Smallest degrees of representations of exceptional groups of Lie type

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1 Introduction

Let G be a finite simple group and $l = 0$ or l a prime number. We denote by $d_l(G)$ the smallest degree of a non-trivial projective representation of G over a field of characteristic l.

In this note we consider those G arising from the exceptional groups of Lie type, i.e., the finite simple groups in the following list:

$$^2B_2(2^{2m+1}),\ ^2G_2(3^{2m+1}),\ G_2(q),\ ^3D_4(q),\ ^2F_4(2^{2m+1}),$$
$$F_4(q),\ ^2E_6(q),\ E_6(q),\ E_7(q),\ E_8(q),$$

where $m \in \mathbb{N}$ and q is an arbitrary power of a prime p.

For a group G in this list let $\tilde{G} := G(q)_{sc}$ be a corresponding finite group of Lie type arising as group of fixed points under a Frobenius map of a simple simply-connected algebraic group. Up to a finite number of exceptions $\tilde{G}$ is the universal covering group of G and $G \cong \tilde{G}/Z(\tilde{G})$. So, in this case, the smallest degrees of non-trivial projective representations of G are equal to the smallest non-trivial degrees of representations of $\tilde{G}$ (which is a perfect group).

As the main result of this note we determine in Section 2 the first few smallest non-trivial degrees of complex representations of the groups $\tilde{G}$, together with their multiplicities. We get these as an application of Deligne-Lusztig theory and Lusztig's classification of irreducible characters of finite groups of Lie type.

The groups with exceptional universal coverings (as well as the Tits group $^2F_4(2)'$) are listed and dealt with in Section 3. This completes the determination of $d_0(G)$ for all groups in the above list.

In Section 4 we collect for the first five types of groups some values $d_l(G)$ for l a prime not equal to p. The information is complete in the first three cases. This improves the known lower bounds for $d_l(G)$, $l \neq p$, given by Landázuri, Seitz and Zalesskii in [LS74] and [SZ93].

Some related results. The case of classical groups and $l = 0$ was considered by Tiep and Zalesskii in [TZ96]. The introduction of that paper also gives a motivation and explains some applications of smallest character degrees.

For the case of alternating groups see [Wag77].

For sporadic simple groups G all $d_l(G)$ are known, up to a few cases for the Baby Monster. There is a not yet published list of these results - which are due to many people - prepared by C. Jansen[1].

For simple groups of Lie type G in characteristic p, minimal non-trivial degrees $d_p(G)$ in defining characteristic are given in [KL90, 5.4].

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2 Small degrees in characteristic 0

In this section we describe the determination of the smallest non-trivial character degrees in characteristic 0 for all simple simply connected exceptional finite groups of Lie type. We can actually compute *all* the degrees of irreducible complex characters and their multiplicities for these groups. We give an overview how this can be achieved.

We use Deligne-Lusztig theory and in particular Lusztig's Jordan decomposition of characters: Let $\mathbf{G}(q)$ be a finite group of Lie type, i.e., the group of $\mathbb{F}_q$-rational points of a connected reductive algebraic group $\mathbf{G}$ defined over a finite field $\mathbb{F}_q$ with q elements. We denote by $\mathbf{G}^*(q)$ its dual group. Then there is a partition of the set of complex irreducible characters,

$$\mathrm{Irr}(\mathbf{G}(q)) = \dot{\bigcup_{(s)}} \mathcal{E}(\mathbf{G}(q), (s)),$$

where (s) runs over the semisimple conjugacy classes of $\mathbf{G}^*(q)$. The sets of characters $\mathcal{E}(\mathbf{G}(q), (s))$, called *Lusztig series*, have a parameterization which only depends on the twisted Dynkin diagram of the connected component $C^0_{\mathbf{G}^*}(s)$ of the centralizer $C_{\mathbf{G}^*}(s)$ and the operation of the factor group $C_{\mathbf{G}^*}(s)/C^0_{\mathbf{G}^*}(s)$ on the set of so called *unipotent characters* of $C^0_{\mathbf{G}^*}(s)(q)$. (These connected centralizers are again connected reductive groups.)

[1] See `http://www.math.rwth-aachen.de/~MOC/mindeg/`.

In particular, if $s, s' \in \mathbf{G}^*(q)$ have $\mathbf{G}^*(q)$-conjugate centralizers, then $\mathcal{E}(\mathbf{G}(q),(s))$ and $\mathcal{E}(\mathbf{G}(q),(s'))$ can be parameterized by the same set. We say that a character in $\mathcal{E}(\mathbf{G}(q),(s))$ and a character in $\mathcal{E}(\mathbf{G}(q),(s'))$, corresponding to the same parameter, belong to the same *character type*.

