The Brauer trees of the exceptional Chevalley groups of type E_6 *

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1 Introduction

In [11], Feit has shown that most trees do not occur as Brauer trees for cyclic blocks of finite groups. In fact, he essentially reduced the problem of finding all occurring Brauer trees to the case of simple groups and their covering groups. The Brauer trees of the classical groups are known by the work of Fong and Srinivasan [14, 13]. Those for the alternating groups can easily be determined by the results in [21]. Up to a few exceptions, the trees for the sporadic simple groups and their covering groups have been determined in [20]. Some series of exceptional groups of Lie type have also been investigated [27, 28, 29, 19, 15, 31]. Finally, some exceptional covering groups of alternating groups and groups of Lie type have been considered [22, 30].

In this note we determine the Brauer trees for the exceptional Chevalley groups of type E_6. The only simple groups that remain to be considered are the exceptional groups of Lie type F_4, ${}^2E_6$, E_7 and E_8.

With two exceptions, the trees occurring in unipotent blocks of E_6 have already been known (see [16]) or are easy to determine. The problem which can not be solved from the results in [16] is to locate a pair of complex conjugate cuspidal unipotent characters on the trees. The first of these remaining problems could be solved by viewing our group as Levi subgroup of a Chevalley group of type E_8. In order to solve the second of these problems, we had to compute some scalar products of unipotent characters with a tensor product of unipotent characters. To be able to do this, the characters usually have to be given explicitly as class functions. We can do with somewhat less information. We show that in the groups we consider the tensor product of two uniform functions is again uniform. This allows us to compute the relevant scalar products from a knowledge of the Deligne-Lusztig characters $R_T^G(1)$. These can be computed explicitly with the tools provided by the CHEVIE-system [17].

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Similar methods could and will be tried for finding the trees for the remaining exceptional groups of Lie type. We give a conjecture relevant to the primes dividing the order of the Coxeter torus.

2 Computation of some unipotent characters

We shall need the values of some unipotent characters of a finite Chevalley group of type E_6. To describe the necessary computations, we begin by defining a reductive algebraic group G, defined over an algebraically closed field k, such that G and its dual group G^* have the following properties: They are of type E_6, have connected centres, simply-connected derived subgroups and central factor groups of adjoint type. The group G/k is determined up to isomorphism by a root datum $(X, \Phi, Y, \Phi^\vee)$ (in the notation of [6, 1.9]). We assume to have chosen $\mathbb{Z}$-bases of the free abelian groups X and Y which are dual with respect to the natural pairing $X \times Y \to \mathbb{Z}$. Then we can describe the root system Φ and the coroot system $\Phi^\vee$ by giving a set $\Delta = \{\alpha_1, \ldots, \alpha_6\}$ of simple roots and the set $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_6^\vee\}$ of corresponding coroots with respect to the chosen bases. We write $\alpha_1, \ldots, \alpha_6$ as rows of a matrix A and $\alpha_1^\vee, \ldots, \alpha_6^\vee$ as rows of a matrix $A^\vee$. Let G be defined in this way by the following two matrices.

$$
\begin{aligned}
A &:= \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \end{pmatrix} \\
A^\vee &:= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned}
\qquad (1)
$$

The above stated properties of the group G can be checked by using the results in [6, 1.9, 4.2, 4.5]: The Cartan matrix is $(\alpha_j, \alpha_i^\vee)_{1\le j,i\le 6}$, and $X/\mathbb{Z}\Phi$ and $Y/\mathbb{Z}\Phi^\vee$ are torsion free.

Now we assume that k is the algebraic closure of a finite prime field $\mathbb{F}_p$. To define a Frobenius morphism F of G, it is sufficient to give the induced action of F on X which is of the form $q \cdot F_0$, where q is a power of p and F_0 is an automorphism of finite order, since this determines the finite group G^F up to isomorphism. Here, we take the identity map for F_0.

We have general computer programs, written by the second author in the languages of the computer algebra systems GAP [26] and Maple [7], which compute for a root datum, given by matrices A and $A^\vee$ as above, and a map F_0, a parametrization of the conjugacy classes of G^F and explicit tables of Deligne-Lusztig characters of the form $R_T^G(1)$ (generically in q). Details of these programs will be explained elsewhere, some of them are already described in [23]. They work for groups G for which all centralizers of semisimple elements are connected. This condition is satisfied in our case, see [6, 3.5.6].

