# Appendix: Practical Computation of Formal Degrees

Frank Lübeck, Aachen

## 1 Introduction

These remarks are an appendix to the preceding paper [6] by Mark Reeder. We will report on the use of the computer algebra package CHEVIE [3] for carrying out some of the computations explained in that paper.

CHEVIE is a collection of computer programs and data bases dealing with finite Coxeter groups, Iwahori-Hecke algebras, finite reductive groups and their representations. It is based on the more general purpose computer algebra systems GAP [7] and Maple [1]. Let us mention that GAP, CHEVIE and the additional programs mentioned in this note are free software, they are available with the complete source code and free of charge.[1] In this note we show that the system can be useful in related fields - like $p$-adic groups - too.

We will use the following notations from the beginning of the introduction of [6]: $G$, $q$, $\hat{G}$, $x$, $A_x$, $\rho$ and $V_{x,\rho}$.

From now on we assume that the representation $V_{x,\rho}$ of $G$ is of Iwahori-spherical type. In the following sections we describe the practical computation of the formal degree $\deg(V_{x,\rho})$.

**Acknowledgement.** I wish to thank Mark Reeder for showing me his article and explaining the background and some details of his work.

## 2 Summary of the method

We recall the basic statements from [6] which describe the calculations we have to do.

**(2.1)** The following notations are also similar to those in [6], except that we will use a ˆ-accent for structures associated to the complex dual group $\hat{G}$.

Let $T$ be a split maximal torus of the $p$-adic group $G$ and $\hat{T}$ a dual maximal torus of $\hat{G}$. We denote $\Sigma'$ a set of Coxeter generators of the affine Weyl group $W'$ of $G$ (,i.e., the Coxeter group associated to the extended Dynkin diagram of $G$). For each proper subset $J \subset \Sigma'$ let $(P_J, U_J, M_J)$ be the corresponding standard parahoric subgroup of $G$. Here $M_J$ is a split finite reductive group over the finite field with $q$ elements.

**(2.2)** Let $V = V_{x,\rho}$ be one of the representations we want to consider. For each $J$ as above the space $V^{U_J}$ of fixed points of $V$ under $U_J$ is finite dimensional; it is a representation of $M_J$ whose irreducible constituents are unipotent discrete series representations.

[1] see http://www.math.rwth-aachen.de/~CHEVIE and the links given there

Following Lusztig [4] these unipotent characters are parameterized by the irreducible characters of the Weyl group $W_J$ of $M_J$ and their degrees are given by the generic degrees of the Iwahori-Hecke algebra associated to $W_J$ with all parameters equal to $q$.

**(2.3)** Writing $Z(\hat{G})$ for the center of $\hat{G}$ and $\phi_W$ for the Poincaré polynomial of a finite Coxeter group $W$ we can describe the formal degree of $V$ by the expression (see [6, (2.2)])

$$\deg(V) := \frac{(-1)^{\mathrm{rank}(G)}}{|Z(\hat{G})|} \sum_{J \subset \Sigma'} (-1)^{|J|} \frac{\dim V^{U_J}}{\phi_{W_J}(q)}.$$

**(2.4)** For our representations $V_{x,\rho}$ let $x = su \in \hat{G}$, $s \in \hat{T}$, be the Jordan decomposition of $x$. Since $\hat{G}$ is simply connected the centralizer $\hat{G}_s$ of $s$ in $\hat{G}$ is a connected reductive group. So the representation $\rho$ of $A_x$ can be identified with a representation of $A_u$ (defined inside $\hat{G}_s$).

Denote $\hat{W}$ the Weyl group of $\hat{G}$, similarly $\hat{W}_s \subset \hat{W}$ the Weyl group of $\hat{G}_s$ and $\hat{X}$ the character group of $\hat{T}$. In our situation $s$ is of finite order and can be identified with an element $s \in \mathrm{Hom}(\hat{X}, \Omega)$ where $\Omega$ is the group of roots of unity in $\mathbb{C}$.