Fix a semisimple $s \in \mathbf{G}^*(q)$. Then the degrees of the characters in $\mathcal{E}(\mathbf{G}(q),(s))$ can be expressed as polynomials in q.

In Lusztig's book [Lus85, 4.26] there is a formula which expresses the characters in $\mathcal{E}(\mathbf{G}(q),(s))$ as linear combinations of so-called almost characters. This is done under the assumption that the center of $\mathbf{G}$ is connected (in this case the centralizers $C_{\mathbf{G}^*}(s)$ are all connected). These almost characters are class functions which have value 0 on $1 \in \mathbf{G}(q)$ or which are linear combinations of Deligne-Lusztig virtual characters; see [Lus85, 3.7] for a formula. The values of the Deligne-Lusztig characters on $1 \in \mathbf{G}(q)$ can be computed from the orders of maximal tori in $\mathbf{G}(q)$, see [DL76, 7.2].

Since simply connected groups $\mathbf{G}$ in general don't have a connected center, we also need the extension of these results given in [Lus88].

To evaluate the linear combinations of Deligne-Lusztig characters which form the almost characters in [Lus85, 3.7] one needs to know the character tables of the Weyl groups of $\mathbf{G}$ and of $C_{\mathbf{G}^*}(s)$, extended by the Frobenius actions, and the fusions of these groups into each other.

To evaluate the degrees of irreducible characters as linear combinations of the values of the almost characters on $1 \in \mathbf{G}(q)$ as given in [Lus85, 4.26] one needs Lusztig's non-abelian Fourier transform matrices. These are described combinatorically in a case by case manner in [Lus85, 4.].

Hence, for computing all character degrees of the groups $\mathbf{G}(q)$ and their multiplicities we first determine the semisimple conjugacy classes of the dual group $\mathbf{G}^*(q)$ and the types of the centralizers of their elements. After distinguishing a finite number of congruence classes for q the number of conjugacy classes with a centralizer of fixed type can also be expressed as a polynomial in q. In [Lüb] we will describe this in detail. We then compute the degrees corresponding to the different types of centralizers as indicated above.

Given all character degrees and their multiplicities as polynomials in q we first specialize this list for a certain number of small q. (We usually have to handle several congruence classes of q modulo some number, which depends on the Dynkin type, separately.) This gives us a guess for the generic relative ordering of the degrees for larger q and also a list of exceptional orderings for small q. The guess is proved by examining the differences of the degree polynomials; we compute a bound for the zeros and just check for positivity by evaluating at all the prime powers below the bound. We remark that in each Lusztig series there is one character whose degree divides all degrees of the other ones in the same series. This reduces the number of necessary comparisons considerably.

The actual computations where done by a collection of computer programs, written by the author. They are based on the systems GAP [S+97] and CHEVIE [GHL+96]. The programs get as input a complete root datum describing a series of finite groups of Lie type. The computations along the lines indicated above can then be done with the available commands. (For technical reasons the programs do not cover the Suzuki and Ree groups, but for these groups the generic character tables are known ([Suz62], [War66], [Mal90]) and can be found in CHEVIE [GHL+96].)

Theorem 2.1 *Let $\tilde{G}$ be a finite group of Lie type arising from a simple simply connected exceptional algebraic group. Then the tables in Section 5 show the (at least) 7 non-trivial character types of $\tilde{G}$ with the smallest degrees. All characters not listed there have strictly bigger degrees than the listed ones. Except for the cases handled in Section 3, the first degree in these tables, which is not equal to 1, gives $d_0(G)$ for the simple finite group $G = \tilde{G}/Z(\tilde{G})$.*

3 Groups with exceptional universal covering

In some cases the simple groups considered in this paper don't have the corresponding simply connected group of Lie type as universal covering group. The question of smallest degree projective representations for these groups is handled in this section.

3.1 The groups $G_2(2)$ and ${}^2F_4(2)$

These simple groups are the derived subgroups of the corresponding groups of Lie type in which they have index 2. The simple groups have trivial Schur multiplier and their character tables can be found in the Atlas [CCN+85].

3.2 The groups $G_2(3)$, $G_2(4)$ and $F_4(2)$

These groups have exceptional non-trivial coverings, whose character tables can be found in the Atlas [CCN+85].