The resulting tables have a format defined in the system CHEVIE [17]. This system allows to compute tensor products and scalar products of given class functions.

Using these explicit computations we can show the following. Here, as usually, a class function on G^F is called uniform, if it lies in the space spanned by the Deligne-Lusztig characters $R_T^G(\theta)$.

Lemma 2.1 *Let G be a group as defined above by the matrices in (1). Then tensor products of uniform class functions of G^F are uniform.*

Proof. We define an equivalence relation $\sim$ on G^F such that the $\sim$-classes are unions of conjugacy classes of G^F and their characteristic functions form a basis of the space of uniform class functions. Then the statement of the lemma is clear because for two such characteristic functions ρ_1 and ρ_2 we have

$$\rho_1 \otimes \rho_2 = \begin{cases} \rho_1 & \text{if } \rho_1 = \rho_2 \\ 0 & \text{if } \rho_1 \neq \rho_2 \end{cases}$$

Let g_1, g_2 be elements of G^F and let $g_i = s_i u_i = u_i s_i$, s_i semisimple, u_i unipotent, $i = 1, 2$, denote their Jordan decompositions. We say that $g_1 \sim g_2$ if and only if there exists an $x \in G^F$ such that $x^{-1} s_1 x = s_2$ and all Green functions of the centralizer $C_G(s_2)^F$ have the same value on $x^{-1} u_1 x$ and on u_2. Note that G^F-conjugate elements are in the same $\sim$-class.

We investigate the space $\mathcal{U}$ of class functions of G^F spanned by the characteristic functions on the $\sim$-classes. We see from the character formula for the Deligne-Lusztig characters $R_T^G(\theta)$ (see [6, 7.2.8]) that the uniform functions are a subspace of $\mathcal{U}$. We can compute the dimension of $\mathcal{U}$ from the parametrization of semisimple conjugacy classes of G^F and the tables of Green functions for all centralizers of semisimple elements, generically, as polynomials in q.

On the other hand we can compute the dimension of the space of uniform functions: It follows from the orthogonality relations of Deligne-Lusztig characters that the different $R_T^G(\theta)$ are parametrized by the G^F-classes of pairs (T, θ) and that they are linearly independent, see [9, 11.15]. Furthermore the G^F-classes of pairs (T, θ) are in bijective correspondence with G^{*F^*}-classes of pairs (T^*, s), where T^* is an F^*-stable maximal torus of G^* and $s \in T^{*F^*}$ ([9, 13.13]). For a fixed semisimple element $s \in G^{*F^*}$ with centralizer $C = C_{G^*}(s)$ the G^{*F^*}-classes of pairs (T^*, s) are described by the C^{F^*}-classes of F^*-stable maximal tori in C. We can determine the semisimple classes of G^{*F^*} and the numbers of rational classes of maximal tori in the corresponding centralizers by using our programs for the dual group G^* (i.e., by changing the rôles of A and $A^\vee$).

The computations show that the dimension of the space of uniform functions equals the dimension of $\mathcal{U}$. So the two spaces themselves are equal. □

In the case of good characteristic for G, i.e., if q is not a power of 2 or 3, one could give a more conceptual proof Lemma 2.1. The idea of the above proof is to show that a column of the table of Deligne-Lusztig characters, which is a linear combination of some other columns, is in fact equal to some other column. This result can also be proved with Lusztig's theory of character sheaves and almost characters. Applying this to other exceptional groups of Lie type, one sees that the statement of Lemma 2.1 does not hold for groups of type F_4 or E_8. (We are indebted to Meinolf Geck for this remark.)

The uniform unipotent almost characters of G^F are the class functions R_φ defined by

$$R_\varphi := \frac{1}{|W|} \sum_{w \in W} \varphi(w) R_{T_w}^G(1),$$

where W denotes the Weyl group of G, φ is an irreducible ordinary character of W, and T_w is an F-stable maximal torus of G in relative position w (see [6, 12.3]). Thus from the knowledge of the $R_T^G(1)$ and the irreducible characters of W it is possible to compute the R_φ's. All unipotent characters of G^F can be expressed as linear combinations of these functions and some other unipotent almost characters which are orthogonal to all uniform functions. The coefficients for the linear combinations are explicitly given by certain Fourier transform matrices ([6, 13.6]).

