Finitely presented algebras defined by homogeneous semigroup relations

Jan Okniński

Aachen, March 2010

Plan of the talk

1. motivating problems
2. algebras defined by homogeneous semigroup presentations
3. special classes of algebras

   i) algebras defined by permutation relations
   ii) Chinese algebras
   iii) algebras yielding solutions of the quantum Yang-Baxter equation

Notation

K will denote a field

X - a free finitely generated monoid (or a free semigroup)

$K\langle X\rangle$ - the corresponding free algebra

a K-algebra R is finitely presented if it is of the form $R = K\langle X\rangle/I$ for a finitely generated ideal I of $K\langle X\rangle$.

Special classes of finitely presented algebras (within a general program on algebras defined by homogeneous semigroup presentations):

i) defined by semigroup relations:
I is an ideal of $K\langle X\rangle$ generated by a set of the form

$$\{w - u \mid (w, u) \in A\}$$

for a subset $A \subseteq X \times X$.
In other words, $R = K[M]$, where M is the monoid defined by $M = X/\rho$, where ρ is the congruence on X generated by the set A.

ii) defined by homogeneous semigroup relations:
R is as in i) but we also assume that $|w| = |u|$ for all $(w, u) \in A$ (words of equal length). So, R has a natural $\mathbb{Z}$-gradation.

Open problems

Assume R is a finitely presented K-algebra

1. Let R be a nil algebra. Is R nilpotent (hence, finite dimensional)? (Latyshev, Zelmanov)

2. Is the Jacobson radical $\mathcal{J}(R)$ a nil ideal? (Amitsur)

3. Is the Jacobson radical $\mathcal{J}(R)$ a locally nilpotent ideal?

4. Does R have only finitely many minimal prime ideals (at least for some important classes of such algebras)?
(This is perhaps too optimistic in general.)

Note: positive answers to some of the questions are known if R is a finitely generated PI-algebra, or it is noetherian or the field is nondenumerable.
A.Belov announced recently an example of a finitely presented nil semigroup S that is infinite.

5. Let $GK(R)$ be the Gelfand-Kirillov dimension of R. Do we always have $GK(R) \in \mathbb{Q} \cup \{\infty\}$?

It is known that $GK(R)$ can be a rational number that is not an integer (A.Belov, I.Ivanov).

6. Assume R is finitely generated and right noetherian (not necessarily finitely presented). Do we have $GK(R) \in \mathbb{Z} \cup \{\infty\}$?

It is known that the growth is subexponential (Stephenson, Zhang). If $R = K[S]$ and S has no free noncommutative subsemigroups and R is defined by a homogeneous presentation then $K[S]$ is PI, so $GK(R)$ is an integer (Gateva-Ivanova, Jespers, J.O.)

7. What happens if (in Problem 6) we assume R is finitely presented?

There exists a finitely generated nil algebra of finite GK-dimension, but of infinite (linear) K-dimension (Lenagan, Smoktunowicz).

8. What is the minimal possible GK-dimension of such an algebra?

Existence of such algebras seems to be unknown even if $GK(R) = 2$.

9. Do we have such examples with R finitely presented?

10. What about algebras defined by homogeneous semigroup presentations?

All these problems seem very difficult in full generality. So, any partial results would be of interest.

Another motivation for studying such problems comes from the interest in the structure and properties of certain important special classes of finitely presented algebras.
In particular:

- algebras defined by permutation relations
- Chinese algebras
- algebras related to the quantum Yang-Baxter equation

Permutation relations (F.Cedo, E.Jespers, J.O.)