The cohomology $H(\mathcal{B}_s^u)$ of the $u$-fixed points of the flag variety $\mathcal{B}_s$ of $\hat{G}_s$ is a $\hat{W}_s \times A_u$-module via the Springer correspondence. Let $H(\mathcal{B}_s^u)_\rho$ be the $\rho$-isotypic part of the $A_u$-action. This $\hat{W}_s$-module can be extended to a $\hat{X}\hat{W}_s$-module $s \otimes H(\mathcal{B}_s^u)_\rho$ where $\hat{X}$ acts via $s$.

We associate to the pair $(x, \rho)$ a representation of the extended affine Weyl group $\hat{X}\hat{W}$ by

$$\hat{V}_{x,\rho} := \epsilon \otimes \mathrm{ind}_{\hat{X}\hat{W}_s}^{\hat{X}\hat{W}} \left(s \otimes H(\mathcal{B}_s^u)_\rho\right),$$

where $\epsilon$ is the sign representation on $\hat{W}$ and trivial on $\hat{X}$.

**(2.5)** Consider the homomorphism $\psi : W' \longrightarrow \hat{X}\hat{W}$ which maps the chosen Coxeter generators of $W$ to the corresponding ones in $\hat{W}$ and the remaining generator to $(\hat{\alpha}_0 \hat{s}_0)$ where $\hat{\alpha}_0$ is the highest short root of $\hat{G}$ and $\hat{s}_0 \in \hat{W}$ the reflection along $\hat{\alpha}_0$.

To compute the formal degrees by the formula in (2.3) we need to know for each $J$ as in (2.1) the character of $W_J$ corresponding to $V^{U_J}$ as described in (2.2).

We know from [6, 8.] (where results of Lusztig and Kato are used) that this representation equals the restriction of $\hat{V}_{x,\rho}$ to $\psi(W_J)$ pulled back to $W_J$ (note that $\psi$ is injective on each $W_J$).

# 3 Remarks on the computations

**(3.1)** In CHEVIE [3] Weyl groups can be entered in the form of a root datum. The groups are realized as permutation groups acting on the set of roots - this makes many efficient algorithms for general permutation groups applicable to them. Elements can be converted, e.g., between permutations, reduced words and matrices acting on the root lattice. This can be used to compute with explicit elements from the corresponding extended affine Weyl group. For all Weyl groups appearing in our application one can find representatives of conjugacy classes, the (complex) character table, reflection subgroups and induce-restrict matrices for the irreducible characters quite fast.

Also the Iwahori-Hecke algebras for finite Coxeter groups and their character tables are available, in particular their generic degrees.

**(3.2)** To compute the restrictions to $\psi(W_J)$ in (2.5) we have written a program which gets as input the root datum of $\hat{G}$, an element $s \in \hat{T} \cong \mathrm{Hom}(X, \Omega)$ and one of the $\psi(W_J)$. It returns the table of multiplicities

$$\left\langle \epsilon \otimes \mathrm{ind}_{\hat{X}\hat{W}_s}^{\hat{X}\hat{W}} \left(s \otimes \chi\right), \ \phi \right\rangle_{\psi(W_J)},$$

for all $\chi \in \mathrm{Irr}(\hat{W}_s)$ and $\phi \in \mathrm{Irr}(\psi(W_J))$.

A simple way to implement this is to apply the induction formula to a list of conjugacy class representatives of $\psi(W_J)$ (which is easy to find according to (3.1)): For $y \in \hat{X}\hat{W}$ the value of the induced character is

$$\mathrm{ind}_{\hat{X}\hat{W}_s}^{\hat{X}\hat{W}} \left(s \otimes \chi\right)(y) = \sum_{w \in R,\ wyw^{-1} \in \hat{X}\hat{W}_s} (s \otimes \chi)(wyw^{-1}),$$

where $R$ is a set of right coset representatives of $\hat{W}_s \backslash \hat{W}$.

This is good enough to handle all the applications needed in [6] although in the worst cases (like $\hat{W}$ of type $E_8$ and $\hat{W}_s$ of Type $A_1A_2A_5$ with index $> 80000$) the computation takes several hours on a currently fast computer. To make the program more efficient and interesting for further applications we have also implemented a version which uses the Mackey formula, similar to [6, 8.2(a)].

