a p-subgroup, and let $H := N_G(P)$.

(i) If $M \in {}_{FG}\mathbf{mod}$ is indecomposable with vertex P then there is a unique indecomposable direct summand N of $\mathrm{Res}_H^G(M)$ with vertex P, up to isomorphism. Moreover, N appears with multiplicity 1 in $\mathrm{Res}_H^G(M)$, and M and N have a common source.

(ii) If $N \in {}_{FH}\mathbf{mod}$ is indecomposable with vertex P then there is a unique indecomposable direct summand M of $\mathrm{Ind}_H^G(N)$ with vertex P, up to isomorphism. Moreover, M appears with multiplicity 1 in $\mathrm{Ind}_H^G(N)$, and N and M have a common source.

(iii) By (i) and (ii), we obtain mutually inverse bijections between

- isomorphism classes of indecomposable FG-modules with vertex P;
- isomorphism classes of indecomposable FH-modules with vertex P.

This is called Green correspondence.

Trivial source modules

Example:

(i) Let $M \in {}_{FG}\mathbf{mod}$ be indecomposable with vertex P and trivial source F_P. Then P acts trivially on the Green correspondent N of M in $N_G(P)$. Thus N can be viewed as an $F[N_G(P)/P]$-module. As such, N is indecomposable and projective.

(ii) Conversely, let $N \in {}_{F[N_G(P)/P]}\mathbf{mod}$ be indecomposable and projective. Then N can be viewed as an $FN_G(P)$-module via inflation. As such, N is indecomposable with vertex P, so its Green correspondent is an indecomposable FG-module with vertex P and trivial source.

(iii) In this way we obtain a bijection between

- isomorphism classes of indecomposable FG-modules with vertex P and trivial source;
- isomorphism classes of indecomposable projective $F[N_G(P)/P]$-modules.

Trivial source modules

We denote by $\mathcal{X}(G)$ the set of all pairs (P, N) where P is a p-subgroup of G and N is an indecomposable projective $F[N_G(P)/P]$-module (or rather its isomorphism class). Then, as explained above, each pair $(P, N) \in \mathcal{X}(G)$ defines an indecomposable FG-module $M_{(P,N)}$ with vertex P and trivial source. Moreover, we have $M_{(P,N)} \cong M_{(Q,L)}$ iff (P, N) and (Q, L) are conjugate in G. Here the conjugation action of G on $\mathcal{X}(G)$ is defined in the usual way, and we denote by $\mathcal{X}(G)/G$ the set of all orbits $[P, N]_G$.

A finitely generated FG-module M is called a trivial source module if all its indecomposable direct summands have a trivial source.

Theorem.

For $M \in {}_{FG}\mathbf{mod}$, the following assertions are equivalent:

(i) M is a trivial source module;

(ii) M is a direct summand of a permutation FG-module;

(iii) $\mathrm{Res}_P^G(M)$ is a permutation FP-module for some (and hence every) Sylow p-subgroup P of G.

For this reason, trivial source modules are also called p-permutation modules.

The Brauer construction

Let $M \in {}_{FG}\mathbf{mod}$ and $P \leq G$. We set

$$M^P := \{m \in M : xm = m \quad \text{for} \quad x \in P\}.$$

Then, for $Q \leq P$,

$$\mathrm{Tr}_Q^P : M^Q \longrightarrow M^P, \quad m \longmapsto \sum_{xQ \in P/Q} xm,$$

is called the relative trace map, and

$$M(P) := M^P / \sum_{Q<P} \mathrm{Tr}_Q^P(M^Q)$$

is called the Brauer construction of M with respect to P. It is obvious that $M(P)$ becomes a finitely generated $F[N_G(P)/P]$-module. Moreover, we have $M(P) = 0$ unless P is a p-group.

An alternative approach

Theorem.

Let $M \in {}_{FG}\mathbf{mod}$ be indecomposable with trivial source, and let $P \leq G$ be a p-subgroup.

(i) Then P is a vertex of M iff P is maximal among the p-subgroups $Q \leq G$ with $M(Q) \neq 0$.

(ii) In this case, $M(P)$ is the indecomposable projective $F[N_G(P)/P]$-module corresponding to M (via the Green correspondence).

