Exam 3(March 24, 2003)

MAS4105 Linear Algebra I (F. Lübeck)

(1) Mark as true or false. (*One point* for each correct answer.)

(a) A homogeneous system of linear equations has no solutions if there are more equations than unknowns. true: ☐ false: ☐

(b) Any system of linear equations with less equations than unknowns has a solution. true: ☐ false: ☐

(c) A system of linear equations with an invertible (square) coefficient matrix has exactly one solution. true: ☐ false: ☐

(d) The set of solutions of a homogeneous system of linear equations with n unknowns over the field F is a subspace of $F^{n\times 1}$. true: ☐ false: ☐

(e) A system of linear equations $Ax = b$ has a solution if and only if A and the augmented matrix $(A|b)$ have the same rank. true: ☐ false: ☐

Now let V and W be vectorspaces over a field F, and let $\varphi:\ V \to W$ be a linear map.

(f) For all $v, v' \in V$ the equation $\varphi(v+v') = \varphi(v) + \varphi(v')$ holds. true: ☐ false: ☐

(g) If φ is invertible then its inverse φ^{-1} is also a linear map from V to W. true: ☐ false: ☐

(h) If β is a basis of V and $\psi:\ V \to W$ is another linear map with $\varphi(v) = \psi(v)$ for all $v \in \beta$ then $\varphi = \psi$. true: ☐ false: ☐

(i) φ is one-to-one if $\mathrm{Im}(\varphi) = \{0\}$. true: ☐ false: ☐

(j) If V and W are finite-dimensional and φ is an isomorphism, then $\dim V \geq \dim W$. true: ☐ false: ☐

(2) (*four points*) Find all solutions of the following system of linear equations.

$$\begin{array}{rcrcrcrcrcr}
2x_1 & + & 3x_2 & + & x_3 & + & 4x_4 & - & 9x_5 & = & 17\\
x_1 & + & x_2 & + & x_3 & + & x_4 & - & 3x_5 & = & 6\\
x_1 & + & x_2 & + & x_3 & + & 2x_4 & - & 5x_5 & = & 8\\
2x_1 & + & 2x_2 & + & 2x_3 & + & 3x_4 & - & 8x_5 & = & 14
\end{array}$$

(3) (*three points*) Prove or disprove that the following maps are linear.

$\varphi:\ \mathbb{R}^2 \to \mathbb{R}^3,\ (a,b) \mapsto (a+b, a-b, 2a+3b)$ $\qquad$ $\psi:\ \mathbb{R}^3 \to \mathbb{R}^3,\ (a,b,c) \mapsto (a^2, b^2, ab)$

For the linear maps among φ and ψ compute the kernel and nullity and the image and rank.

(4) (*three points*) Is there a linear map $\varphi:\ \mathbb{R}^{2\times 1} \to \mathbb{R}^{2\times 1}$ such that $\varphi(\begin{pmatrix}1\\1\end{pmatrix}) = \begin{pmatrix}1\\0\end{pmatrix}$ and $\varphi(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}1\\1\end{pmatrix}$? (Give a reason for your answer.) If the answer is yes, what is $\varphi(\begin{pmatrix}13\\5\end{pmatrix})$?