Matrix generators for the Ree groups ${}^2G_2(q)$.

Gregor Kemper, Frank Lübeck, and Kay Magaard*

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For the purposes of [K] and [KM] it became necessary to have 7×7 matrix generators for a Sylow-3-subgroup of the Ree groups ${}^2G_2(q)$ and its normalizer. For example in [K] we wanted to show that in a seven dimensional representation the Jordan canonical form of any element of order nine is a single Jordan block of size 7. In [KM] we develop group recognition algorithms. At some stage we need to identify a certain subset of group elements with the Sylow-3-subgroup of ${}^2G_2(q)$. This is most easily done using a faithful matrix representation of small dimension. In this note we provide matrix generators for two distinct Sylow-3-subgroups of ${}^2G_2(q)$, thereby providing a matrix generating set for the whole group. Starting with the Steinberg generators for a seven dimensional representation of $G_2(q)$ we construct our matrices following Carter [C], chapters 12 and 13. The matrices for the Steinberg generators of $G_2(q)$ were computed with the help of a computer program developed by the second author.

For our setup we let $G = G_2(K)$, where K is the algebraic closure of a finite field of characteristic 3. Let F a Frobenius endomorphism of G whose set of fixed points G^F is a Ree group of type ${}^2G_2(q)$, $q = 3^{2m+1}$. Let T be an F-invariant maximal torus of G, and let B and B^- be F-invariant Borel subgroups intersecting in T with unipotent radicals U respectively U^-. By N we denote the normalizer $N_G(T)$. Let Φ be the root system of G with respect to T and $\{a, b\}$ its base given by B, where a is a short and b a long root. Now U respectively U^- is generated by subgroups X_r respectively X_{-r}, where $r \in \Phi^+$ (the set of positive roots). The groups X_r are isomorphic to the additive group of the field K. We denote the elements of X_r by $X_r(t)$ where $t \in K$.

The reductive group G has an irreducible 7-dimensional representation over K (with highest weight $(1, 0)$) which can be found as follows: In characteristic 3, the 14-dimensional adjoint representation V of G has a 7-dimensional irreducible submodule. This submodule is spanned by those elements of the Chevalley basis of V which are labeled by short roots. The restriction of this representation to ${}^2G_2(q)$ remains irreducible.

The second author has implemented a computer program which computes for an arbitrary Chevalley group explicit matrices for the root elements $X_r(t)$ in its adjoint representation with respect to a Chevalley basis. Using this for type G_2, we obtain the images of the $X_r(t)$ in the 7-dimensional representation with high weight $(1, 0)$ by cutting out the appropriate 7×7-blocks of the $X_r(t)$. (By abuse of notation we also denote these images by $X_r(t)$ in the sequel.)

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These programs use the computer algebra packages GAP [GAP] and CHEVIE [GHLMP]. They work along the construction of the Chevalley groups as explained in Carters book [C]. It is planned to make them available to interested users as part of a larger package [L] for computing characters and highest weight representations of reductive groups.

The following list gives the matrices $X_r(t)$ for G. The reader can check that they satisfy the Steinberg relations [C] 12.2.1, where the structure constants are chosen as in the table on page 211 of [C]. (We denote zero entries by dots.)

$$X_a(t) = \begin{pmatrix} 1 & t & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & t & 2t^2 & . & . \\ . & . & . & 1 & t & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & 2t \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-a}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ t & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & 2t & 1 & . & . & . \\ . & . & 2t^2 & 2t & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & 2t & 1 \end{pmatrix},$$

$$X_b(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & t & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & 2t & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-b}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & t & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & 2t & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix},$$

$$X_{a+b}(t) = \begin{pmatrix} 1 & . & 2t & . & . & . & . \\ . & 1 & . & t & . & 2t^2 & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & t & . \\ . & . & . & . & 1 & . & t \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-a-b}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ 2t & . & 1 & . & . & . & . \\ . & 2t & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & 2t^2 & . & 2t & . & 1 & . \\ . & . & . & . & t & . & 1 \end{pmatrix},$$

$$X_{2a+b}(t) = \begin{pmatrix} 1 & . & . & 2t & . & . & 2t^2 \\ . & 1 & . & . & 2t & . & . \\ . & . & 1 & . & . & t & . \\ . & . & . & 1 & . & . & 2t \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}, \quad X_{-2a-b}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ t & . & . & 1 & . & . & . \\ . & 2t & . & . & 1 & . & . \\ . & . & t & . & . & 1 & . \\ 2t^2 & . & . & t & . & . & 1 \end{pmatrix},$$

$$X_{3a+b}(t) = \begin{pmatrix} 1 & . & . & . & 2t & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & t \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-3a-b}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ 2t & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & t & . & . & . & 1 \end{pmatrix},$$

$$X_{3a+2b}(t) = \begin{pmatrix} 1 & . & . & . & . & 2t & . \\ . & 1 & . & . & . & . & t \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-3a-2b}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ 2t & . & . & . & . & 1 & . \\ . & t & . & . & . & . & 1 \end{pmatrix}.$$