3.3 The group ${}^2E_6(2)$

The simply connected group of Lie type is a 3-fold covering of this simple group, but its Schur multiplier is an abelian group of type $2^2 \cdot 3$. Clearly each character of the universal covering has a central subgroup of order 2 in the kernel. Furthermore the three factor groups of type $6.{}^2E_6(2)$ are isomorphic via an outer automorphism. (See the corresponding information in the Atlas [CCN+85].) So, we are interested in the smallest character degrees of $6.{}^2E_6(2)$. Unfortunately the character table of this group is not yet known but we can nevertheless determine the two smallest non-trivial character degrees.

From our list in Section 5 we get the smallest degrees of characters of $6.{}^2E_6(2)$ having the central subgroup of order 2 in the kernel. The Atlas contains the table of $2.{}^2E_6(2)$ and so gives the smallest degrees of characters having the central subgroup of order 3 in the kernel. The smallest non-trivial degrees found so far are 1938 and 2432.

The following argument shows that the smallest degree of a character which is faithful on the center is ≥ 22464. It was pointed out to me by Jürgen Müller. One uses the fact that a character which is faithful on the center, restricted to a subgroup containing the center can only decompose into characters with the same property. From the information in the Atlas we can show that the simple factor ${}^2E_6(2)$ contains a sporadic group Fi_{22} which lifts to a non-split extension $6.Fi_{22}$ in $6.{}^2E_6(2)$. Using the character table of this subgroup, which is in the Atlas, we see that the degree of a character which is faithful on the center and smaller than 61776 must be a multiple of 1728. Considering similarly a subgroup of type $2.F_4(2)$ and comparing, one finds that the degree of such a character must be at least 22464.

3.4 The table of results

Here is a table of smallest character degrees of the groups considered in this section. Bold entries indicate that these degrees do not occur in the lists given in Section 5.

$G_2(2)'$	$6, 7, 7, 7, 14, 21, 21, 21$
${}^2F_4(2)'$	$26, 26, 27, 27, 78, 104, 104$
$3.G_2(3)$	$14, \mathbf{27}, \mathbf{27}, 64, 64, 78$
$2.G_2(4)$	$\mathbf{12}, 65, 78, \mathbf{104}, \mathbf{104}, 300$
$2.F_4(2)$	$\mathbf{52}, 833, 1105, 1105, 1326, 1377$
$6.{}^2E_6(2)$	$1938, \mathbf{2432}$

4 Small degrees in prime characteristic $l \nmid q$

In this section we collect some values $d_l(G)$ with l a prime not dividing q for exceptional groups of Lie type. In the cases $G_2(q)$ and ${}^2G_2(3^{2m+1})$ we find all $d_l(G)$ and this result improves the lower bounds given in [LS74] and [SZ93] for these cases. But in the higher rank cases we cannot determine $d_l(G)$.

4.1 The Suzuki groups ${}^2B_2(2^{2m+1})$

For Suzuki groups G the ordinary character tables and all decomposition numbers are known, see [Bur79] and [His93, D.1]. From this information and the ordinary character degrees we can compute the modular character degrees in all cases as polynomials in q. Comparing these polynomials as in Section 2 we find that

$$d_l(G) = d_0(G), \text{ for all primes } l \neq 2.$$

4.2 The Ree groups ${}^2G_2(3^{2m+1})$

First let $l \neq 2, 3$. Then the l-Sylow group of a Ree group $G = {}^2G_2(3^{2m+1})$ is cyclic and so the l-modular decomposition numbers are encoded in the Brauer trees which are all determined in [His93, D.2]. In these cases we find again

$$d_l(G) = d_0(G), \text{ for all primes } l \neq 2, 3.$$

For $l = 2$ the decomposition matrix of the principal block was determined in [Fon74] and [LM80]. All other 2-blocks contain only ordinary characters which all have the same degree, so their decomposition matrices are all trivial. Computing the degrees of the Brauer characters in the principal block we find

$$d_2(G) = d_0(G) - 1.$$

4.3 The groups $G_2(q)$

In this case the minimal degrees are determined and explicitly given in [His93, 9.2]. We have for $G = G_2(q)$:

$$d_l(G) = \begin{cases} d_0(G) - 1, & \text{if either } l = 2,\ p = 3,\ q > 3 \text{ or } l = 3 \text{ and } 3 \mid (q-1) \\ d_0(G), & \text{else} \end{cases}$$

4.4 The groups ${}^3D_4(q)$, q odd

A lot of information about the decomposition numbers of triality groups $G = {}^3D_4(q)$ with odd q in non-defining characteristic was determined in [Gec91]. For $l \neq 2$ all decomposition matrices are explicitly given up to a few entries. For the missing entries there are at least lower bounds.

