Nilpotency of group ring units symmetric with respect to an involution

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Arithmetic of group rings and related structures

Outline

1. Involutions in group rings
   - Classical and K-linear involutions
   - Symmetric and skew-symmetric elements
   - Symmetric and unitary units
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   - Main question for symmetric units
3. Group identities for symmetric units
   - The classical involution: an overview
   - K-linear involutions: the main results

Canonical and K-linear involutions

Let G be a group endowed with an involution $\star$.
Let us consider the K-linear extension of $\star$ to KG by setting

$$\Big(\sum_{g\in G} a_g g\Big)^{\star} := \sum_{g\in G} a_g g^{\star}.$$

This extension, which we denote again by $\star$, is an involution of KG wich fixes the ground field K elementwise.
As well-known, any group G has a natural involution which is given by the map $* : g \mapsto g^{-1}$.

Definition

Let KG be the group algebra of a group G over a field K. If G is endowed with an involution $\star$, its linear extension to the group algebra KG is called a *K-linear involution* of KG. In particular, if $\star = *$ the induced involution is called *the classical involution*.

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Main question for symmetric and skew-symmetric elements

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If A is an arbitrary algebra with involution, let us define in the same manner A^+ and A^-.

Theorem [Amitsur,1968]

Let A be an algebra with involution. If A^+ or A^- satisfies a polynomial identity, then A satisfies a polynomial identity.

Lie properties for KG^+ and KG^-

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Assume that KG is endowed with a K-linear involution. If KG^+ and/or KG^- satisfies a Lie identity, what can you say about KG?

Lie nilpotency for KG^+ and KG^-

Theorem [Giambruno-Sehgal, 1993]

Let KG be the group algebra of a group G without 2-elements over a field K of characteristic $p \neq 2$ endowed with the classical involution. Then KG^+ or KG^- is Lie nilpotent if, and only if, KG is Lie nilpotent.

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- The previous result does not hold without the assumption on the order of the elements of G.
- Lee (1999) completed the classification with regard to KG^+ by showing that the result is heavily effected by the presence of Q_8 in G.
- Giambruno-Sehgal (2007) completed the classification with regard to KG^-.

Theorem [Giambruno-Polcino Milies-Sehgal, 2009]

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- Lee-Sehgal-Spinelli (2009) completed the classification for arbitrary G.
- Giambruno-Polcino Milies-Sehgal (2010) studied the question for KG^-, when G is a torsion group without 2-elements.

Lie n-Engel condition for KG^+ and KG^-

Theorem [Lee, 2000]

Let KG be the group algebra of a group G with no 2-elements over a field K of characteristic $p \neq 2$ endowed with the classical involution. Then KG^+ or KG^- is Lie n-Engel, for some n, if, and only if, KG is Lie m-Engel, for some m.

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Remark

Let KG be the group algebra of a group G over a field K of characteristic $p \neq 2$. If KG^- is Lie solvable, then KG^+ is Lie solvable and

$$dl_L(KG^+) \leq dl_L(KG^-) + 1.$$

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Theorem [Lee-Sehgal-S., 2009]

Let KG be the group algebra of a group G without 2-elements over a field K of characteristic $p \neq 2$ endowed with the classical involution. If P is finite, then KG^+ is Lie solvable if, and only if, KG is Lie solvable.

Theorem [Lee-Sehgal-S., 2009]

Let K be a field of characteristic $p > 2$, and let G be a group such that P contains an infinite subgroup of bounded exponent, and G contains no nontrivial elements of order dividing $p^2 - 1$. Then the following statements are equivalent:

- KG^- is Lie solvable;
- KG^+ is Lie solvable;
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Theorem [Lee-Sehgal-S., 2009]

Let K be a field of characteristic $p > 2$ and G a group containing elements of infinite order, but no 2-elements. If KG^+ is Lie solvable, then KG is Lie solvable.

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In this spirit is the following

Third Question

Do the Lie properties satisfied by the symmetric elements reflect the group identities satisfied by the symmetric units of the group ring?

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Theorem [Giambruno-Sehgal-Valenti, 1998]

Let KG be the group algebra of a torsion group G over an infinite field K of characteristic $p \neq 2$.

