# Isomorphisms of group algebras of finite almost simple groups

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March 25, 2010

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# Principal Approach

For $\mathbb{F} = \mathbb{C}$ the principal approach used in the proof is as follows:

- As $H$ has only one representation of degree 1, $H$ is perfect
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# Tools

Modifying a result by R. Rasala (1977) for the symmetric groups the following proposition has been established in my Diplomarbeit.

# Tools

Proposition: Let

$d_0 = 1,\ d_1 = n-1,\ d_2 = \frac{1}{2}n(n-3),\ d_3 = \frac{1}{2}(n-1)(n-2)$
$d_4 = \frac{1}{6}n(n-1)(n-5),\ d_5 = \frac{1}{6}(n-1)(n-2)(n-3)$ ,
$d_6 = \frac{1}{3}n(n-2)(n-4)$
$d_7 = \frac{1}{24}n(n-1)(n-2)(n-7)$,
$d_8 = \frac{1}{24}(n-1)(n-2)(n-3)(n-4)$.

Then:

- If $7 \leq n \leq 9$, then $d_0, d_1$ are the two smallest degrees of $cdA_n$.
- If $10 \leq n \leq 14$, then $d_0, d_1, d_2, d_3$ are the four smallest degrees of ordinary irreducible representaitons of $A_n$.
- If $15 \leq n \leq 21$, then $d_0, d_1, d_2, d_3, d_4, d_5, d_6$ are the seven smallest degrees of ordinary irreducible representaitons of $A_n$.
- If $22 \leq n$, then $d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8$ are the nine smallest degrees of ordinary irreducible representaitons of $A_n$.

There is exactly one representation of each degree listed above.

# The Complex Case

The proof of the theorem gives reason to consider the following question.

Let $G$ be a finite simple group and $H$ be a group with $\mathbb{C}G \cong \mathbb{C}H$. Are $G$ and $H$ isomorphic groups?

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*(Preliminary Version) $\mathbb{C}G$ determines $G$ if $G$ is an simple alternating group, a sporadic simple group or a simple group of Lie type over a prime field.*

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# Tools

The proof makes use of the list of minimal characters of groups of Lie-Type (Tiep, Zalesskii, Luebeck)

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# Related Questions

Does the complex group algebra determine almost simple groups up to isomorphism?

Concerning the techniques used in the proof for the alternating groups the following holds:
$\mathbb{C}S_n$ determines $S_n$ up to isomorphism.(Some small $n$ remain to be checked)

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# Related Questions

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2. Does the above statement hold for all primes?

The first statement is true for almost all simple groups of Lie-Type by a result of Kimmerle (using Humphrey's result on the number of blocks of simple groups of Lie type in defining characteristic).

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