This is more complicated to program, but when it becomes difficult like enumerating double coset representatives and handling the cases where the intersection of some conjugate of $\hat{X}\hat{W}_s$ with $\psi(W_J)$ is not a reflection subgroup, we just fall back on using general programs for permutation groups. This version turned out to handle the above mentioned worst cases in a few minutes computation time.

**(3.3)** To evaluate the sum in (2.3) we use the method in (3.2) for *maximal* $J$ only. The following facts can be seen from [2, 70.6 and 70.24]. If $J' \subset J \subset \Sigma'$ then the $M_{J'}$-module $V^{U_{J'}}$ is the Harish-Chandra restriction (or truncation) of the $M_J$-module $V^{U_J}$. This restriction is completely described in terms of the restrictions of the corresponding characters of the Weyl groups $W_J$ to $W_{J'}$. The term $\phi_{W_J}(q)\mathrm{dim}V^{U_{J'}}/\phi_{W_{J'}}(q)$ equals the degree of the Harish-Chandra induction of $V^{U_{J'}}$ to $M_J$ (note that $\phi_{W_J}(q)$ is the maximal divisor of the order of $M_J$ which is prime to $q$).

So, knowing the $V^{U_J}$ for maximal $J$ (in form of the corresponding character of $W_J$) we get the remaining terms in the sum (2.3) from the induce-restrict matrices of parabolic subgroups of these $W_J$. Since this is computed quickly by CHEVIE it is not necessary to implement the use of Alvis-Curtis duality as described in [6, 3.]. Only in the final step we need the generic degrees of the Iwahori-Hecke algebras of the $W_J$ with maximal $J$. These are available in CHEVIE.

**(3.4)** Putting together (3.2) and (3.3) we can evaluate for each irreducible character $\chi$ of $\hat{W}_s$ the sum in (2.3) but using $\epsilon \otimes \mathrm{ind}_{\hat{X}\hat{W}_s}^{\hat{X}\hat{W}} \left(s \otimes \chi\right)$ instead of $V_{x,\rho}$ in (2.5).

The formal degree $\deg V_{x,\rho}$ is then a linear combination of these rational functions according to the decomposition of $H(\mathcal{B}_s^u)_\rho$ into irreducible $\hat{W}_s$-modules.

The structure of $H(\mathcal{B}_s^u)_\rho$ as $\hat{W}_s$-module can be computed by an algorithm described by Lusztig in [5, 24.]. This algorithm needs as input the Springer correspondence for $\hat{G}_s$. We have implemented this algorithm as well as a data base of the (generalized) Springer correspondence - which was mainly determined by Lusztig, Shoji and Spaltenstein. (For certain technical reasons these programs are not yet distributed with CHEVIE.)

**(3.5)** To summarize, having CHEVIE available it was relatively straight forward to write some additional programs which do the computations described in [6]: We implemented an arithmetic for elements in the extended affine Weyl group, some programs to deal with the rational functions appearing in this application and (most important and a bit tricky in detail) we wrote programs to carry out the steps (2.5) and (2.3).

# 4 An example

To give the reader an idea how it looks like to use the programs mentioned above let us consider the case $G$ of type $G_2$ which is left out in the tables in [6].

Here $\hat{G}$ has 4 conjugacy classes having empty intersection with any proper Levi subgroup: The unipotent classes $G_2$ and $G_2(a_1)$ and two mixed classes of elements $su$ with $\hat{G}_s$ of type $A_1\tilde{A}_1$ or $A_2$, respectively, and $u$ regular unipotent in $\hat{G}_s$. The corresponding component groups are - in the same order - $S_1$, $S_3$, $S_2$ (symmetric) and $C_3$ (cyclic). They have 0, 1, 1 and 2 irreducible characters corresponding to the 4 cuspidal unipotent representations of $G_2(q)$. (Divide the degrees of the cuspidal unipotent representations by the Poincaré polynomial of $W(G_2)$ to find the formal degrees in these cases.) The remaining ones correspond to Iwahori-spherical representations which we handle now. We start to define $G$.

```
gap> G2 := CoxeterGroup("G", 2);;
```