This means, in particular, that $(M_{(P,N)})(P) \cong N$, for all $(P, N) \in \mathcal{X}(G)$.

The Green ring

The Green ring $A(FG)$ is defined as the Grothendieck ring of the category ${}_{FG}\mathbf{mod}$ with respect to **split** short exact sequences. Addition in $A(FG)$ comes from direct sums, and multiplication comes from tensor products. By the Krull-Schmidt Theorem, $A(FG)$ is free as a $\mathbb{Z}$-module, and a basis is given by the isomorphism classes of indecomposable FG-modules. Thus $A(FG)$ is not finitely generated as a $\mathbb{Z}$-module, in general.

Since direct sums and tensor products of permutation modules are again permutation modules, the isomorphism classes of indecomposable trivial source FG-modules generate a subring $T(FG)$ of $A(FG)$, the trivial source ring of FG. Then $T(FG)$ is a finitely generated free $\mathbb{Z}$-module, and the isomorphism classes of the indecomposable trivial source FG-modules $M_{(P,N)}$ $([P,N]_G \in \mathcal{X}(G)/G)$ form a basis.

Examples

Since every finite G-set defines a permutation FG-module (and thus a trivial source module) we obtain a ring homomorphism $B(G) \longrightarrow T(FG)$ where $B(G)$ denotes the Burnside ring of G.

Since every (trivial source) module defines a Brauer character, we obtain a ring homomorphism $T(FG) \longrightarrow R(FG)$.

Proposition.

(i) If G is a p-group then $T(FG)$ is isomorphic to the Burnside ring $B(G)$.

(ii) If G is a p'-group (i. e. $|G| \not\equiv 0 \pmod{p}$) then $T(FG)$ is isomorphic to the ring $R(FG)$ of Brauer characters.

Examples

Let $G = S_3$ and $p = 2$.

$P = S_2$: Since $N_G(P)/P = 1$ the only indecomposable projective $F[N_G(P)/P]$-module is trivial and corresponds to the trivial FG-module $M_1 = F$.

$P = 1$ gives the indecomposable projective modules M_2 and M_3, both of dimension 2; M_2 is the projective cover of the trivial FG-module, and M_3 is both simple and projective.

It is easy to see that

$$[M_2]^2 = 2[M_2], \quad [M_2][M_3] = 2[M_3] \quad \text{and} \quad [M_3]^2 = [M_2] + [M_3].$$

Examples

$G = S_3$ and $p = 3$.

$P = A_3$, i. e. $N_G(P)/P = S_3/A_3 \cong S_2$ has 2 simple modules, the trivial and the alternating one. Both are also projective. They correspond to the trivial FG-module M_1 and the alternating FG-module M_2.

$P = 1$ gives the indecomposable projective FG-modules M_3 and M_4, both of dimension 3. Here M_3 is the projective cover of the trivial FG-module, and M_4 is the projective cover of the alternating FG-module.

It is easy to see that

$$[M_2]^2 = [M_1], \quad [M_2][M_3] = [M_4], \quad [M_2][M_4] = [M_3],$$

$$[M_3]^2 = 2[M_3] + [M_4] = [M_4]^2 \quad \text{and} \quad [M_3][M_4] = [M_3] + 2[M_4].$$

Let $H \leq G$. Then restriction from ${}_{FG}\textbf{mod}$ to ${}_{FH}\textbf{mod}$ gives rise to a ring homomorphism

$$\mathrm{res}_H^G : A(FG) \longrightarrow A(FH)$$

sending $T(FG)$ into $T(FH)$, and induction from ${}_{FH}\textbf{mod}$ to ${}_{FG}\textbf{mod}$ gives rise to a group homomorphism

$$\mathrm{ind}_H^G : A(FH) \longrightarrow A(FG)$$

whose image is an ideal in $A(FG)$. Moreover, ind_H^G sends $T(FH)$ into $T(FG)$, and $\mathrm{ind}_H^G(T(FH))$ is an ideal in $T(FG)$.

The Dress Induction Theorem

Using trivial source modules, the Green correspondence and Brauer's Induction Theorem, one can prove the following:

Dress Induction Theorem.