Algebras defined by permutation relations:

$$A = K\langle a_1, a_2, \ldots, a_n \mid a_1 a_2 \cdots a_n = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)},\ \sigma \in H\rangle,$$

where H is a subset of the symmetric group Sym_n of degree n. So $A = K[S_n(H)]$ where

$$S_n(H) = \langle a_1, a_2, \ldots, a_n \mid a_1 a_2 \cdots a_n = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)},\ \sigma \in H\rangle,$$

the monoid with "the same" presentation as the algebra. By $G_n(H)$ we denote the group defined by this presentation. So

$$G_n(H) = \mathrm{gr}(a_1, a_2, \ldots, a_n \mid a_1 a_2 \cdots a_n = a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)},\ \sigma \in H).$$

Two obvious examples are:
i) the free K-algebra $K[S_n(\{1\})] = K\langle a_1, \ldots, a_n\rangle$ with $H = \{1\}$ and $S_n(\{1\}) = FM_n$, the rank n free monoid,
ii) the commutative polynomial algebra $K[S_2(\mathrm{Sym}_2)] = K[a_1, a_2]$ with $H = Sym_2$ and $S_n(H) = FaM_2$, the rank 2 free abelian monoid.

For $M = S_n(Sym_n)$, the latter can be extended as follows: the algebra $K[M]$ is the subdirect product of the commutative polynomial algebra $K[a_1, \ldots, a_n]$ and a primitive algebra $K[M]/K[Mz]$, with $z = a_1 a_2 \cdots a_n$, a central element. (Since the latter is isomorphic to a monomial algebra, this uses the recent result of J.Bell, P. Colak, or independently J.O.)

Theorem

Let $H = \mathrm{gr}((1, 2, \ldots, n))$. The monoid $S_n(H)$ is cancellative and it has a group G of fractions of the form $G = S_n(H)\langle a_1 \cdots a_n \rangle^{-1} \cong F \times C$, where $F = \mathrm{gr}(a_1, \ldots, a_{n-1})$ is a free group of rank $n-1$ and $C = \mathrm{gr}(a_1 \cdots a_n)$ is a cyclic infinite group.

Corollary

The algebra $K[S_n(H)]$ is a domain and it is semiprimitive.

If $n = 3$, this can be treated as $S_n(Alt_n)$.

By $\rho = \rho_S$ we denote the least cancellative congruence on a semigroup S.

If η is a congruence on S then

$$I(\eta) = \mathrm{lin_K\{s - t \mid s, t \in S, (s, t) \in \eta\}}$$

is the kernel of the natural epimorphism $K[S] \longrightarrow K[S/\eta]$. So $K[S]/I(\eta) = K[S/\eta]$.

For a ring R, we denote by
$\mathcal{J}(R)$ its Jacobson radical
$\mathcal{B}(R)$ its prime radical.

Theorem

Assume $n \geq 4$. *Let* $M = S_n(Alt_n)$, $z = a_1 a_2 \cdots a_n \in M$ *and* $G = G_n(Alt_n)$. *Then*

(i) $C = \{1, a_1 a_2 a_1^{-1} a_2^{-1}\}$ *is a central subgroup of* G *and* G/C *is a free abelian group of rank* n. *Moreover* $D = \mathrm{gr}(a_i^2 \mid i = 1, \ldots, n)$ *is a central subgroup of* G *with* $G/(CD) \cong (\mathbb{Z}/2\mathbb{Z})^n$.

(ii) $K[G]$ *is a noetherian PI-algebra. If* K *has characteristic* $\neq 2$, *then* $\mathcal{J}(K[G]) = 0$. *If* K *has characteristic* 2, *then* $\mathcal{J}(K[G]) = (1 - a_1 a_2 a_1^{-1} a_2^{-1})K[G]$ *and* $\mathcal{J}(K[G])^2 = 0$.

(iii) *The element* z^2 *is central in* M *and* $z^2 M$ *is a cancellative ideal of* M *such that* $G \cong (z^2 M)\langle z^2 \rangle^{-1}$. *Furthermore,* $K[M/\rho]$ *is a noetherian PI-algebra and* $\mathcal{J}(K[M])$ *is nilpotent.*

(iv) $\mathcal{J}(K[M]) = I(\eta)$ *for a congruence* η *on* M *and* $\mathcal{J}(K[M])$ *is a finitely generated ideal.*

(v) $\mathcal{J}(K[M]) = 0$ *if and only if* n *be even and* $\mathrm{char}(K) \neq 2$.

