The Brauer trees of the exceptional Chevalley groups of types F_4 and $^2E_6$ *

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1 Introduction

In this note we determine the Brauer trees of the cyclic unipotent blocks of the Chevalley groups $F_4(q)$, and, except for one tree in case $q \not\equiv -1 (\mathrm{mod}\ 3)$, those of the twisted Chevalley groups $^2E_6(q)$. The results show that the statements of [6, Theorem 5.1] also hold for the groups considered here.

Our work is a contribution to the program of finding the Brauer trees for all finite groups. As far as finite groups of Lie type are concerned, only the Brauer trees of E_7 and E_8 remain to be found (and the one tree in $^2E_6(q)$ which we were not able to determine here).

The methods are much the same as those used in the computation of the Brauer tress for groups of type E_6 in [6]. In particular, we make essential use of the unipotent character tables for these groups.

For performing some computations we have used Maple [4] and GAP [7], as well as CHEVIE [5], which is based on these two systems.

The results for $F_4(q)$ had previously been obtained by Elmar Wings [9] in case $q \equiv 1 (\mathrm{mod}\ 12)$ and by Donald White [8] in case $q = 2$. Also, Klaus Lux has computed the Brauer trees of $^2E_6(2)$ (unpublished), using the explicit character table available in GAP. The results in this paper have been announced at the conference "Representation theory of finite groups," Bad Honnef, August 26–30, 1996.

2 Results

The notation for the characters in the following theorems is taken from [3, pp. 479,481]. The exceptional node of the Brauer tree is indicated by a black circle. The corresponding characters are always non-unipotent.

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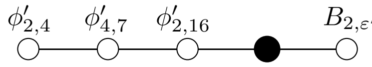
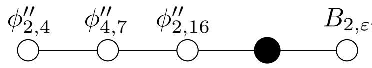
Theorem 2.1 *Let G be a simple algebraic group of type F_4, defined and split over a finite field $\mathbb{F}_q$, and let F denote the corresponding Frobenius morphism. Let $\ell > 3$ be a prime not dividing q but dividing $|G^F|$ and write e for the multiplicative order of q modulo ℓ. Then $e \in \{1, 2, 3, 4, 6, 8, 12\}$.*

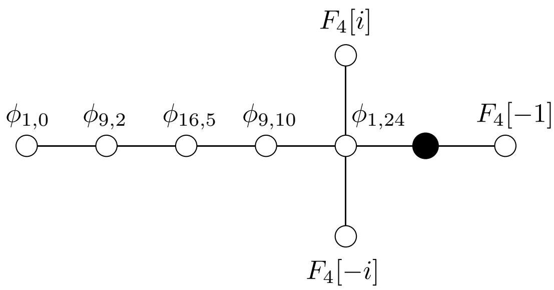
(1) *If $e \in \{1, 2, 3, 6\}$, there is no unipotent ℓ-block with a non-trivial cyclic defect group.*

(2) *If $e = 4$, there are exactly two unipotent ℓ-blocks of G^F with non-trivial cyclic defect groups. Their Brauer trees are as follows.*

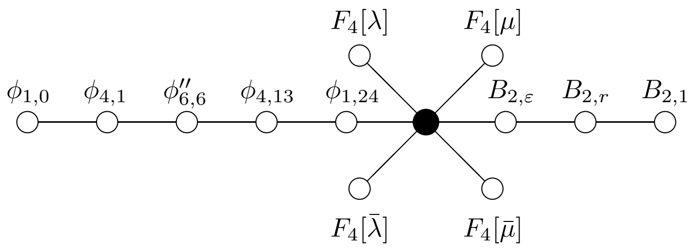




(3) *If $e = 8$, the principal ℓ-block of G^F is the only unipotent ℓ-block with a non-trivial cyclic defect group. It has the following Brauer tree.*



(4) *If $e = 12$, the principal ℓ-block of G^F is the only unipotent ℓ-block with a non-trivial cyclic defect group. It has the following Brauer tree.*



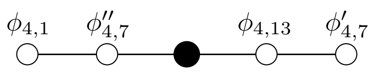
Here, $\{\lambda, \mu\} = \{\theta, i\}$ (in other words, the planar embedding of the tree is not determined).