If we now compute a tensor product χ of two uniform unipotent characters we know from Lemma 2.1 that χ is again uniform. In particular the scalar product of χ with all non-uniform unipotent almost characters is zero. So we can determine all multiplicities of unipotent characters in χ by computing its scalar product with the R_φ's. We used the computer algebra system CHEVIE [17] for these computations.

Lemma 2.2 *Let G be a connected reductive algebraic group, defined and split over a finite field $\mathbb{F}_q$, such that the centre $Z(G)$ is connected and $G/Z(G)$ is isomorphic to the simple group G_{ad} of adjoint type E_6. Let F be the corresponding Frobenius morphism.*

Then the cuspidal unipotent characters $E_6[\theta]$ and $E_6[\theta^2]$ of G^F do not appear as constituents of the tensor product $\phi_{6,1} \otimes \phi_{6,25}$ (notation of [6, p. 480]).

Proof. Since $Z(G)$ is connected, we have $G^F/Z(G^F) = G^F/Z(G)^F \cong G_{ad}^F$. Since the unipotent characters of G^F and those of G_{ad}^F correspond to each other via inflation, the result is true for G^F if and only if it is true for G_{ad}^F. So it is enough to prove the result for any one of the groups satisfying the assumptions of the lemma. We have checked it for the group defined by (1) with the methods described above. □

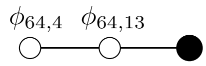
3 Results

The notation for the characters in the following theorem is taken from [6, p. 480]. The exceptional node of the Brauer tree is indicated by a black circle. The corresponding characters are always non-unipotent.

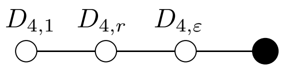
Theorem 3.1 *Let G be a simple algebraic group of type E_6, defined and split over a finite field $\mathbb{F}_q$, and let F denote the corresponding Frobenius morphism. Let $\ell > 3$ be a prime not dividing q but dividing $|G^F|$ and write e for the multiplicative order of q modulo ℓ. Then $e \in \{1, \ldots, 6, 8, 9, 12\}$.*

(1) If $e \in \{1, 6\}$, there is no unipotent block with a non-trivial cyclic defect group.

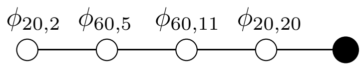
(2) If $e = 2$, there is exactly one unipotent ℓ-block of G^F with a non-trivial cyclic defect group. Its Brauer tree is as follows.



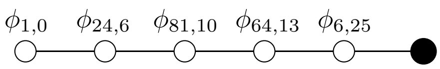
(3) If $e = 3$, there is exactly one unipotent ℓ-block of G^F with a non-trivial cyclic defect group. Its Brauer tree is as follows.

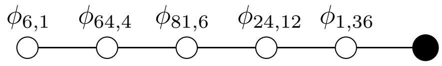


(4) If $e = 4$, there is exactly one unipotent ℓ-block of G^F with a non-trivial cyclic defect group. Its Brauer tree is as follows.

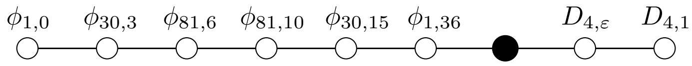


(5) If $e = 5$, there are exactly two unipotent ℓ-blocks of G^F with non-trivial cyclic defect groups. Their Brauer trees are as follows.

