Constructing Maximal Subgroups of Classical Groups

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Introduction

By max(G) we denote the set of maximal subgroups of G, up to G-conjugacy.

G is *almost simple* if there exists a nonabelian simple group T with $T \trianglelefteq G \leq \mathrm{Aut}(T)$.

If MAGMA knows the maximal subgroups of all almost simple groups whose socles are composition factors of a permutation group G, then it can compute max(G), the subgroups of G and the automorphism group of G. [Cannon/Holt]

Almost Simple Groups

$A_n \trianglelefteq$ G $\leq$ S_n, n $>$ 6.
Theory of max(G) well understood. [O'Nan–Scott Theorem, Liebeck/Praeger/Saxl]. max(G) known explicitly up to $n = 1000$. Current work of CMRD will extend to $n = 2500$.

Use black box constructive recognition [Bratus/Pak or Beals *et al*]: make maximals in natural representation. Construct the intransitive/imprimitive maximals generically, then use database of primitive groups for the rest.

T $\trianglelefteq$ G $\leq$ Aut(T), T sporadic.
With exception of the Monster, the maximal subgroups are known.

If T has "moderate" permutation representation, then standard generators are known, as are words for the generators of each maximal subgroup. [Wilson *et al*].

T $\trianglelefteq$ G $\leq$ Aut(T), T exceptional.
Families of maximal subgroups reasonably well understood. max(G) known explicitly in small cases.

Very few low degree permutation representations, so can treat as sporadic.

Classicals
Up to now, the classical groups have been treated as sporadic.

This talk describes new methods for constructing max(G) for almost simple groups G with socle PSL(d, q), PSp(d, q) or PSU(d, q).

We omit the orthogonal groups for now: not many low degree permutation representations, and many different cases.

Algorithmic Overview

Input: An almost simple permutation group G, with $T := \mathsf{Soc}(G) \cong \mathsf{PSL}(d,q)$, $\mathsf{PSp}(d,q)$ or $\mathsf{PSU}(d,q)$.
Output: A set $\{H \cap T : H \leq_{\mathsf{max}} G, T \not\leq H\}$, up to G-conjugacy.

1. We use constructive recognition to find an isomorphism $\lambda : G \to S$, where S is the natural permutation representation. [Kantor/Seress, Brooksbank]

2. We use a homomorphism $\mu : C \to \mathsf{Soc}(S)$, where $C \in \{\mathsf{SL}(d,q), \mathsf{Sp}(d,q), \mathsf{SU}(d,q)\}$.

3. We construct the maximal subgroups of $\mathsf{SL}(d,q)$, $\mathsf{Sp}(d,q)$ or $\mathsf{SU}(d,q)$ in their natural representation, up to conjugacy in $\mathsf{GL}(d,q)$, $\mathsf{GSp}(d,q)$, or $\mathsf{GU}(d,q)$. [More on this later]

With a small number of exceptions:
(a) If $H \leq_{\mathsf{max}} G$ then $H \cap T \leq_{\mathsf{max}} T$
(b) If two maximal subgroups of G are conjugate, they are conjugate in $\mathsf{PGL}(d,q)$, $\mathsf{PGSp}(d,q)$ or $\mathsf{PGU}(d,q)$ [Aschbacher].

4. We use information about G/T to construct the appropriate G-conjugacy class representations [Kleidman/Liebeck].

5. Finally, we pull back the subgroups to G.

Maximal subgroups of classical groups

Aschbacher's theorem divides the maximal subgroups of a classical group G, other than 8-dimensional orthogonal groups of $+$ type, into various classes.

Groups in all but the last class of this classification are called *geometric*. The final class is denoted $\mathcal{S}$ and roughly consists of absolutely irreducible groups that are almost simple modulo scalars.

A group which is a maximal member of one of these classes is called *AS-maximal*. For most dimensions it is known when a geometric AS-maximal is maximal [Kleidman/Liebeck].

The groups in $\mathcal{S}$ are known for $d \leq 250$. [Hiss/Malle, Lübeck].

Some results

Our algorithms construct geometric AS-maximals in the natural representation.

Input: Form type (L, S or U), d, q, Aschbacher class.
Output: Sets of generating matrices for the maximal subgroups in that Aschbacher class.

Theorem Let $K = \mathrm{SL}(d,q)$, $\mathrm{Sp}(d,q)$ or $\mathrm{SU}(d,q)$ in its natural representation. Then generators for all of the geometric maximal subgroups of K, up to conjugacy in $\mathrm{GL}(d,q)$, $\mathrm{GSp}(d,q)$ or $\mathrm{GU}(d,q)$ can be constructed in time $O(d^{3+\epsilon}\log^3 q)$, for any real $\epsilon > 0$. $\square$

We ensure that the groups that we construct preserve our chosen classical form.

Theorem Let $G \leq \mathsf{GL}(d,q)$ be absolutely irreducible and preserve a classical form, and let F be a matrix of a classical form of the same type. Then in time $O(d^3 \log q)$ a matrix X can be constructed such that G^X preserves a form with matrix F. $\square$.

An implementation

We have implemented most of our algorithms in MAGMA.

MAGMA V2.10 can compute max(G) for any almost simple G with socle:

PSL($2, p^e$)	p prime, $e \leq 3$	
PSL($3, p$)	PSU($3, p$)	PSp($4, p$)

MAGMA V2.11 will compute the maximal subgroups of any almost simple G with socle PSL($2, q$), PSL($3, q$), PSL($4, q$), PSL($5, p$), for q any prime power and p prime.

When constructive recognition implementations for unitary and symplectic groups become available, MAGMA will be able to compute max(G) for any almost simple G with socle PSU($3, q$) or PSp($4, q$) (q odd or $G \neq$ Aut(PSp($4, q$))).