Chinese algebras (J.Jaszunska, J.O)

For a positive integer n we consider the monoid $M = \langle a_1, \dots, a_n \rangle$ (called Chinese monoid) defined by the relations

$$a_j a_i a_k = a_j a_k a_i = a_k a_j a_i \text{ for } i \leq k \leq j.$$

It is called the Chinese monoid of rank n. It is known that every element of M has a unique presentation of the form
$x = b_1 b_2 \cdots b_n$, where

$$\begin{aligned} b_1 &= a_1^{k_{11}} \\ b_2 &= (a_2 a_1)^{k_{21}} a_2^{k_{22}} \\ b_3 &= (a_3 a_1)^{k_{31}} (a_3 a_2)^{k_{32}} a_3^{k_{33}} \\ &\dots \\ b_n &= (a_n a_1)^{k_{n1}} (a_n a_2)^{k_{n2}} \cdots (a_n a_{n-1})^{k_{n(n-1)}} a_n^{k_{nn}}, \end{aligned}$$

with all exponents nonnegative (J. Cassaigne, M. Espie, D. Krob, J.-C. Novelli and F. Hivert, 2001). We call it the canonical form of the element $x \in M$.

In particular, $M = M_n$ has polynomial growth of degree $n(n+1)/2$.

Combinatorial properties of M were studied in detail in [CEKNH].

If $n = 2$ then $M = \langle a, b : ba \text{ is central } \rangle$ and the structure of $K[M_2]$ is easy to understand.

Theorem

(i) *ba is a central and regular element in $K[M_2]$.*

(ii) *$M_2\langle ba\rangle^{-1} \cong B \times \mathbb{Z}$, where $B = \langle p, q : qp = 1\rangle$, the so called bicyclic monoid.*

(ii) *$K[B]$ is primitive and $J = K[B](1 - pq)K[B]$ is an ideal of $K[B]$ such that $K[B]/J \cong K[x, x^{-1}]$. Moreover $J \cong M_\infty(K)$, the algebra of infinite matrices with finitely many nonzero entries.*

(iii) *$K[M_2]$ is prime and semiprimitive.*

Notice that $K[M]$ is not noetherian and it does not satisfy any polynomial identity.

Key tool in case $n \geq 3$ - minimal prime ideals

Theorem

(i) *Minimal primes in $K[M]$ are of the form $I(\eta)$ for certrain certain homogeneous congruences on M.*

(ii) *They are in a one to one correspondence with diagrams of certain special type.*

(iii) *There are finitely many such ideals. Their number is equal to T_n, where the Tribonacci sequence is defined by the linear recurrence*
$T_0 = T_1 = T_2 = 1, \;\; T_{n+1} = T_n + T_{n-1} + T_{n-2}$ *for* $n \geq 2$.

(iv) *We have $\mathcal{J}(K[M]) = \mathcal{B}(K[M]) \neq 0$.*

(v) *The monoid M embeds into the algebra $K[M]/\mathcal{B}(K[M])$.*

Corollary

(i) *A new representation of M as a submonoid of the product $B^d \times \mathbb{Z}^e$ for some $d, e > 1$, where B stands for the bicyclic monoid, is derived. Namely, there exists an embedding $M \hookrightarrow \mathbb{N}^c \times (B \times \mathbb{Z})^d$, where $c + 2d = nT_n$.*

(ii) *Consequently (using a result of Adian), M satisfies the identity.*

$$xy^2xxyxy^2x = xy^2xyxxy^2x.$$

Work in progress: $\mathcal{J}(K[M])$ is nilpotent.

Idea of the proof.

1. certain equalities of the form $\alpha K[M]\beta = 0$ for $\alpha, \beta \in K[M]$ are established. They are used to introduce two finite families of ideals of $K[M]$ such that every prime ideal P of $K[M]$ contains one of these ideals. Each of these ideals is of the form $I(\rho)$ for a congruence ρ on M. It is also shown that ρ is a homogeneous congruence on M, which means that $(s,t) \in \rho$ implies that s, t have equal length.

2. A much more involved construction allows us to continue this process by showing that every prime P contains an ideal of the form $I(\rho_2)$ for some homogeneous congruence ρ_2 containing ρ. Continuing this process, we construct a finite tree D whose vertices correspond to certain homogeneous congruences on M and such that $\rho \subseteq \rho'$ if the vertex in D corresponding to ρ is above that corresponding to ρ'. Moreover, ideals corresponding to vertices lying in different branches of the tree D are incomparable under inclusion.