With our choice of structure constants the automorphism F has a particularly nice form. Let π be the involution on Φ which permutes $\pm a$ and $\pm b$, $\pm(a+b)$ and $\pm(3a+b)$, and $\pm(2a+b)$ and $\pm(3a+2b)$. Let $\theta = 3^m$. Now

$$F(X_r(t)) = X_{\pi(r)}(t^{\lambda(\pi(r))\theta}),$$

where $\lambda(\pi(r))$ is 1 if $\pi(r)$ is short and 3 if $\pi(r)$ is long, see for example Proposition 12.4.1 of [C] and the discussion on page 225.

Now every element of U is of the form

$$X_a(t_1)X_b(t_2)X_{a+b}(t_3)X_{2a+b}(t_4)X_{3a+b}(t_5)X_{3a+2b}(t_6)$$

with unique elements $t_1, \ldots, t_6 \in K$. Following the proof of [C] Proposition 13.6.3(vii) we confirm, using the computer program Maple, that

$$\begin{aligned} &F(X_a(t_1)X_b(t_2)X_{a+b}(t_3)X_{2a+b}(t_4)X_{3a+b}(t_5)X_{3a+2b}(t_6)) = \\ &\quad X_b(t_1^{3\theta})X_a(t_2^{\theta})X_{3a+b}(t_3^{3\theta})X_{3a+2b}(t_4^{3\theta})X_{a+b}(t_5^{\theta})X_{2a+b}(t_6^{\theta}) = X_a(t_2^{\theta})X_b(t_1^{3\theta}) \cdot \\ &\quad\; X_{a+b}(t_1^{3\theta}t_2^{\theta} + t_5^{\theta})X_{2a+b}(t_1^{3\theta}t_2^{2\theta} + t_6^{\theta})X_{3a+b}(-(t_1t_2)^{3\theta} + t_3^{3\theta})X_{3a+2b}(-t_1^{6\theta}t_2^{3\theta} + t_4^{3\theta}). \end{aligned}$$

Comparing the coefficients of the factors and using that for $x \in K$ we have $x^{3\theta^2} = x^q = x$ iff $x \in F_q$, we get the parametrization of the F-stable elements of U.

Set $t = t_2$, $u = t_5$ and $v = t_6$. Then $U^F = \{x_S(t,u,v) \mid t,u,v \in F_q\}$, where

$$x_S(t,u,v) = \begin{pmatrix} 1 & t^\theta & -u^\theta & (tu)^\theta - v^\theta & f_1(t,u,v) & f_2(t,u,v) & f_3(t,u,v) \\ . & 1 & t & u^\theta + t^{\theta+1} & -t^{2\theta+1} - v^\theta & f_4(t,u,v) & f_5(t,u,v) \\ . & . & 1 & t^\theta & -t^{2\theta} & v^\theta + (tu)^\theta & f_6(t,u,v) \\ . & . & . & 1 & t^\theta & u^\theta & (tu)^\theta - v^\theta \\ . & . & . & . & 1 & -t & u^\theta + t^{\theta+1} \\ . & . & . & . & . & 1 & -t^\theta \\ . & . & . & . & . & . & 1 \end{pmatrix}$$

with

$$\begin{aligned} \theta &= 3^m, \\ f_1(t,u,v) &= -u - t^{3\theta+1} - (tv)^\theta, \\ f_2(t,u,v) &= -v - (uv)^\theta - t^{3\theta+2} - t^\theta u^{2\theta}, \\ f_3(t,u,v) &= t^\theta v - u^{\theta+1} + t^{4\theta+2} - v^{2\theta} - t^{3\theta+1}u^\theta - (tuv)^\theta, \\ f_4(t,u,v) &= -u^{2\theta} + t^{\theta+1}u^\theta + tv^\theta, \\ f_5(t,u,v) &= v + tu - t^{2\theta+1}u^\theta - (uv)^\theta - t^{3\theta+2} - t^{\theta+1}v^\theta, \\ f_6(t,u,v) &= u + t^{3\theta+1} - (tv)^\theta - t^{2\theta}u^\theta. \end{aligned}$$

Using this we confirm, see [C] on page 236, that the group law for U^F is as follows:

$$x_S(t_1,u_1,v_1)x_S(t_2,u_2,v_2) = x_S(t_1+t_2, u_1+u_2-t_1t_2^{3\theta}, v_1+v_2-t_1^2t_2^{3\theta} - t_2u_1 + t_1t_2^{3\theta+1}).$$

