Final Exam (April 29, 2003)

MAS4105 Linear Algebra I (F. Lübeck)

(1) Mark as true or false. (*One point* for each correct answer.)

Let V be a vectorspace of dimension $n \in \mathbb{N}$ over a field F.

(a) The intersection of two subspaces of V is again a subspace of V. true: ☐ false: ☐

(b) The union of two subspaces of V is again a subspace of V. true: ☐ false: ☐

(c) A linearly independent subset of V with n elements is a generating set of V. true: ☐ false: ☐

(d) Any generating set of V contains at most n elements. true: ☐ false: ☐

(e) The span of any subset of V is a subspace of V. true: ☐ false: ☐

Let $A = \begin{pmatrix} -1 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$, $B = \begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_{3\times 4}(\mathbb{R})$
and $C = \begin{pmatrix} 1 & -1 & 3 \\ 3 & -1 & -1 \\ 0 & -3 & 1 \\ 0 & -3 & 4 \end{pmatrix} \in M_{4\times 3}(\mathbb{R})$. We write B^t for the transpose of B and I_3 for the 3×3 identity matrix over $\mathbb{R}$.

(f) The product AC is defined and has entry -4 in position $1, 1$. true: ☐ false: ☐

(g) The product CA is defined and has entry -2 in position $1, 1$. true: ☐ false: ☐

(h) BC and CB are both defined, but $BC \neq CB$. true: ☐ false: ☐

(i) $BB^t \in M_{4\times 4}(\mathbb{R})$. true: ☐ false: ☐

(k) $A^3 - A = A(A^2 - I_3) = (A^2 - I_3)A$. true: ☐ false: ☐

Now let F be any field. The linear equations in the following questions have coefficients in F.

(l) A system of linear equations with less equations than unknowns has always a solution. true: ☐ false: ☐

(m) The set of solutions of a homogeneous system of linear equations with m equations and n unknowns is a subspace of F^n. true: ☐ false: ☐

(n) A system of linear equations $Ax = b$ has a solution if and only if the rank of the augmented matrix $(A|b)$ is larger than the rank of A. true: ☐ false: ☐

(o) There are systems of linear equations which have no solution. true: ☐ false: ☐

(p) If S is the set of solutions of a system of linear equations $Ax = b$ and $s \in S$, then $(-s) + S$ is the set of solutions of the homogeneous system $Ax = 0$. true: ☐ false: ☐

Now let F be a field, V a vectorspace over F of dimension n with ordered basis β, and W a vectorspace over F of dimension m with ordered basis γ. Furthermore let $\varphi : V \to V$ and $\psi : V \to W$ be linear maps.

(q) $[\varphi]_\beta$ is an $n \times n$ matrix. true: ☐ false: ☐

(r) For $v \in V$ we have $[\psi \circ \varphi(v)]_\gamma = [\psi]_\beta^\gamma [\varphi]_\beta [v]_\beta$. true: ☐ false: ☐

(s) $[\varphi]_\beta$ is invertible if and only if φ is one-to-one. true: ☐ false: ☐

(t) If ψ is invertible then $m = n$. true: ☐ false: ☐

(u) $\varphi(0) = 0$. true: ☐ false: ☐

(2) (*three points*) Let $P(\mathbb{R})$ be the vectorspace of polynomials over $\mathbb{R}$. Show that the subset

$$W = \{a + ax + ax^3 \mid a \in \mathbb{R}\}$$

is a subspace of $P(\mathbb{R})$. What is the dimension of W?

(3) (*three points*) Let $V = \mathbb{R}^3$ and $S = \{(1,-1,0),(2,0,0),(4,-2,0),(-1,-1,-1)\} \subseteq V$. Show that S is a generating set of V. Find a subset of S that is a basis of V (don't forget the explanation).

(4) (*four points*) Compute the reduced row echelon form of the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 & 2 & 2 \\ -1 & -1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 4 \end{pmatrix} \in M_{3\times 5}(\mathbb{R}).$$

Determine a basis of the nullspace $N(A)$.

(5) (*three points*) Compute the inverse of the matrix $A = \begin{pmatrix} 0 & -1 & 1 \\ 2 & -8 & 5 \\ -1 & 4 & -3 \end{pmatrix}$.

(6) (*three points*) Find all solutions of the following system of linear equations.

$$\begin{array}{rcrcrcrcl} -4x_1 & + & 3x_2 & - & x_3 & + & x_4 & = & 3 \\ -2x_1 & + & 5x_2 & - & 5x_3 & + & 3x_4 & = & 5 \\ x_1 & + & x_2 & - & 2x_3 & + & x_4 & = & 1 \end{array}$$

(7) (*four points*) Consider $V = \mathbb{R}^3$. Let α be the standard ordered basis of V and $\beta = ((1,1,1),(1,1,0),(1,0,0))$. Show that β is also an ordered basis of V. Determine $[Id_V]_\beta^\alpha$ and $[Id_V]_\alpha^\beta$. What is $[(3,2,1)]_\alpha$ and $[(3,2,1)]_\beta$? How are these related?