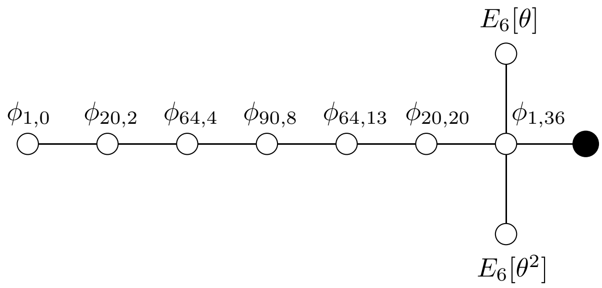




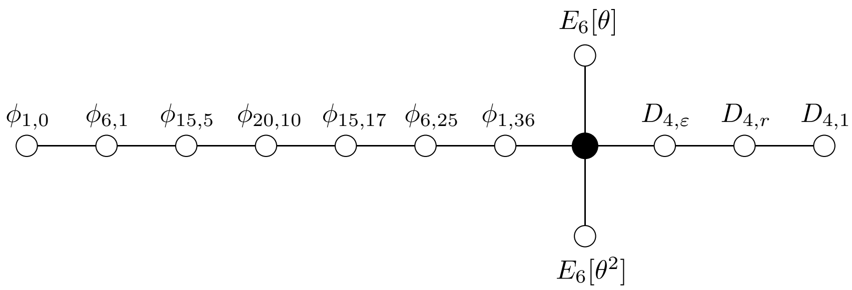
(6) If $e = 8$, the principal ℓ-block of G^F is the only unipotent ℓ-block with a non-trivial cyclic defect group. It has the following Brauer tree.



(7) If $e = 9$, the principal ℓ-block of G^F is the only unipotent ℓ-block with a non-trivial cyclic defect group. It has the following Brauer tree.



(8) If $e = 12$, the principal ℓ-block of G^F is the only unipotent ℓ-block with a non-trivial cyclic defect group. It has the following Brauer tree.



We finally formulate a result which allows to determine the Brauer trees of the non-unipotent blocks.

Theorem 3.2 *Let G be the algebraic group of type E_6 defined in Section 2 by the two matrices A and $A^\vee$ displayed in (1). Let $\ell > 3$ be a prime not dividing q and suppose that $e > 3$, where e denotes the multiplicative order of q modulo ℓ. Finally, let $s \in G^{*F}$ be a semisimple ℓ'-element.*

Then there is a bijection (Jordan decomposition of characters) between $\mathcal{E}_\ell(G^F, s)$ and $\mathcal{E}_\ell(C_{G^*}(s)^F, 1)$ preserving ℓ-blocks with cyclic defect groups. Moreover, this bijection induces a tree isomorphism between the (planar embedded) Brauer trees of corresponding blocks, which are therefore Morita equivalent.

Note that the Brauer trees for the unipotent blocks of the centralizers occurring in Theorem 3.2 are all known by [13, 14, 15]. The theorem is probably also true without the restriction on e. The proofs will be given in the next section.

4 Proofs

4.1 Proof of Theorem 3.1

Since the unipotent characters of G^F correspond bijectively to the unipotent characters of G_{ad}^F via restriction (to the image of G^F in G_{ad}^F), we may assume that $G = G_{ad}$ is the simple group of adjoint type E_6.

If e is one of 5, 8, 9 or 12, then a Sylow ℓ-subgroup of G^F is cyclic. It follows by inspection that a unipotent character of G^F either has ℓ-defect 0, or else its degree is not divisible by ℓ. Thus the unipotent characters which lie in blocks with non-trivial cyclic defect groups are exactly those whose degree is not divisible by ℓ. They can be read off from the table in [6, p. 480]. Their distribution into blocks as well as the non-unipotent characters in a block can be determined from the results in [3, Theorem 5.24(1), Tables 1,2].

This theorem also allows to find the unipotent ℓ-blocks of G^F in the cases $e = 6$ or $e \leq 4$ if $\ell \neq 5$. For $\ell = 5$ one can use [4, Théorème 2.1]. The defect groups are described in [3, Theorem 5.24(2)] for $\ell \neq 5$ and in [4, Théorème 2.1] for $\ell = 5$. It turns out that in each of the cases $e \in \{2, 3, 4\}$ there is a unique unipotent ℓ-block with a non-trivial cyclic defect group. There is none such in the remaining cases $e \in \{1, 6\}$.

The branch of the real stem of the Brauer tree corresponding to the principal series characters can be determined with the results of Dipper [10, Corollary 4.10] and Geck [16, Theorem 12.5(iv)], or, more easily, by Harish-Chandra induction of suitable characters from proper Levi subgroups. These methods already determine the trees for $e \in \{3, 4, 5, 8\}$.