This is not sufficient to find the minimal non-trivial modular degrees. But if we also find some good enough *upper* bounds for the missing entries we can check that the reductions of the complex representations of degree $d_0(G)$ modulo l contain a large irreducible constituent which is the smallest non-trivial degree representation in characteristic l. And their degrees are determined by the known parts of the decomposition matrices.

The idea to find upper bounds for entries of a decomposition matrix is to construct projective representations modulo l and to decompose them into (ordinary) irreducible characters. These multiplicities give upper bounds for decomposition numbers.

Some useful projective characters for this purpose are already given in [Gec91, 3.3 and 4.2], namely the (modified) generalized Gelfand-Graev characters of G. Others we find by considering tensor products of certain defect-0 characters with other irreducible characters. Such characters and their scalar products with the irreducible characters can be found and computed with a computer using CHEVIE, see [GHL+96]. This system contains the ordinary character table of the groups ${}^3D_4(q)$ and programs for computing with such tables without specializing the parameter q.

We find for $l \neq 2$ as result:

$$d_l(G) = \begin{cases} d_0(G) - 1 & \text{if } l \mid q+1 \\ d_0(G) & \text{else} \end{cases}$$

For $l = 2$ much less information on decomposition numbers is known, see again [Gec91], and our ad hoc method was not good enough to find $d_2(G)$.

4.5 The groups ${}^2F_4(2^{2m+1})$

For the groups $G = {}^2F_4(2^{2m+1})$ again all Brauer trees are known, see [His93, D.3]. But here for many primes l the l-Sylow subgroups are not cyclic and then only very few information about the decomposition numbers is known. On the other hand, for primes $l \neq 2, 3$ which divide ϕ_{12}, ϕ'_{24} or ϕ''_{24} (see notation in 5.5) we do have enough information from the Brauer trees and find that in these cases

$$d_l(G) = d_0(G).$$

4.6 The other exceptional groups

For exceptional groups of larger rank only few results on decomposition numbers are known. They seem not to be enough to determine the corresponding $d_l(G)$.

In the examples above we found, that the l-modular representation of smallest degree always is a component of the reduction modulo l of a non-trivial representation of smallest degree in characteristic 0. So, let us add a remark here, that one can at least estimate how such a characteristic 0 representation can decompose modulo l.

Comparing the lower bounds for $d_l(G)$ given by Landázuri, Seitz and Zalesskii in [LS74] and [SZ93] with our $d_0(G)$ we see that the bounds are always bigger than $d_0(G)/2$. This shows that the reduction of a characteristic 0 representation of degree $d_0(G)$ modulo l can only have one non-trivial irreducible constituent. In some cases it is also relatively easy to find an upper bound for the number of trivial constituents in this reduction. For example in the cases $G = E_r(q)$, $r = 6, 7, 8$, one can use the projective character, coming from Harish-Chandra inducing the regular representation of a maximally split torus, to find that there are at most r trivial constituents.

5 Tables of small degrees in characteristic 0

In this section we give our main result from 2.1 in the form of tables. The degree polynomials are given in factored form, ϕ_i denotes the i-th cyclotomic polynomial in q. We consider all finite groups of exceptional Lie type $\tilde{G} = \mathbf{G}(q)_{sc}$ which arise from simple simply connected algebraic groups. There is a subsection for each of the series listed in the introduction. Sometimes we have to distinguish certain congruence classes of q.

Each table lists the character types with smallest possible degrees in increasing order (leaving out the trivial representation). The degrees of representations are given in columns labeled *Degree*. For convenience we also give the degrees of these polynomials in the column labeled d. The columns labeled *Mult.* show the number of characters in the corresponding character type.

If a character lies in the Lusztig series $\mathcal{E}(\mathbf{G}(q), (s))$ we indicate the type of the centralizer $C_{\mathbf{G}^*}(s)(q)$ in the column labeled $C(s)$. Most such centralizers occurring in our tables are connected (there is an exception in 5.9). In this case the characters in $\mathcal{E}(\mathbf{G}(q), (s))$ can be parameterized by the unipotent characters of $C_{\mathbf{G}^*}(s)(q)$. In the column labeled *Char.* we give such a parameter to indicate the corresponding character type. The notation for the unipotent characters is very close to that of Carter's book [Car85, Ch.13] respectively to that of the articles cited above for the Suzuki and Ree groups.