Replacing positive roots by negative roots and proceeding as above we get that $U^{-F} = \{x'_S(t,u,v) \mid t,u,v \in F_q\}$, where

$$x'_S(t,u,v) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ t^\theta & 1 & . & . & . & . & . \\ t^{\theta+1} - u^\theta & t & 1 & . & . & . & . \\ (tu)^\theta + v^\theta & -u^\theta & -t^\theta & 1 & . & . & . \\ g_1(t,u,v) & (tu)^\theta - v^\theta & -t^{2\theta} & -t^\theta & 1 & . & . \\ g_2(t,u,v) & g_3(t,u,v) & t^{2\theta+1} + v^\theta & t^{\theta+1} - u^\theta & -t & 1 & . \\ g_4(t,u,v) & g_5(t,u,v) & g_6(t,u,v) & (tu)^\theta + v^\theta & u^\theta & -t^\theta & 1 \end{pmatrix}$$

with

$$\begin{aligned} \theta &= 3^m, \\ g_1(t,u,v) &= t^{3\theta+1} + t^{2\theta}u^\theta - (tv)^\theta - u, \\ g_2(t,u,v) &= t^{3\theta+2} + t^{\theta+1}v^\theta - u^\theta t^{2\theta+1} - (uv)^\theta + tu - v, \\ g_3(t,u,v) &= -t^{\theta+1}u^\theta - u^{2\theta} + tv^\theta \\ g_4(t,u,v) &= t^{4\theta+2} + u^\theta t^{3\theta+1} + (tuv)^\theta - v^{2\theta} - u^{\theta+1} + t^\theta v, \\ g_5(t,u,v) &= t^{3\theta+2} + t^\theta u^{2\theta} - (uv)^\theta + v, \\ g_6(t,u,v) &= -t^{3\theta+1} - (tv)^\theta + u. \end{aligned}$$

Here the group law is as follows:

$$x'_S(t_1,u_1,v_1)x'_S(t_2,u_2,v_2) = x'_S(t_1+t_2, u_1+u_2+t_1t_2^{3\theta}, v_1+v_2-t_1^2t_2^{3\theta}+t_2u_1+t_1t_2^{3\theta+1}).$$

We note that the form of the group law in U^F differs from that of U^{-F} by two minus signs. We also remark here that if we let our matrices $X_r(t)$ act on the right, rather than on the left, then the form of the group law for U^F changes to that of U^{-F} and vice versa.

Following [C] Lemma 12.1.1 we define $n_r(t)$ as $X_r(t)X_{-r}(-t^{-1})X_r(t)$ and $h_r(t) = n_r(t)n_r(-1)$. Now every element of T is of the form $h_a(t_1)h_b(t_2)$. Then by Lemma 13.7.1 we have $F(h_a(t_1))F(h_b(t_2)) = h_b(t_1^{3\theta})h_a(t_2^\theta)$. So by Theorem 13.7.4 an element of T is F-invariant iff $t_1 = t_2^\theta$ and $t_2 = t_1^{3\theta}$; i.e. all the $h_a(t)h_b(t^{3\theta})$, where $t \in F_q$, are invariant. Let $t = t_1$, then $T^F = \{h(t) \mid t \in F_q^*\}$, where

$$h(t) = \begin{pmatrix} t^\theta & . & . & . & . & . & . \\ . & t^{1-\theta} & . & . & . & . & . \\ . & . & t^{2\theta-1} & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & t^{1-2\theta} & . & . \\ . & . & . & . & . & t^{\theta-1} & . \\ . & . & . & . & . & . & t^{-\theta} \end{pmatrix}.$$

Finally we note that N^F is generated by T^F and the matrix

$$n := n_{a+b}(1)n_{3a+b}(1) = \begin{pmatrix} . & . & . & . & . & . & -1 \\ . & . & . & . & . & -1 & . \\ . & . & . & . & -1 & . & . \\ . & . & . & -1 & . & . & . \\ . & . & -1 & . & . & . & . \\ . & -1 & . & . & . & . & . \\ -1 & . & . & . & . & . & . \end{pmatrix}.$$

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Authors addresses:
Gregor Kemper
Department of Mathematics and Statistics
Queen's University
Kingston, Ontario K7L 3N6
Canada
Gregor.Kemper@IWR.Uni-Heidelberg.De

Frank Lübeck
Lehrstuhl D für Mathematik
RWTH Aachen
Templergraben 64
D-52062 Aachen
Germany
Frank.Luebeck@Math.RWTH-Aachen.De

Kay Magaard
Department of Mathematics
Wayne State University
Detroit, Mi 48202
USA
kaym@math.wayne.edu