Theorem 2.2 *Let G be a simple algebraic group of type E_6, defined over a finite field $\mathbb{F}_q$, and let F denote the corresponding Frobenius morphism. Suppose that G^F is of twisted type ${}^2E_6$. Let $\ell > 3$ be a prime not dividing q but dividing $|G^F|$ and write e for the multiplicative order of q modulo ℓ. Then $e \in \{1, 2, 3, 4, 6, 8, 10, 12, 18\}$.*

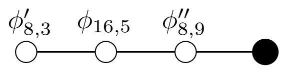
(1) *If $e \in \{2, 3\}$, there is no unipotent ℓ-block with a non-trivial cyclic defect group.*

(2) *If $e = 1$, there are exactly two unipotent ℓ-blocks of G^F with non-trivial cyclic defect groups. Their Brauer trees are as follows.*

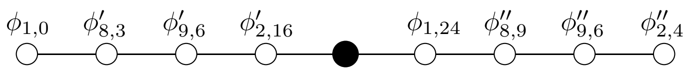
(3) *If $e = 4$, there is exactly one unipotent ℓ-block of G^F with a non-trivial cyclic defect group. Its Brauer tree is as follows.*



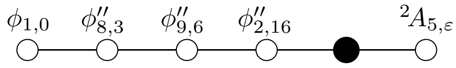
(4) *If $e = 6$, there is exactly one unipotent ℓ-block of G^F with a non-trivial cyclic defect group. Its Brauer tree is as follows.*

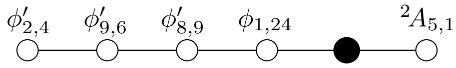


(5) *If $e = 8$, the principal ℓ-block of G^F is the only unipotent ℓ-block with a non-trivial cyclic defect group. It has the following Brauer tree.*

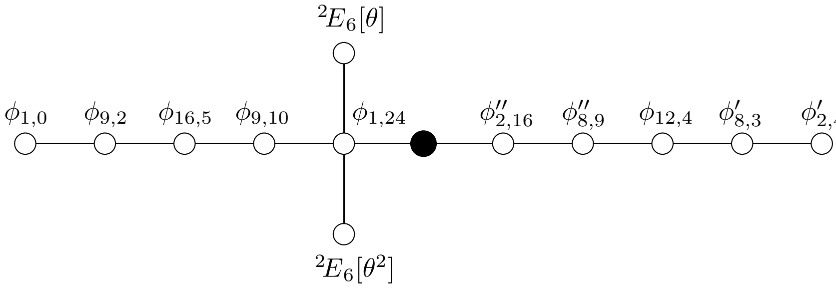


(6) *If $e = 10$, there are exactly two unipotent ℓ-blocks of G^F with non-trivial cyclic defect groups. Their Brauer trees are as follows.*



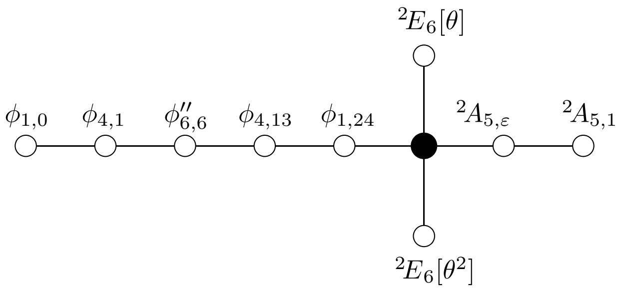


(7) *If $e = 12$, the principal ℓ-block of G^F is the only unipotent ℓ-block with a non-trivial cyclic defect group. If $q \equiv -1(\mathrm{mod}\ 3)$, it has the following Brauer tree.*



If $q \not\equiv -1(\mathrm{mod}\ 3)$, there are two possibilities for the Brauer tree. It either is the one given above, or it is the tree with the same real stem but the nodes corresponding to the non-real characters linked to $\phi''_{2,16}$.