(a) If $p = 0$, $\mathcal{U}^+(KG)$ is GI if, and only if, G is either abelian or Hamiltonian 2-group.

(b) If $p > 2$, $\mathcal{U}^+(KG)$ is GI if, and only if, KG is PI and either $Q_8 \not\subseteq G$ and G' is of bounded exponent p^k for some $k \geq 0$ or $Q_8 \subseteq G$ and
- P is a normal subgroup of G and G/P is a Hamiltonian 2-group;
- G is of bounded exponent $4p^s$ for some $s \geq 0$.

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- Sehgal-Valenti (2006) studied the non-torsion case.

Special group identities

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Theorem [Lee, 2003]

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$$\mathcal{U}^+(KG) \text{ is nilpotent} \iff KG^+ \text{ is Lie nilpotent.}$$

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Theorem [Lee-Spinelli, 2010]

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Theorem [Lee-Spinelli, 2009]

Let KG be the group algebra of a torsion group G over an infinite field K of characteristic $p \neq 2$ endowed with the classical involution. If P is infinite and G does not contain elements whose order divides $p^2 - 1$,

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Theorem [Giambruno-Polcino Milies-Sehgal, 2009]

Let KG be the group algebra of a torsion group G over an infinite field K of characteristic $p \neq 2$ endowed with a K-linear involution. Then $\mathcal{U}^+(KG)$ is GI if, and only if,

(a) KG is semiprime and G is either abelian or an SLC-group, or

(b) KG is not semiprime, P is a normal subgroup of G, G has a p-abelian normal subgroup of finite index and either
  - G' is a p-group of bounded exponent or
  - G/P is an SLC-group and G contains a normal $*$-invariant p-subgroup B of bounded exponent such that P/B is central in G/P and the induced involution acts as the identity on P/B.

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Definition

A group G endowed with an involution $\star$ is said to be a *special LC-group*, or *SLC-group*, if it is an LC-group, it has a unique nonidentity commutator z and, for all $g \in G$, we have $g^\star = g$ if $g \in \zeta(G)$ and, otherwise, $g^\star = zg$.

Theorem [Jespers-Ruiz Marin, 2006]

Let R be a commutative ring of characteristic different from 2, and G a nonabelian group endowed with an involution $\star$. Then RG^+ is commutative if, and only if, G is an SLC-group.

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- Let $N \trianglelefteq G$ and $\star$-invariant. If $\mathcal{U}^+(KG)$ satisfies w, then $\mathcal{U}^+(K(G/N))$ satisfies w.

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- Let $N \trianglelefteq G$ and $\star$-invariant. If $\mathcal{U}^+(KG)$ satisfies w, then $\mathcal{U}^+(K(G/N))$ satisfies w.
- $\mathcal{U}^+(K(G/P))$ is nilpotent and $K(G/P)$ is semiprime.
- By [GPMS] G/P is abelian or G/P is an SLC-group.

G finite

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By [GPMS] G is locally finite. Hence it is relevant to study the case in which G is finite.

Lemma

Let G be a finite group. If $\mathcal{U}^+(KG)$ is nilpotent, then G is nilpotent and G/P is either abelian or an SLC-group.

G/P abelian

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Theorem [Lee-Sehgal-S., 2010]

Let G/P be abelian. If $\mathcal{U}^+(KG)$ is nilpotent, then G is nilpotent and p-abelian (hence, $\mathcal{U}(KG)$ is nilpotent).

G/P is an SLC-group

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Theorem [Lee-Sehgal-S., 2010]

Let G/P be an SLC-group. Then $\mathcal{U}^+(KG)$ is nilpotent if, and only if, G is nilpotent and G has a finite normal $*$-invariant p-subgroup N such that G/N is an SLC-group.

Main Theorem

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Theorem [Lee-Sehgal-S., 2010]

Let K be an infinite field of characteristic $p > 2$ and G a torsion group having an involution $*$, and let KG have the induced involution. Suppose that $\mathcal{U}(KG)$ is not nilpotent. Then $\mathcal{U}^+(KG)$ is nilpotent if, and only if, G is nilpotent and G has a finite normal $*$-invariant p-subgroup N such that G/N is an *SLC*-group.

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By using the results of [LSS1], one has

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