For $M \in {}_{FG}\mathbf{mod}$, one can write

$$[M] = \sum_H a_H [\mathrm{Ind}_H^G(M_H)] \quad \text{in} \quad A(FG),$$

with integers a_H and FH-modules M_H where H ranges over the subgroups of G such that $H/O_p(H)$ is elementary (i. e. a direct product of a cyclic group and a q-group, for some $q \in \mathbb{P}$).

Here $[M]$ denotes the isomorphism class of M.

There is also a canonical induction formula for the trivial source ring $T(FG)$, due to Robert Boltje, which we cannot mention here.

Recall that $T(FG)$ is a $\mathbb{Z}$-order of rank

$$\sum_P \ell(N_G(P)/P)$$

where P ranges over a transversal for the conjugacy classes of p-subgroups of G. Thus

$$\mathbb{C}T(FG) := \mathbb{C} \otimes_{\mathbb{Z}} T(FG)$$

is a finite-dimensional $\mathbb{C}$-algebra. Let us determine the species of $\mathbb{C}T(FG)$, i. e. the $\mathbb{C}$-algebra homomorphisms $\mathbb{C}T(FG) \longrightarrow \mathbb{C}$.

Species

We set

$$\mathcal{Y}(G) := \{(P, g) : P \leq G \; p\text{–subgroup}, \; g \in N_G(P)_{p'}\}.$$

Then G acts on $\mathcal{Y}(G)$ via conjugation, and we denote by $\mathcal{Y}(G)/G$ the set of G-orbits $[P, g]_G$.

Every pair $(P, g) \in \mathcal{Y}(G)$ defines a species $s_{(P,g)}$ of $\mathbb{C}T(FG)$ as the composition

$$\mathbb{C}T(FG) \longrightarrow \mathbb{C}T(F[N_G(P)/P]) \longrightarrow \mathbb{C}R(F[N_G(P)/P]) \longrightarrow \mathbb{C};$$

here the map $\mathbb{C}T(FG) \longrightarrow \mathbb{C}T(F[N_G(P)/P])$ is induced by the Brauer construction $M \longmapsto M(P)$, the map $\mathbb{C}T(F[N_G(P)/P]) \longrightarrow \mathbb{C}R(F[N_G(P)/P])$ sends every trivial source module to its Brauer character, and the map $\mathbb{C}R(F[N_G(P)/P]) \longrightarrow \mathbb{C}$ evaluates each Brauer character at gP.

Proposition.

(i) Every species of $\mathbb{C}T(FG)$ arises in the way described above.

(ii) For $(P,g),(Q,h) \in \mathcal{Y}(G)$, we have $s_{(P,g)} = s_{(Q,h)}$ iff (P,g) and (Q,h) are conjugate in G.

It follows easily that the dimension of $\mathbb{C}T(FG)$ equals $|\mathcal{Y}(G)/G|$. Thus the species of $\mathbb{C}T(FG)$ define an isomorphism

$$\mathbb{C}T(FG) \cong \prod_{[P,g]_G \in \mathcal{Y}(G)/G} \mathbb{C}.$$

In particular, $\mathbb{C}T(FG)$ is a finite-dimensional commutative semisimple $\mathbb{C}$-algebra. Moreover, the species of $\mathbb{C}T(FG)$ can be viewed as the projections in the isomorphism above.

Example

$G = S_3 = \langle a, b : a^3 = 1 = b^2,\ bab = a^2 \rangle$ and $p = 2$.

Then $\mathcal{Y}(G)/G = \{[S_2, 1]_G, [1, 1]_G, [1, a]_G\}$.

The species table is as follows:

	M_1	M_2	M_3
$(S_2, 1)$	1	0	0
$(1, 1)$	1	2	2
$(1, a)$	1	2	-1

Example

$G = S_3 = \langle a, b : a^3 = 1 = b^2,\ bab = a^2 \rangle$ and $p = 3$.

Then $\mathcal{Y}(G)/G = \{[A_3, 1]_G, [A_3, b]_G, [1, 1]_G, [1, b]_G\}$.