3. The leaves of this tree correspond to prime ideals of $K[M]$.

4. The main result shows that the leaves of D are in a one to one correspondence with the minimal prime ideals of $K[M]$. The proof provides us with a procedure to construct every such prime P. In particular, every minimal prime P has a remarkable form $P = I(\rho_P)$, where ρ_P is the congruence on M defined by $\rho_P = \{(s,t) \in M \times M \colon s - t \in P\}$. In particular, $K[M]/P \simeq K[M/\rho_P]$, so $K[M]/P$ inherits the natural $\mathbb{Z}$-gradation and therefore this algebra is again defined by a homogeneous semigroup presentation.

5. The construction implies that every M/ρ_P is contained in a product $B^i \times \mathbb{Z}^j$ for some i, j, where B is the bicyclic monoid.

6. We then show that M embeds into the product $\prod_P K[M]/P$, where P runs over the set of all minimal primes in $K[M]$. Hence M embeds into some $B^d \times \mathbb{Z}^e$. However, the algebra $K[M]$ is not semiprime if $n \geq 3$.

7. The description of minimal primes P of $K[M]$ allows us to prove that every $K[M]/P$ is semiprimitive. Consequently, $\mathcal{B}(K[M]) = \mathcal{J}(K[M])$.

Algebras yielding solutions of the quantum Yang-Baxter equation

Definition

Let X be a nonempty set and $r : X^2 \longrightarrow X^2$ a bijective map. One says that r is a set theoretic solution of the Yang-Baxter equation if it satisfies on X^3 the braid relation

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23}$$

where r_{ij} stands for the mapping $X^3 \longrightarrow X^3$ obtained by application of r in the components i, j.
The solution is often denoted by (X, r).

Denote by $\tau : X^2 \longrightarrow X^2$ the map defined by $\tau(x, y) = (y, x)$. It can be verified that r is a set theoretic solution of the Yang-Baxter equation if and only if $R = \tau \circ r$ is a solution of the quantum Yang-Baxter equation, that is,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where R_{ij} denotes R acting on the i-th and j-th component.
The quantum Yang-Baxter equation appeared in a paper by Yang in statistical mechanics and it turned out to be one of the basic equations in mathematical physics. It lies at the foundations of the theory of quantum groups. One of the important problems is to discover all solutions of the quantum Yang-Baxter equation.
Drinfeld (1992) posed the question of finding all set theoretic solutions.

We shall assume $X = \{x_1, \dots, x_n\}$ is a finite set. For a solution (X, r), let

$$\sigma_i, \gamma_i \colon \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$$

be the maps defined by

$$r(x_i, x_k) = (x_{\sigma_i(k)}, x_{\gamma_k(i)}).$$

Definition

Let $X = \{x_1, x_2, \dots, x_n\}$, with $n > 1$. A set theoretic solution (X, r) is called:

(1) involutive if $r^2 = id_{X \times X}$,

(2) left (right) non-degenerate if all maps σ_i (all γ_i, respectively) are bijections,

(3) square free if $r(x_i, x_i) = (x_i, x_i)$ for every i.

Theorem (Jespers, JO)

An involutive solution (X, r) is left non-degenerate if and only if it is right non-degenerate.

Related monoids, groups and algebras

(X, r) will stand for an involutive non-degenerate solution.

Definition

- Let $S(X, r)$ and $G(X, r)$ be the monoid and the group defined by the presentation

$$\langle x_1, \ldots, x_n \mid x_i x_j = x_k x_l \text{ if } r(x_i, x_j) = (x_k, x_l) \rangle.$$

  It is called the structure monoid, resp. the structure group, of the solution (X, r), or the monoid and the group of I-type associated to the solution (X, r).
- $K[S(X, r)]$ is the algebra defined by the above presentation.

Proposition (Gateva-Ivanova, Van den Bergh)

Let (X, r) be an involutive non-degenerate solution. Then $S(X, r)$ and $G(X, r)$ have a presentation with $n(n-1)/2$ (quadratic) relations.