Now let $e = 9$. The real stem of the Brauer tree is completely determined by the remarks above. It remains to determine the positions of the two complex conjugate characters $E_6[\theta]$ and $E_6[\theta^2]$ on the tree. (The two characters correspond to a pair of complex conjugate eigenvalues of the Frobenius map on a certain cohomology group (see [24, (7.3)]), and thus are indeed complex conjugate.) The degree of $E_6[\theta]$ is congruent to -1 modulo ℓ. It can thus be joined only to a character whose degree is congruent to 1 modulo ℓ, since two adjacent characters add up to a projective character (the sum of the exceptional characters is considered to correspond to a node). Thus $E_6[\theta]$ can only be joined to one of $\phi_{64,4}$, $\phi_{64,13}$ or $\phi_{1,36}$.

We have

$$\frac{\phi_{64,4}(1)}{2(E_6[\theta])(1)} = \frac{3\Phi_6(q)^2\Phi_{12}(q)}{2q^3\Phi_1(q)^6\Phi_2(q)\Phi_5(q)} < \frac{3\Phi_6(q)^2\Phi_{12}(q)}{2q^8\Phi_1(q)^6} < 1$$

for all q, and so $E_6[\theta]$ is connected to one of $\phi_{64,13}$ or $\phi_{1,36}$. Suppose that $E_6[\theta]$ is joined to $\phi_{64,13}$. Then $\Psi := \phi_{64,13} + E_6[\theta]$ is a projective character.

Let H denote a simple group of Lie type E_8, defined over $\mathbb{F}_q$ with corresponding Frobenius map F. Consider the ℓ-block B of H^F corresponding to the 9-cuspidal pair (M, λ), where M is the 9-split Levi subgroup of H of type $A_2(q)$ and λ is the Steinberg character of M^F (see [3, Table 1, Case 68]). The unipotent characters of H^F lying in B can be read off from [3, Table 2]. Let L be a 1-split Levi subgroup of H of type E_6. We identify the unipotent characters of L^F with those of G^F. Since Ψ is projective, so is $R_L^H(\Psi)$, and also the restriction to B of $R_L^H(\Psi)$. It follows that

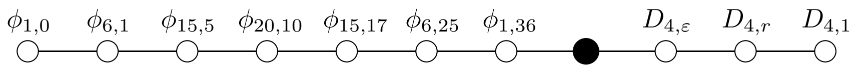
$$\phi_{2800,25} + \phi_{700,28} + E_6[\theta], \phi'_{1,3} + E_6[\theta], \phi_{1,6}$$

is a projective character of H^F in B. In particular, the non-real character $E_6[\theta], \phi_{1,6}$ is joined to one of the real characters $\phi_{2800,25}$ or $\phi_{700,28}$ on the Brauer tree of B. This is a contradiction, since

$$2(E_6[\theta], \phi_{1,6})(1) > \max\{\phi_{2800,25}(1), \phi_{700,28}(1)\}$$

for all q. Hence $E_6[\theta]$ is joined to $\phi_{1,36}$ on the Brauer tree for G^F.

Finally, suppose that $e = 12$. The non-exceptional characters on the real stem either are in the principal series or in the D_4-series. By the remarks above, the principal series branch of the tree is known, and by [18, Theorem 3.5], the exceptional node is joined to the Steinberg character. Inducing the cuspidal unipotent character of the Levi subgroup of type D_4, we obtain a projective character $D_{4,\varepsilon} + 2D_{4,r} + D_{4,1}$, and so $D_{4,r}$ is connected to $D_{4,\varepsilon}$ and $D_{4,1}$. Since the degree of $D_{4,\varepsilon}$ is larger than the degree of $D_{4,r}$, the end node of the tree is $D_{4,1}$, whereas $D_{4,\varepsilon}$ is connected to the exceptional node. We thus have the following real stem of the Brauer tree.



It remains to determine the positions of the two complex conjugate characters $E_6[\theta]$ and $E_6[\theta^2]$ on the tree. With similar arguments as in the case $e = 9$ one shows that $E_6[\theta]$ can only be joined to one of $\phi_{6,1}$, $\phi_{20,10}$, $\phi_{6,25}$, $D_{4,r}$ or to the exceptional node.

