Group of units

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Let R be a ring with unity and $U(R)$ its group of units. Let

$$\begin{aligned} \Delta U &= \{a \in U(R) \quad | \quad [U(R) : C_{U(R)}(a)] < \infty\}, \\ \bigtriangledown(R) &= \{a \in R \quad | \quad [U(R) : C_{U(R)}(a)] < \infty\}, \end{aligned}$$

which are called the *FC*-radical of $U(R)$ and *FC*-subring of R, respectively. The *FC*-subring $\bigtriangledown(R)$ is invariant under the automorphisms of R and contains the center of R.

The investigation of the *FC*-radical ΔU and the *FC*-subring $\nabla(R)$ was proposed by *H. Zassenhaus* (+ *S. K. Sehgal*)

They described the *FC*-subring of a $\mathbb{Z}$-order as a unital ring with a finite $\mathbb{Z}$-basis and a semisimple quotient ring.

V. Bovdi. Twisted group rings whose units form an FC-group. *Canad. J. Math.*, 47(2):274–289, 1995.

An infinite subgroup H of $U(R)$ is said to be an ω-subgroup if the left annihilator of each nonzero Lie commmutator $[x, y] = xy - yx$ in R contains only a finite number of elements of the form $1 - h$, where $h \in H$ and $x, y \in R$.

$U(R)$ of the following infinite rings R contain ω-subgroups:

(i) Let A be an algebra over an infinite field F. Then the subgroup $U(F)$ is an ω-subgroup.

(ii) Let $R = KG$ be the group ring of an infinite group G over the ring K. Since the left annihilator of any $z \in KG$ contains only a finite number of elements of the form $g - 1$, where $g \in G$, so G is an ω-subgroup.

(iii) Let $R = F_\lambda G$ be an infinite twisted group algebra over the field F with an F-basis $\{u_g \mid g \in G\}$. Then the subgroup

$$\overline{G} = \{\lambda u_g \mid \lambda \in U(F), g \in G\}$$

is an ω-subgroup.

(iv) If A is an algebra over a field F, and A contains a subalgebra D such that $1 \in D$ and D is either an infinite field or a skewfield, then every infinite subgroup of $U(D)$ is an ω-subgroup.

Theorem

Let R be an algebra over a field F such that the group of units $U(R)$ contains an ω-subgroup, and let $\nabla(R)$ be the FC-subalgebra of R. Then the set of algebraic elements A of $\nabla(R)$ is a locally finite algebra, the Jacobson radical $\mathfrak{J}(A)$ is a central locally nilpotent ideal in $\nabla(R)$ and $A/\mathfrak{J}(A)$ is commutative.

Theorem

Let R be an algebra over a field F such that the group of units $U(R)$ contains an ω-subgroup. Then

- (i) *the elements of the commutator subgroup of $t(\Delta U)$ are unipotent and central in ΔU;*
- (ii) *if all elements of $\nabla(R)$ are algebraic then ΔU is nilpotent of class 2;*
- (iii) *ΔU is a solvable group of length at most 3, and the subgroup $t(\Delta U)$ is nilpotent of class at most 2.*

Theorem

Let R be an algebra over an infinite field F. Then

(i) *any algebraic unit over F belongs to the centralizer of $\nabla(R)$;*

(ii) *if R is generated by algebraic units over F, then $\nabla(R)$ belongs to the center of R.*

Theorem

Let R be an algebra over an infinite field F, and let $t(\Delta U)$ be the torsion subgroup of ΔU. Then

(i) *$t(\Delta U)$ is abelian and ΔU is a nilpotent group of class at most 2;*

(ii) *if every unit of R is an algebraic element over F, then ΔU is central in $U(R)$.*

Let G be a group

KG be the group ring over a commutative ring K

$U(KG)$ be the group of units of KG

Let (X, ρ) be a metric space with a metric ρ. For any $a, b, c \in X$, the Gromov product $\langle b, c\rangle_a$ of b and c with respect to $a \in X$ is defined as

$$\langle b, c\rangle_a = \frac{1}{2}(\rho(b, a) + \rho(c, a) - \rho(b, c)).$$

The metric space is called δ-hyperbolic ($\delta \geq 0$) if

$$\langle a, b\rangle_d \geq \min\left\{\langle a, c\rangle_d, \langle b, c\rangle_d\right\} - \delta \qquad (a, b, c, d \in X).$$