The species table is as follows:

	M_1	M_2	M_3	M_4
$(A_3, 1)$	1	1	0	0
(A_3, b)	1	-1	0	0
$(1, 1)$	1	1	3	3
$(1, b)$	1	-1	1	-1

The isomorphism

$$\mathbb{C}T(FG) \cong \prod_{[P,g]_G \in \mathcal{Y}(G)/G} \mathbb{C}$$

implies that the primitive idempotents of $\mathbb{C}T(FG)$ are in bijection with $\mathcal{Y}(G)/G$. We denote the primitive idempotent corresponding to $[P,g]_G \in \mathcal{Y}(G)/G$ by $e_{(P,g)}$. Thus $s_{(P,g)}(e_{(P,g)}) = 1$. One can prove the following formula for these idempotents:

$$e_{(P,g)} = \frac{1}{|\langle g\rangle| \cdot |N_G(P,gP)|} \sum_{L,\phi} \phi(g^{-1})|L|\mu_{L,\langle P,g\rangle}[\mathrm{Ind}_L^G(F_{L,\phi})]$$

where L ranges over the subgroups of $\langle P,g\rangle$ with $PL = \langle P,g\rangle$ and ϕ ranges over $\widehat{\langle g\rangle}$. Moreover, $F_{L,\phi}$ denotes the 1-dimensional FL-module whose Brauer character is obtained from ϕ via inflation to $\langle P,g\rangle$ and restriction to L.

Example

Let $G = S_3$ and $p = 2$. Then

$$e_{(S_2,1)} = [M_1] - \frac{1}{2}[M_2],$$

$$e_{(1,1)} = \frac{1}{6}[M_2] + \frac{1}{3}[M_3],$$

$$e_{(1,a)} = -\frac{1}{3}[M_2] + \frac{1}{3}[M_3].$$

Example

Let $G = S_3$ and $p = 3$. Then

$$e_{(A_3,1)} = \frac{1}{2}[M_1] + \frac{1}{2}[M_2] - \frac{1}{6}[M_3] - \frac{1}{6}[M_4],$$

$$e_{(A_3,b)} = \frac{1}{2}[M_1] - \frac{1}{2}[M_2] - \frac{1}{2}[M_3] + \frac{1}{2}[M_4],$$

$$e_{(1,1)} = \frac{1}{6}[M_3] + \frac{1}{6}[M_4],$$

$$e_{(1,b)} = \frac{1}{2}[M_3] - \frac{1}{2}[M_4].$$

The formulae for the primitive idempotents in $\mathbb{C}T(FG)$ can be used to show that $|G|$ is determined by $\mathbb{C}T(FG)$. They can also be used to prove the following:

Proposition.

0 and 1 are the only idempotents in $T(FG)$.

Let ζ be a primitive $|G|$th root of unity in $\mathbb{C}$. We set

$$\mathbb{Z}[\zeta]T(FG) := \mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} T(FG).$$

The spectrum

As before, we denote by $\mathrm{Spec}(R)$ the spectrum of a commutative ring R. Also, we set

$$\mathrm{Spec}_0(R) := \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathrm{char}(R/\mathfrak{p}) = 0\}$$

and

$$\mathrm{Spec}_q(R) := \{\mathfrak{q} \in \mathrm{Spec}(R) : \mathrm{char}(R/\mathfrak{q}) = q\},$$

for $q \in \mathbb{P}$.

Our aim is to determine these sets of prime ideals, for $R := \mathbb{Z}[\zeta]T(FG) := \mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} T(FG)$. There are 3 parts:

- $\mathrm{Spec}_0(\mathbb{Z}[\zeta]T(FG))$,
- $\mathrm{Spec}_p(\mathbb{Z}[\zeta]T(FG))$ where $p := \mathrm{char}(F)$,
- $\mathrm{Spec}_q(\mathbb{Z}[\zeta]T(FG))$ where $p \neq q \in \mathbb{P}$.

Residue characteristic 0

We recall that

$$\mathcal{Y}(G) := \{(P,g) : P \leq G \ p\text{−subgroup}, \ g \in N_G(P)_{p'}\}$$

and set

$$\mathfrak{P}(P,g,0) := \{x \in \mathbb{Z}[\zeta]T(FG) : s_{(P,g)}(x) = 0\},$$

for $(P,g) \in \mathcal{Y}(G)$. Then

$$\mathrm{Spec}_0(\mathbb{Z}[\zeta]T(FG)) = \{\mathfrak{P}(P,g,0) : (P,g) \in \mathcal{Y}(G)\}.$$

Moreover, we have $\mathfrak{P}(P,g,0) = \mathfrak{P}(Q,h,0)$ iff the pairs (P,g) and (Q,h) in $\mathcal{Y}(G)$ are conjugate in G.