Several nice properties of the algebra, similar to the polynomial algebra, were proved (using methods of Tate and Van den Bergh).

Theorem (Gateva-Ivanova,Van den Bergh)

Assume S of I-type. Then the global dimension of $K[S]$ is n, $K[S]$ is a maximal order in a division ring, it satisfies some homological "regularity" conditions (it is Cohen-Macaulay, Artin-Schelter regular, and a Koszul algebra).

The homological properties were used to show that the algebra $K[S]$ is a domain. Even more:

Theorem (Gateva-Ivanova,Van den Bergh)

$K[S]$ is a Noetherian PI-domain. The growth function of $K[S]$ is the same as of the polynomial ring in n indeterminates. In particular, $S = S(X, r)$ has a group of quotients. It follows that $SS^{-1} = G(X, r)$. Moreover, $G(X, r)$ is a central localization of S.

Examples of solutions

Simplest examples

Example

- $S = \mathrm{FaM}_n$. Then $X = \{x_1, \ldots, x_n\}$ and $\sigma_i = id$ for every i. So $G_r = \{id\}$. $G(X, r) = \mathrm{Fa}_n$, free abelian group of rank n. This is called a trivial solution.
- $S = \langle x, y \mid x^2 = y^2 \rangle$. Then $S \subseteq \mathrm{FaM}_2 \rtimes \mathrm{Sym}_2$ and $\sigma_i = (12)$ for $i = 1, 2$. So $G_r = \mathrm{Sym}_2$. Also $G(X, r)$ is the group defined by the same (group) presentation.
- $S = \langle x_1, x_2, x_3 \mid x_1x_2 = x_2x_1, x_3x_1 = x_2x_3, x_3x_2 = x_1x_3 \rangle$. Then $\sigma_1 = \sigma_2 = id$ and $\sigma_3 = (12)$. So $G_r = \langle (12) \rangle$ and $G(X, r) \cong \mathrm{Fa}_2 \rtimes \mathrm{Sym}_2$.

Certain special classes of solutions have been constructed, however it turns out to be very hard to find new classes.

Theorem (Rump; Gateva-Ivanova; Jespers and JO)

Let $S = S(X, r)$ be a square free monoid of I-type. Then

- *after renumbering the generators* $S = \{x_1^{i_1} \cdots x_n^{i_n} \mid i_j \geq 0\}$, *("normal form" of elements of S).*
- *Every height one prime ideal of $K[S]$ that intersects S is of the form $f_i K[S]$, where f_i is a normal element of S that is a product (in some order) of all $x_j \in X_i$.*
- *The other height one primes of $K[S]$ are of the form $Q \cap K[S]$, where Q is a height one prime of $K[G]$, $G = SS^{-1}$ (compare: K.Brown on primes in maximal orders $K[G]$).*

References

[CEKNH] J. Cassaigne, M. Espie, D. Krob, J.-C. Novelli and F. Hivert, The Chinese monoid, Int. J. Algebra and Comput. 11 (2001), 301–334.
[CJO1] F. Cedó, E. Jespers and J. Okniński, Finitely presented algebras and groups defined by permutation relations, J. Pure Appl. Algebra, in press.
[CJO2] F. Cedó, E. Jespers and J. Okniński, Algebras and groups defined by permutation relations of alternating type.
[D] V.G. Drinfeld, On some unsolved problems in quantum group theory, Lect. Notes in Math., vol. 1510, pp. 1–8, 1992.
[G1] T. Gateva-Ivanova, A combinatorial approach to the set-theoretic solutions of the Yang-Baxter equation, J. Math. Phys. 45 (2004), 3828–3858.

[GV] T. Gateva-Ivanova and M. Van den Bergh, Semigroups of I-type, J. Algebra 206 (1998), 97–112.
[JaO] J. Jaszuńska and J. Okniński, Structure of the Chinese algebras, in preparation.
[JO2] E. Jespers and J. Okniński, Noetherian Semigroup Algebras, Springer, 2007.
[R] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, Adv. Math. 193 (2005), 40–55.
[TV] J. Tate and M. Van den Bergh, Homological properties of Sklyanin algebras, Invent. Math. 124 (1996), 619–647.