Let G be a finitely generated group and let S be a finite set of generators for G. The Cayley graph $\mathfrak{C}(G, S)$ of the group G with respect to the set S is the metric graph whose vertices are in one-to-one correspondence with the elements of G. Their edges (labeled s) of length 1 are joining g to gs for each $g \in G$ and $s \in S$. The group G is called *hyperbolic*

(see M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.)

if its Cayley graph $\mathfrak{C}(G, S)$ is a δ-hyperbolic metric space for some $\delta \geq 0$. It is well known (see M. Gromov) that this definition does not depend on the choice of the generating set S.

Problem

The natural question is the following one: When does the group of units $U(KG)$ of the group ring KG of a group G over the commutative ring K with unity is hyperbolic.

For several particular cases this problem was solved in

- $K = \mathbb{Z}$ and G is polycyclic by finite
  S. O. Juriaans, I. B. S. Passi, and D. Prasad. Hyperbolic unit groups. *Proc. Amer. Math. Soc.*, 133(2):415–423 (electronic), 2005.
- G is a finite group, K the ring of integers of a quadratic extension $\mathbb{Q}[\sqrt{d}]$ of the field $\mathbb{Q}$ of rational numbers, where d is a square-free integer $d \neq 1$.
  S. O. Juriaans, I. B. S. Passi, and A. C. Souza Filho. Hyperbolic unit groups and quaternion algebras. *Proc. Indian Acad. Sci. Math. Sci.*, 119(1):9–22, 2009.
- G is a finite group K is a field of a positive characteristic.

  E. Iwaki and S. O. Juriaans. Hypercentral unit groups and the hyperbolicity of a modular group algebra. *Comm. Algebra*, 36(4):1336–1345, 2008.

- Iwaki E, Juriaans S O and Souza Filho A C, Hyperbolicity of semigroup algebras, J. Algebra 319(12) (2008) 5000 - 5015
- Juriaans S O, Polcino Milies C and Souza Filho A C, Alternative algebras with quasihyperbolic unit loops, http://arXiv.org/abs/0810.4544

The idea of proof is the using the following properties of hyperbolic groups

M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

Theorem

If G is a hyperbolic group, then:

(i) $C_\infty \times C_\infty$ *does not embed as a subgroup of* G*;*

(ii) *if* $g \in G$ *has infinite order, then* $[C_G(g) : \langle g \rangle]$ *is finite;*

(iii) *torsion subgroups of* G *are finite of bounded order.*

(iv) G *is virtually free if and only if its boundary has dimension zero;*

(v) *if* G *is quasi-isometric to a free group, then* G *is virtually free. If, moreover,* G *is torsion-free, then it is free.*

Here we give a more general characterization.

Theorem

Let G be a group, such that the torsion part $t(G) \neq \{1\}$. Let K be a commutative ring of $char(K) = 0$ with unity. If the group of units $U(KG)$ of the group ring KG is hyperbolic, then one of the following conditions holds:

(i) $G \in \{C_5, C_8, C_{12}\}$ *or* G *is finite abelian of* $exp(G) \in \{2, 3, 4, 6\}$;

(ii) G *is a Hamiltonian* 2*-group;*

(iii) $G \in \{H_{3,2}, H_{3,4}, H_{4,2}, H_{4,4}\}$, *where* $H_{s,n} = \langle a, b \mid a^s = b^n = 1, a^b = a^{-1} \rangle$;

(iv) $G = t(G) \rtimes \langle \xi \rangle$, *where* $t(G)$ *is either a finite Hamiltonian* 2*-group or a finite abelian group of* $exp(t(G)) \in \{2, 3, 4, 6\}$ *and* $\langle \xi \rangle \cong C_\infty$. *Moreover, if* $t(G)$ *is abelian, then conjugation by* ξ *either inverts all elements from* $t(G)$ *or leave them fixed.*

Theorem

Let KG be the group algebra of a group G over a field K of positive characteristic, such that the torsion part $t(G) \neq \{1\}$. The group of units $U(KG)$ is hyperbolic if and only if when K is a finite field and G is a finite group.