We write elements $s \in \hat{T}$ as rank($G$)-tuples of rational numbers. These numbers represent elements of $\mathbb{Q}/\mathbb{Z} \cong \Omega$ (via $x \mapsto \exp(2\pi i x)$) which give the images of chosen basis elements of the $\mathbb{Z}$-lattice $\hat{X}$. First consider $s = 1$. The following command produces data which describe $\hat{G}_s$ and for all $\chi \in \mathrm{Irr}(\hat{W}_s)$ its possible contribution to a formal degree and to the corresponding $K$-type (see [6, 14.]).

```
gap> fd_G2 := FormalDegreeSummands(G2, [0, 0]);;
```

Let's look at the labels for the irreducible characters of $\hat{W}_s$.

```
gap> DisplayCharsCoxeter(fd_G2[1]);
Characters of:
G2    2 > 1
1    phi_{1,0}
2    phi_{1,6}
```

```
3    phi_{1,3}'
4    phi_{1,3}''
5    phi_{2,1}
6    phi_{2,2}
```

The summands for the formal degrees are in `fd_G2[3]`. We know the cohomology of the $\mathcal{B}_u$ from another program, mentioned above. We can read off the formal degrees as follows. (Text after # is a comment.)

```
gap> # phi_{1,0}   for x in class G_2, rho=1
gap> fd_G2[3][1];
phi1^2*phi5 / phi2^2*phi3*phi6
gap> # phi_{1,3}'  for x in class G_2(a_1), rho(1)=2
gap> fd_G2[3][3];
1/3*q*phi1^2 / phi2^2*phi3
gap> # phi_{1,0}+phi_{2,1} for x in class G_2(a_1), rho=1
gap> fd_G2[3][1] + fd_G2[3][5];
1/6*q*phi1^2 / phi2^2*phi3
```

A small utility gives equations ("linear" over $\mathbb{Q}/\mathbb{Z}$) for an $s$ with centralizer of type dual to $W_J$. We use a solution to find the summands for the formal degrees in the case $\hat{G}_s$ of type $A_1\tilde{A}_1$.

```
gap> PrintArray(EquationsSemisimpleElement(G2, [0,1]));
[ [   0,   1 ],
  [   2,  -1 ] ]
gap> fd_A1A1 := FormalDegreeSummands(G2, [1/2, 0]);;
```

Here we only need the information for the trivial character of $\hat{W}_s$. The corresponding $K$-type is in the last column of the table below and we show the formal degree for this class and $\rho = 1$.

```
gap> Display(fd_A1A1[2]);
K-type summands: ^W_s of type ~A1xA1 and W of type G2
            | 11,11 11,2 2,11 2,2
---------------------------------
phi_{1,0}   |     1    .    .   .
phi_{1,6}   |     .    .    .   1
phi_{1,3}'  |     .    .    1   .
phi_{1,3}'' |     .    1    .   .
phi_{2,1}   |     .    1    1   .
phi_{2,2}   |     1    .    .   1

gap> fd_A1A1[3][4];
1/2*q*phi1^2 / phi2^2*phi6
```

And similarly for $\hat{G}_s$ of type $A_2$.

```
gap> fd_A2 := FormalDegreeSummands(G2, [1/3, 0]);;
gap> fd_A2[3][3];
1/3*q*phi1^2 / phi3*phi6
```

# References


[1] B. W. Char, K. O. Geddes, G. H. Gonnet, B. L. Leong, M. B. Monagan, and S. M. Watt. *Maple V, Language Reference Manual.* Springer-Verlag, 1991. 1

[2] C. W. Curtis and I. Reiner. *Methods of representation theory, vol. I, II.* Wiley, New York, 1981/1987. (3.3)

[3] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer. CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras. *AAECC*, 7:175–210, 1996. 1, (3.1)

[4] G. Lusztig. *Characters of reductive groups over a finite field*, volume 107 of *Annals of Mathematical Studies.* Princeton University Press, 1985. (2.2)

[5] G. Lusztig. Character sheaves V. *Adv. in Math.*, 61:103–155, 1986. (3.4)

[6] M. Reeder. Formal degrees and $L$-packets of unipotent discrete series representations of exceptional $p$-adic groups. *in preparation*, 1998. 1, 1, 2, (2.1), (2.3), (2.5), (3.2), (3.3), (3.5), 4

[7] Martin Schönert et al. *GAP – Groups, Algorithms, and Programming.* Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, third edition, 1993–1997. 1