5.1 The Suzuki groups ${}^2B_2(q^2)$, $q = \sqrt{2^{2m+1}}$

Smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$1/2\,\sqrt{2}q\phi_1\phi_2$	3	1	${}^2B_2(q^2)$	${}^2B_2[b]$
$1/2\,\sqrt{2}q\phi_1\phi_2$	3	1	${}^2B_2(q^2)$	${}^2B_2[a]$
$\phi_1\phi_2\phi_8''$	4	$1/4\,q(q+\sqrt{2})$	$A_0(q^2)$	χ_6
q^4	4	1	${}^2B_2(q^2)$	St
ϕ_8	4	$1/2\,(q-\sqrt{2})(q+\sqrt{2})$	$A_0(q^2)$	χ_5
$\phi_1\phi_2\phi_8'$	4	$1/4\,q(q-\sqrt{2})$	$A_0(q^2)$	χ_7

(where $\phi_8' = q^2+\sqrt{2}q+1$ and $\phi_8'' = q^2-\sqrt{2}q+1$)

5.2 The Ree groups ${}^2G_2(q^2)$, $q=\sqrt{3^{2m+1}}$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
ϕ_{12}	4	1	$A_1(q^2)$	ξ_2
$1/6\,\sqrt{3}q\phi_1\phi_2\phi_{12}'$	5	1	${}^2G_2(q^2)$	ξ_8
$1/6\,\sqrt{3}q\phi_1\phi_2\phi_{12}'$	5	1	${}^2G_2(q^2)$	ξ_6
$1/6\,\sqrt{3}q\phi_1\phi_2\phi_{12}''$	5	1	${}^2G_2(q^2)$	ξ_7
$1/6\,\sqrt{3}q\phi_1\phi_2\phi_{12}''$	5	1	${}^2G_2(q^2)$	ξ_5
$1/3\,\sqrt{3}q\phi_1\phi_2\phi_4$	5	1	${}^2G_2(q^2)$	ξ_{10}

(where $\phi_{12}' = q^2-\sqrt{3}q+1$ and $\phi_{12}'' = q^2+\sqrt{3}q+1$)
with the following exception for small q:
$q=\sqrt{3}$:

Degree	Mult.	$C(s)$	Char.
1	1	${}^2G_2(q^2)$	ξ_8
1	1	${}^2G_2(q^2)$	ξ_6
7	1	$A_1(q^2)$	ξ_2
7	1	${}^2G_2(q^2)$	ξ_7
7	1	${}^2G_2(q^2)$	ξ_5
8	1	${}^2G_2(q^2)$	ξ_{10}
8	1	${}^2G_2(q^2)$	ξ_9
8	1	$A_0(q^2)$	η_i^+

5.3 The groups $G_2(q)$

Case $q \equiv 1 \pmod 6$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$\phi_2\phi_6$	3	1	$A_2(q)$	[3]
$\phi_3\phi_6$	4	1	$A_1(q) \times \tilde{A}_1(q)$	$[2], [2]$
$1/6\, q\phi_1^2\phi_6$	5	1	$G_2(q)$	$G_2[1]$
$1/6\, q\phi_2^2\phi_3$	5	1	$G_2(q)$	$\phi_{2,1}$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta]$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta^2]$

with the following exception for small q:

$q = 7$:

Degree	Mult.	$C(s)$	Char.
344	1	$A_2(q)$	[3]
1806	1	$G_2(q)$	$G_2[1]$
2451	1	$A_1(q) \times \tilde{A}_1(q)$	$[2], [2]$
4256	1	$G_2(q)$	$\phi_{2,1}$
5376	1	$G_2(q)$	$G_2[\theta]$
5376	1	$G_2(q)$	$G_2[\theta^2]$

Case $q \equiv 2 \pmod 6$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$\phi_1\phi_3$	3	1	${}^2A_2(q)$	[3]
$1/6\, q\phi_1^2\phi_6$	5	1	$G_2(q)$	$G_2[1]$
$1/6\, q\phi_2^2\phi_3$	5	1	$G_2(q)$	$\phi_{2,1}$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta]$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta^2]$
$1/3\, q\phi_3\phi_6$	5	1	$G_2(q)$	$\phi'_{1,3}$
$1/3\, q\phi_3\phi_6$	5	1	$G_2(q)$	$\phi''_{1,3}$

with the following exception for small q:

$q = 2$:

