On Duo Group Rings

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Introduction and Preliminaries

- **Some Notations**
  1. R – Associative ring with 1.
  2. kG – Group ring over a commutative ring k.
  3. $Q_8 = \langle a, b | a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$.

- **Definition 1.1**
  Let R be an associative ring with identity. Call R left (right) duo if every left (right) ideal is an ideal, and call R duo if it is both left and right duo. Define R to be reversible if $\alpha\beta = 0$ implies $\beta\alpha = 0$, and symmetric if $\alpha\beta\gamma = 0$ implies $\alpha\gamma\beta = 0$ for all $\alpha, \beta, \gamma \in R$. Finally, say that R has the “SI” property if $\alpha\beta = 0$ implies $\alpha R\beta = 0$ for all $\alpha, \beta \in R$.

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- **Remark** Let k be a commutative ring with identity and G be any group. Using the standard involution $*$ on the group ring kG, defined by $(\sum a_i g_i)^* = \sum a_i g_i^{-1}$ for all $a_i \in k$ and $g_i \in G$, we see that kG is left duo if and only if it is right duo.
- Marks [5] has clarified the relationships among duo, reversible and symmetric rings. Moreover, he proved the following result [5, Proposition 6].
- **Proposition 1.2** Let k be a commutative ring with identity, and let G be a finite group. Then the group ring kG is reversible if and only if kG has the "SI" property.

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- We note that this result remains valid for an arbitrary group G.
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- **Example 1.3** Let $Q_8 = \langle a, b | a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle$ be the quaternion group of order 8. The integral group ring $\mathbb{Z}Q_8$ is a reversible ring, but not a duo ring.
- **Proof.** It follows from Theorem 3.1 in [3] that the rational group algebra $\mathbb{Q}Q_8$ is reversible. As a subring of $\mathbb{Q}Q_8$, $\mathbb{Z}Q_8$ is clearly reversible.
- $R = \mathbb{Z}Q_8$ is not a duo ring since the left ideal $R(a + 2b)$ generated by $a + 2b$ is not a right ideal.

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Introduction and Preliminaries

- In this talk, we shall study when the group rings RG of torsion groups over integral domains are duo.
- If $R = K$ is a field, then that KG is a duo ring if and only if KG is reversible.
- If G is a non-abelian torsion group and R is an ID, then RG is a duo ring if and only if R is a filed and RG is reversible for the following cases: (1) char $R \neq 0$, and (2) char $R = 0$, and $S \subseteq R \subseteq K_S$, where S is a ring of algebraic integers and K_S is its quotient field.

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- It was shown in [3, 4] that if RG of a torsion group G over a commutative ring R is reversible, then either G is an abelian group, or G is a Hamiltonian group. In addition, if $R = K$ is a field then the characteristic of K is 0 or 2.
- Next assume that $char(R) = 0$ or 2 and G is a Hamiltonian group, i.e. $G = Q_8 \times E_2 \times E_2'$.
- We first deal with the group algebra KG case where K is a filed. The general case RG will be handled in the last section.

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Group Algebras KG

1. Let $G = Q_8$ and K be a field of characteristic 0. The following result can be derived from some known results in group rings along with the Wedderburn decomposition of KQ_8.
2. **Theorem 2.1** The following statements are equivalent:
   - (1) $R = KQ_8$ is duo.
   - (2) $R = KQ_8$ is reversible.
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- We now characterize when KQ_8 is duo, where K is a field with characteristic 2.
- **Theorem 2.2** The following statements are equivalent:
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- **Theorem 2.5** Let K be a field and let G be a torsion group. Then KG is a duo ring if and only if one of the following conditions holds.

  (1) G is Abelian.

  (2) $G = Q_8 \times E_2 \times E_2'$ is Hamiltonian, the characteristic of K is 0 and the equation $1 + x^2 + y^2 = 0$ has no solutions in any cyclotomic field $K(\xi_d)$ for any odd d which is an order of an element of E_2'.

  (3) $G = Q_8 \times E_2'$, the characteristic of K is 2 and the equation $1 + x + x^2 = 0$ has no solutions in any cyclotomic field $K(\xi_d)$ for any odd d which is an order of an element of E_2'.

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1. **Corollary 2.6** KG is duo if and only if KG is reversible.

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Group rings over integral domains

1. We remark that if G is a non-abelian torsion group and R is a commutative ring with identity for which RG is duo, then RQ_8 is also duo. The question of when RG is duo essentially reduces to that of when RQ_8 is duo, where R is an integral domain.
2. As mentioned early, the integral group ring $\mathbb{Z}Q_8$ is a reversible ring, but not a duo ring [1, Example 1.1] while $\mathbb{Q}Q_8$ is a duo ring. A natural question which arises is as follows:
3. **Question 3.1** Is there any ring R between $\mathbb{Z}$ and $\mathbb{Q}$ (in addition to $\mathbb{Q}$ the field of all rational numbers), such that RQ_8 is duo.

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1. We remark that if G is a non-abelian torsion group and R is a commutative ring with identity for which RG is duo, then RQ_8 is also duo. The question of when RG is duo essentially reduces to that of when RQ_8 is duo, where R is an integral domain.
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1. Theorem 3.5 shows that there does not exist such R. In other words, $\mathbb{Q}$ is the smallest integral domain R of characteristic zero such that RQ_8 is duo.
2. We need the following two lemmas. The first is a well known result in number theory and it is a consequence of [6, Theorem 5.14].
3. **Lemma 3.2** $1+x^2+y^2 \equiv 0 \pmod{p}$ is solvable in $\mathbb{Z}$ for every prime p.
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- **Corollary 3.4** Let R be an integral domain such that RQ_8 is duo. Then char $R = 2$ or char $R = 0$. In the latter case, we have $1 + x^2 + y^2 \in \mathcal{U}(R)$, for all $x, y \in R$.
- **Theorem 3.5** [2, Theorem 2.4] Let R be an integral domain such that RQ_8 is duo. Then the following statements hold.

  (1) If $char(R) \neq 0$, then R must be a field.

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with all $c_i \in \mathbb{Z}$ and $c_0 \neq 0$. Then

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2. Proposition 3.6 Let $S = \mathbb{Q}[x]$ be the polynomial ring over the rational field, and S_P be the localization of S at the maximal ideal $P = \langle x \rangle$. Then $R = S_P$ is a local integral domain of characteristic 0, but not a field, such that RQ_8 is duo.

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1. Theorem 3.5 together with Theorem 2.5 gives a description of when RG is duo.
2. Theorem 3.7 If R is an integral domain of $char(R) \neq 0$ and G is a non-abelian torsion group, then the following statements are equivalent:

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1. Recall that Proposition 3.6 indicates that there exists an integral domain R, which is not a field, such that RQ_8 is duo. In that case, a necessary condition for RQ_8 to be duo is given in Corollary 3.4, i.e. $1+x^2+y^2 \in \mathcal{U}(R)$ for all $x, y \in R$. We are not aware of any example of an integral domain R of $char(R) = 0$ satisfying this necessary condition for which RQ_8 is not duo. We close this talk by proposing the following question.

2. **Question 3.10** Assume that R is an integral domain of $char(R) = 0$ such that $1+x^2+y^2 \in \mathcal{U}(R)$ for all $x, y \in R$. Is RQ_8 duo?

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Thank you!

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