We have $\phi_{6,1}(1)/(2(E_6[\theta])(1)) < 1$ for all q, and

$$\frac{\phi_{20,10}(1)}{2(E_6[\theta])(1)} = \frac{1}{4}\frac{\Phi_6(q)^2\Phi_9(q)}{\Phi_1(q)^6\Phi_2(q)^4} < 1$$

and

$$\frac{D_{4,r}(1)}{2(E_6[\theta])(1)} = \frac{3}{4}\frac{\Phi_3(q)^2\Phi_9(q)}{\Phi_1(q)^2\Phi_2(q)^4\Phi_4(q)^2} < 1$$

for $q > 2$. Thus if $q > 2$ the two cuspidal characters can only be joined to either the exceptional node or to the node corresponding to $\phi_{6,25}$. By Lemma 2.2 we know that $E_6[\theta]$ and $E_6[\theta^2]$ do not appear as constituents in the tensor product $\phi_{6,1} \otimes \phi_{6,25}$. Since $\phi_{1,0} + \phi_{6,1}$ is the character of the projective cover of the trivial module, the tensor product $(\phi_{1,0} + \phi_{6,1}) \otimes E_6[\theta]$ contains the projective cover of the irreducible module corresponding to $E_6[\theta]$. Since $\phi_{6,25}$ is not a constituent of this tensor product, $E_6[\theta]$ cannot be joined to $\phi_{6,25}$ on the Brauer tree. We use Fischer's explicit character table of $E_6(2)$ [12], the methods of [20], and the CAS-system [25] to prove that the theorem is also true for $q = 2$. This completes the proof for the case that the order of q modulo ℓ is 12.

4.2 Proof of Theorem 3.2

Let $s \in {G^*}^F$ be a semisimple ℓ'-element. We first assume that $C_{G^*}(s)$ is a 1-split Levi subgroup of G^*. Let L denote a 1-split Levi subgroup of G, dual to $C_{G^*}(s)$. Then Harish-Chandra induction from L^F to G^F induces a collection of Morita equivalences between the block algebras of G^F in $\mathcal{E}_\ell(G^F, s)$ and those of $C_{G^*}(s)^F$ in $\mathcal{E}_\ell(C_{G^*}(s)^F, 1)$ (see [2, p. 62]). Since the defect group of a block is cyclic, if and only if the block has finite representation type, the theorem is proved in this case.

Now assume that $C_{G^*}(s)$ is not a 1-split Levi subgroup. Then the following holds.

Lemma 4.1 *For any pair (L, L^*) of dual Levi subgroups with $s \in L^*$, there exists a bijection $\mathcal{E}(C_{L^*}(s)^F, 1) \to \mathcal{E}(L^F, s)$, $\lambda \mapsto \chi^L_{s,\lambda}$, satisfying the following conditions:*

(1) $\chi^L_{s,\lambda}(1) = |{L^*}^F : C_{L^*}(s)^F|_{q'}\lambda(1)$.

(2) If $t \in Z(L^*)^F$ and $\hat{t}$ is the linear character of L^F defined by t and the duality, then $\chi^L_{st,\lambda} = \hat{t}\chi^L_{s,\lambda}$.

(3) For all $\gamma \in \mathcal{E}(C_{G^*}(s)^F, 1)$ and $\lambda \in \mathcal{E}(C_{L^*}(s)^F, 1)$, we have

$$\varepsilon_G\varepsilon_L\langle R^G_L\chi^L_{s,\lambda}, \chi^G_{s,\gamma}\rangle = \varepsilon_{C_{G^*}(s)}\varepsilon_{C_{L^*}(s)}\langle R^{C_{G^*}(s)}_{C_{L^*}(s)}\lambda, \gamma\rangle,$$

where ε_G denotes the $\mathbb{F}_q$-rank of G.

Proof. Since G and G^* both have connected centre, the same is true for all Levi subgroups of G and G^* [6, Proposition 8.1.4]. It follows that the centralizers of semisimple elements in all Levi subgroups of G and of G^* are connected [6, Theorem 4.5.9].