We set

$$\mathcal{Y}_p(G) := \{(P,g) \in \mathcal{Y}(G) : [P,g] = P\}$$

and

$$\mathfrak{P}(P,g,\mathfrak{p}) := \{x \in \mathbb{Z}[\zeta]T(FG) : s_{(P,g)}(x) \in \mathfrak{p}\},$$

for $(P,g) \in \mathcal{Y}_p(G)$ and $\mathfrak{p} \in \mathrm{Spec}_p(\mathrm{Z}[\zeta])$. Then

$$\mathrm{Spec}_p(\mathbb{Z}[\zeta]T(FG)) = \{\mathfrak{P}(P,g,\mathfrak{p}) : (P,g) \in \mathcal{Y}_p(G),\ \mathfrak{p} \in \mathrm{Spec}_p(\mathbb{Z}[\zeta])\}.$$

Moreover, we have $\mathfrak{P}(P,g,\mathfrak{p}) = \mathfrak{P}(Q,h,\mathfrak{q})$ iff $\mathfrak{p} = \mathfrak{q}$, and the pairs (P,g) and (Q,h) in $\mathcal{Y}_p(G)$ are conjugate in G.

Let $p \neq q \in \mathbb{P}$. We set

$$\mathcal{Y}_q(G) := \{(P, g) \in \mathcal{Y}(G) : g \in N_G(P)_{q'}\}$$

and

$$\mathfrak{P}(P, g, \mathfrak{q}) := \{x \in \mathbb{Z}[\zeta]T(FG) : s_{(P,g)}(x) \in \mathfrak{q}\},$$

for $(P, g) \in \mathcal{Y}_q(G)$ and $\mathfrak{q} \in \mathrm{Spec}_q(\mathbb{Z}[\zeta])$. Then

$$\mathrm{Spec}_q(\mathbb{Z}[\zeta]T(FG)) = \{\mathfrak{P}(P, g, \mathfrak{q}) : (P, g) \in \mathcal{Y}_q(G),\ \mathfrak{q} \in \mathrm{Spec}_q(\mathbb{Z}[\zeta])\}.$$

Moreover, we have $\mathfrak{P}(P, g, \mathfrak{q}) = \mathfrak{P}(Q, h, \mathfrak{r})$ iff $\mathfrak{q} = \mathfrak{r}$ $(\mathfrak{q}, \mathfrak{r} \in \mathrm{Spec}(\mathbb{Z}[\zeta]))$, and the pairs (P, g) and (Q, h) in $\mathcal{Y}_q(G)$ are conjugate in G.

It is possible to go down from $\mathrm{Spec}(\mathbb{Z}[\zeta]T(FG))$ to $\mathrm{Spec}(T(FG))$, by using Galois theory. We don't do this here.

Little seems to be known about torsion units in $T(FG)$.
The following result is not difficult:

Proposition.

Let G and H be finite abelian groups such that $T(FG) \cong T(FH)$. Then $G \cong H$.

References

1. D. J. Benson, Representations and cohomology, Cambridge University Press, Cambridge 1991
2. R. Boltje, Representation rings of finite groups, their species and idempotent formulae, *J. Algebra*, to appear
3. S. Bouc and J. Thévenaz, The primitive idempotents of the p-permutation ring, *J. Algebra*, to appear
4. M. Broué, On Scott modules and p-permutation modules: an approach through the Brauer morphism, *Proc. Amer. Math. Soc.* **93** (1985), 401-408
5. C. W. Curtis and I. Reiner, Methods of representation theory, Wiley Interscience, New York 1987
6. M. Deiml, Zur Darstellungstheorie von Darstellungsringen, Ph. D. thesis, Jena 1997
7. M. Müller, Zum Isomorphieproblem von Darstellungsringen, Ph. D. thesis, Jena 2008