(8) *If $e = 18$, the principal ℓ-block of G^F is the only unipotent ℓ-block with a non-trivial cyclic defect group. It has the following Brauer tree.*



3 Proofs

3.1 Preliminaries

The distribution of the unipotent characters into ℓ-blocks, as well as the corresponding defect groups, can be determined from the results in [1, Theorem 5.24(1), Tables 1,2] and [2, Théorème 2.1]. The real stems of the Brauer trees are found by Harish-Chandra induction of suitable projective characters from Levi subgroups (except that additional arguments are needed for the proof of Parts (3) and (5) of Theorem 2.2). The computation of the Harish-Chandra induced characters can of course be reduced to computations in the Weyl groups. These were done with the `InductionTable` command of the CHEVIE-system [5].

The following results on some scalar products of unipotent characters are crucial for finding the location of the non-real characters on the trees. The computations were also done with CHEVIE. We remark that we do not have complete unipotent character tables for the groups of type F_4 and ${}^2E_6$, but the known character values are sufficient for our purposes.

Lemma 3.1 (a) *In $F_4(q)$ we have the following scalar products between unipotent characters:*

$$(\phi_{4,1} \otimes F_4[\theta], \phi_{4,13}) = 0,$$

$$(\phi_{4,1} \otimes F_4[i], \phi_{4,13}) = 0.$$

If $q \equiv 3 (\mathrm{mod}\ 12)$,

$$(B_{2,1} \otimes F_4[i], \phi_{16,5}) = 0,$$

$$(B_{2,1} \otimes F_4[i], F_4[-i]) = \frac{1}{2}(q+1).$$

(b) *In ${}^2E_6(q)$ we have the following scalar products between unipotent characters:*

$$(\phi_{4,1} \otimes {}^2E_6[\theta], \phi_{4,13}) = 0,$$

$$(\phi_{4,1} \otimes {}^2E_6[\theta], \phi''_{2,16}) = 0.$$

If $q \equiv -1 (\mathrm{mod}\ 3)$, and if ${}^2E_6(q) = G^F$ with G simply connected, then

$$(\phi_{4,1} \otimes {}^2E_6[\theta], {}^2E_6[\theta]) = 2.$$

3.2 Proof of Theorem 2.1

Let $e = 8$. It remains to determine the positions of the two complex conjugate characters $F_4[i]$ and $F_4[-i]$ on the tree. The degree of $F_4[i]$ is congruent to -1 modulo ℓ. It can thus be joined only to a character whose degree is congruent to 1 modulo ℓ, since two adjacent characters add up to a projective character (the sum of the exceptional characters is considered to correspond to a single node). Thus $F_4[i]$ can only be joined to one of $\phi_{16,5}$, $\phi_{1,24}$ or $F_4[-1]$. However, we have

$$\frac{F_4[-1](1)}{2(F_4[i](1))} = \frac{\left(q^4 - q^2 + 1\right)\left(q^2 + 1\right)^2}{2\left(q^2 - q + 1\right)^2 (q+1)^4} < 1$$

for all q, and

$$\frac{\phi_{16,5}(1)}{2(F_4[i](1))} = \frac{\left(q^4 - q^2 + 1\right)\left(q^2 + 1\right)^2}{2\left(q^2 + q + 1\right)^2 (q-1)^4} < 1$$

for all $q > 3$, and so $F_4[i]$ is connected to $\phi_{1,24}$ for $q > 3$. To rule out the possibility $\phi_{16,5}$ in case $q = 3$, we argue as follows. Since $B_{2,1}$ is a defect zero character, the tensor product $B_{2,1} \otimes F_4[i]$ is projective. Now, by Lemma 3.1(a), $F_4[-i]$ occurs in this projective character with a positive multiplicity, whereas $\phi_{16,5}$ does not occur at all. Hence $F_4[-i]$ cannot be joined to $\phi_{16,5}$ on the Brauer tree. It follows from [8] that the given tree is correct in case $q = 2$.