Since $C_{G^*}(s)$ is not a 1-split Levi subgroup, it is of type A or D_4 (see [8, Table III] for a classification of the semisimple conjugacy classes of a finite group of simple, simply-connected Lie type E_6). In particular, every proper

Levi subgroup of $C_{G^*}(s)$ is of type A. By [9, Theorems 13.23 and 13.25(ii)], this implies that all parts of the lemma are true. □

Let L^* be an e-split Levi subgroup of G^*. Inspecting the list of e-split Levi subgroups of G for $e > 3$, we see that no two distinct characters in $\mathcal{E}(C_{L^*}(s)^F, 1)$ have the same degree. This, Lemma 4.1, and the fact that $C_{G^*}(s)$ is connected implies that [5, Hypothesis 2.1] is satisfied for all pairs of e-split Levi subgroups (L, L^*). Hence we may apply [5, Theorems 3.4, 3.6] for the determination of the cyclic ℓ-blocks in $\mathcal{E}_\ell(G^F, s)$. Note that $\ell \in \Gamma(G, F)$ [5, Notation 1.1], since $\ell > 3$. Note also that these theorems are stated under the condition that [5, Hypothesis 2.1] is satisfied for all Levi subgroups. In our case, however, it suffices to assume this hypothesis for all e-split Levi subgroups. This follows from an analysis of the proofs of [5, Theorems 3.4 and 3.6] and the fact that if d is an integer with $\ell \mid \Phi_d(q)$ and $\Phi_d(q) \mid |G^F|$, then $d = e$. The latter is due to our assumption $\ell > 3$ and $e > 3$. By [5, Remark 3.7], we get the desired bijection between the blocks in $\mathcal{E}_\ell(G^F, s)$ and $\mathcal{E}_\ell(C_{G^*}(s)^F, 1)$.

To prove the second part of the theorem, we argue as follows. Let b be a unipotent ℓ-block of $C_{G^*}(s)$ with a cyclic defect group. Choose a 1-split Levi subgroup in $C_{G^*}(s)$ of the form $C_{L^*}(s)$, where L^* is a 1-split Levi subgroup of G^*, and the following two conditions are satisfied:

(1) L^* and $C_{L^*}(s)$ are of type A.

(2) Harish-Chandra induction of suitable unipotent characters of $C_{L^*}(s)$ to $C_{G^*}(s)$ determines the Brauer tree of b.

In our case this is always possible.

Let L denote a 1-split Levi subgroup of G, dual to L^*. Suppose that B is the ℓ-block of $\mathcal{E}_\ell(G^F, s)$ which corresponds to b via the Jordan decomposition of characters. Then Lemma 4.1(3) shows that the trees of b and B are equal (by Harish-Chandra induction of suitable characters of $\mathcal{E}(L^F, s)$ to G^F). The fact that the planar embedded Brauer tree determines the Morita equivalence class of a cyclic block is well known. This completes the proof of Theorem 3.2.

5 Brauer trees and ℓ-adic cohomology

Inspection of the known Brauer trees for Coxeter primes ℓ (to be described below) reveals a striking pattern. There seems to be a close connection between the arrangement of the irreducible characters on the tree and their appearance as composition factors in the ℓ-adic cohomology groups of the Deligne-Lusztig varieties associated to a Coxeter element.

Let G be a simple algebraic group of adjoint type, defined over some finite field $\mathbb{F}_q$ of characteristic p. Let F denote the Frobenius morphism associated to this $\mathbb{F}_q$-rational structure of G and let G^F denote the corresponding finite group of Lie type. Let ℓ be a prime dividing the order of the maximal torus T^F of G^F, where T denotes the Coxeter torus of G as defined by Lusztig in [24, (1.15)], but not dividing the order of the Weyl group of G. Then a Sylow ℓ-subgroup of G^F

is cyclic and contained in T^F up to conjugation. The set of unipotent characters in the principal ℓ-block B_0 of G^F is equal to the set of irreducible constituents of $R_T^G(1)$ [3, Theorem 5.24(a)]. With the notation from [24], this is the set of irreducible characters of the representations of G^F on the ℓ-adic cohomology groups with compact support $H_c^i(X, \bar{\mathbb{Q}}_\ell)$ of a suitable Deligne-Lusztig variety X associated to the pair (G, T). If r denotes the dimension of the variety X, then the cohomology groups $H_c^i(X, \bar{\mathbb{Q}}_\ell)$ are 0 except for $r \leq i \leq 2r$ [24, (2.9)].

Let $\delta \geq 1$ be the smallest integer such that F^δ acts as the identity on the Weyl group G. Then F^δ acts semisimply on the cohomology groups $H_c^i(X, \bar{\mathbb{Q}}_\ell)$, and the eigenspaces of F^δ are exactly the irreducible $\bar{\mathbb{Q}}_\ell G^F$-submodules [24]. The eigenvalues of F^δ are of the form $\zeta q^{\delta m/2}$, where $\zeta \in \bar{\mathbb{Q}}_\ell$ is a root of unity and m is a non-negative integer. Thus the unipotent irreducible characters in the principal ℓ-block of G^F are labelled by such pairs (ζ, m).