Now suppose that $e = 12$. We have to determine the positions of the two non-real characters $F_4[\theta]$ and $F_4[i]$ and their complex conjugates on the tree. With similar arguments as above one shows that they can only be joined to $\phi_{4,13}$ or to the exceptional node (this time there are no exceptions on the size of q).

By Lemma 3.1(a) we know that none of $F_4[\theta]$, $F_4[\theta^2]$, $F_4[i]$ or $F_4[-i]$ does occur as constituents in the tensor product $\phi_{4,1} \otimes \phi_{4,13}$. Since $\phi_{1,0} + \phi_{4,1}$ is the character of the projective cover of the trivial module, the tensor product $(\phi_{1,0} + \phi_{4,1}) \otimes F_4[\theta]$ contains the projective cover of the irreducible module corresponding to $F_4[\theta]$. Since $\phi_{4,13}$ is not a constituent of this tensor product, $F_4[\theta]$ cannot be joined to $\phi_{4,13}$ on the Brauer tree. Exactly the same argument works for $F_4[i]$. This completes the proof.

3.3 Proof of Theorem 2.2

Let $e = 4$. We have to show that the ordinary irreducible characters $\phi_{4,1}$ and $\phi'_{4,7}$ are located at leaves of the Brauer tree, i.e., that they are irreducible modulo ℓ. By standard arguments involving trivial source modules and cyclic block theory, this is a consequence of the following fact. Let L^F denote the Levi subgroup of G^F of type $A_2(q^2)+A_1(q)$. The restriction of $R_L^G(1)$ to our block has ordinary character $\phi_{4,1} + \phi'_{4,7}$.

Now let $e = 8$. It follows as above that $\phi''_{2,4}$ is irreducible modulo ℓ. Namely, let L^F denote the Levi subgroup of type ${}^2D_4(q)$. Then the restriction to the principal block of $R_L^G(1)$ equals $\phi_{1,0} + \phi''_{2,4}$. The two branches of the tree are easily determined by Harish-Chandra inducing projective characters.

Next let $e = 12$. Here, we assume that $q \equiv -1 (\mathrm{mod}\ 3)$. By considering degrees and their residues, it follows that ${}^2E_6[\theta]$ can only be joined to $\phi_{1,24}$ or to $\phi''_{2,16}$, provided that $q > 3$. Now $\phi_{4,1}$ is a defect zero character, and hence $\phi_{4,1} \otimes {}^2E_6[\theta]$ is projective.

Let G denote the non-abelian simple composition factor of our group ${}^2E_6(q)$. Assume for the moment that the latter arises from a simply connected algebraic group, i.e., has shape $3.G$. Then, as ${}^2E_6[\theta]$ occurs in this tensor product by Lemma 3.1(b), whereas $\phi''_{2,16}$ does not, it follows that ${}^2E_6[\theta]$ is joined to $\phi_{1,24}$. (If q is any prime power and ${}^2E_6(q)$ arises from an algebraic group of adjoint type, then the scalar product $(\phi_{4,1} \otimes {}^2E_6[\theta], {}^2E_6[\theta])$ equals zero.) For $q = 2$ one can use computations with the character table, which is available in GAP [7], to prove the assertion.

If ${}^2E_6(q)$ arises from an adjoint algebraic group, i.e., has shape $G.3$, the Brauer tree of the principal block is the same as that of G, since every unipotent character of $G.3$ restricts irreducibly to G. The Brauer tree of the principal block of G in turn is the same as that of the principal block of $3.G$.

Finally let $e = 18$. Here the proof is exactly the same as in the corresponding case for groups of type F_4, using Lemma 3.1(b). We may omit further details.

Acknowledgment

We thank Gunter Malle for his permission to use his notes containing the real stems of the Brauer trees given here.

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