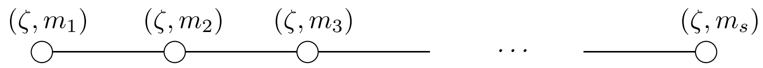
Theorem 5.1 *Let G, ℓ and the notation be as above. Suppose that $G^F \neq {}^2E_6(q), E_7(q), E_8(q)$, and if $G^F = F_4(q)$, assume that $q \equiv 1 (\mathrm{mod}\ 12)$.*

Let $\Gamma^\bullet$ denote the graph obtained from the Brauer tree of B_0 by removing the non-unipotent (exceptional) node and all edges incident to it. Then the following holds:

(1) The connected components of $\Gamma^\bullet$ are straight lines.

(2) Two unipotent characters with labels (ζ, m) and (ζ', m') lie on the same connected component, if and only if $\zeta = \zeta'$. Thus the connected components of $\Gamma^\bullet$ are labelled by roots of unity.

(3) Let $(\zeta, m_1), \dots, (\zeta, m_s)$ be all the labels of the characters lying on the connected component $\Gamma^\bullet_\zeta$ corresponding to ζ, and assume that $m_1 < m_2 < \dots < m_s$. Then $\Gamma^\bullet_\zeta$ is of the form



Furthermore, the node with label (ζ, m_1) is connected to the exceptional node.

(4) The distance (in the graph theoretical sense) of a character χ from the exceptional node is i, if and only if χ is the character of a representation occurring in $H_c^{r+i-1}(X, \bar{\mathbb{Q}}_\ell)$, $1 \leq i \leq r+1$.

(5) The connected components corresponding to the roots of unity ± 1 form the real stem of the tree.

Proof. All parts follow by inspection of the known Brauer trees. For references see the introduction. □

We conjecture that the theorem is also true in the cases where the trees are not known yet. We hardly dare to make any conjecture about the planar embedding of the tree, although, of course, one might speculate that the edges surrounding the exceptional node are ordered according to the ordering of the corresponding roots of unity in the complex numbers after a suitable identification of the roots of unity in $\bar{\mathbb{Q}}_\ell$ with those in $\mathbb{C}$.

By the results of Lusztig [24, (7.3)], the statements (3) and (4) of the theorem are equivalent. Also, part (5) is known to be true in all cases, since the real stems of all such trees are known.

By [24, (6.7), (7.3)], two unipotent characters of G^F occurring in $H_c^*(X, \bar{\mathbb{Q}}_\ell)$ with labels (ζ, m) and (ζ', m') lie in the same Harish-Chandra series, if and only if $\zeta = \zeta'$. Thus, statement (2) is equivalent to the following.

(2′) The nodes on a connected component of $\Gamma^\bullet$ are exactly the characters in the intersection of B_0 with one Harish-Chandra series of ordinary characters of G^F.

It is quite certain that in order to give a conceptual proof of the theorem and the conjecture, one will have to make use of properties of the ℓ-adic cohomology groups $H_c^i(X, \mathbb{Z}_\ell)$ and $H_c^i(X, \mathbb{Z}/\ell^n\mathbb{Z})$.

We expect, however, that before the appearance of such a conceptual proof, the method used in this paper for the $e = 12$ case will provide a positive answer to our conjecture, at least in the remaining cases for type F_4 and for ${}^2E_6$ and E_7. The conjecture is already known to be true for $F_4(2)$ and ${}^2E_6(2)$.

Not much is known about the planar embeddings of the trees. There is nothing to prove, of course, in a classical group, since the tree is a real stem. If $\Gamma^\bullet$ has four connected components, then, by complex conjugation, the planar embedding is also as conjectured. The only non-trivial cases occur in F_4, E_7, E_8, ${}^2G_2$ and ${}^2F_4$. In the latter two cases, the trees have been determined and some of the possible orderings around the exceptional node have been ruled out as possible planar embeddings [19]. The conjectural embeddings are still among the possible ones.

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