Degree	Mult.	$C(s)$	Char.
1	1	$G_2(q)$	$G_2[1]$
6	1	$G_2(q)$	$G_2[\theta]$
6	1	$G_2(q)$	$G_2[\theta^2]$
7	1	${}^2A_2(q)$	$[3]$
7	1	$G_2(q)$	$G_2[-1]$
14	1	$G_2(q)$	$\phi'_{1,3}$
14	1	$G_2(q)$	$\phi''_{1,3}$
14	1	${}^2A_2(q)$	$[2,1]$

Case $q \equiv 3 \pmod 6$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$\phi_3\phi_6$	4	1	$A_1(q) \times \tilde{A}_1(q)$	$[2], [2]$
$1/6\, q\phi_1^2\phi_6$	5	1	$G_2(q)$	$G_2[1]$
$1/6\, q\phi_2^2\phi_3$	5	1	$G_2(q)$	$\phi_{2,1}$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta]$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta^2]$
$1/3\, q\phi_3\phi_6$	5	1	$G_2(q)$	$\phi'_{1,3}$
$1/3\, q\phi_3\phi_6$	5	1	$G_2(q)$	$\phi''_{1,3}$

with the following exception for small q:
$q = 3$:

Degree	Mult.	$C(s)$	Char.
14	1	$G_2(q)$	$G_2[1]$
64	1	$G_2(q)$	$G_2[\theta]$
64	1	$G_2(q)$	$G_2[\theta^2]$
78	1	$G_2(q)$	$G_2[-1]$
91	1	$A_1(q) \times \tilde{A}_1(q)$	$[2], [2]$
91	1	$G_2(q)$	$\phi'_{1,3}$
91	1	$G_2(q)$	$\phi''_{1,3}$

Case $q \equiv 4 \pmod 6$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$\phi_2\phi_6$	3	1	$A_2(q)$	[3]
$1/6\, q\phi_1^2\phi_6$	5	1	$G_2(q)$	$G_2[1]$
$1/6\, q\phi_2^2\phi_3$	5	1	$G_2(q)$	$\phi_{2,1}$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta]$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta^2]$
$1/3\, q\phi_3\phi_6$	5	1	$G_2(q)$	$\phi'_{1,3}$
$1/3\, q\phi_3\phi_6$	5	1	$G_2(q)$	$\phi''_{1,3}$

with the following exception for small q:
$q = 4$:

Degree	Mult.	$C(s)$	Char.
65	1	$A_2(q)$	[3]
78	1	$G_2(q)$	$G_2[1]$
300	1	$G_2(q)$	$G_2[\theta]$
300	1	$G_2(q)$	$G_2[\theta^2]$
350	1	$G_2(q)$	$\phi_{2,1}$
364	1	$G_2(q)$	$\phi'_{1,3}$
364	1	$G_2(q)$	$\phi''_{1,3}$

Case $q \equiv 5 \pmod 6$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$\phi_1\phi_3$	3	1	${}^2A_2(q)$	[3]
$\phi_3\phi_6$	4	1	$A_1(q) \times \tilde{A}_1(q)$	[2], [2]
$1/6\, q\phi_1^2\phi_6$	5	1	$G_2(q)$	$G_2[1]$
$1/6\, q\phi_2^2\phi_3$	5	1	$G_2(q)$	$\phi_{2,1}$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta]$
$1/3\, q\phi_1^2\phi_2^2$	5	1	$G_2(q)$	$G_2[\theta^2]$

with the following exception for small q:

$q = 5$:

Degree	Mult.	$C(s)$	Char.
124	1	${}^2A_2(q)$	$[3]$
280	1	$G_2(q)$	$G_2[1]$
651	1	$A_1(q) \times \tilde{A}_1(q)$	$[2], [2]$
930	1	$G_2(q)$	$\phi_{2,1}$
960	1	$G_2(q)$	$G_2[\theta]$
960	1	$G_2(q)$	$G_2[\theta^2]$

5.4 The groups ${}^3D_4(q)$

For the computation the congruence classes of q modulo 4 must be distinguished. The results only depend on q modulo 2.

Case $q \equiv 0 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$q\phi_{12}$	5	1	${}^3D_4(q)$	$\phi'_{1,3}$
$1/2\, q^3\phi_1^2\phi_{12}$	9	1	${}^3D_4(q)$	${}^3D_4[1]$
$1/2\, q^3\phi_1^2\phi_3^2$	9	1	${}^3D_4(q)$	${}^3D_4[-1]$
$1/2\, q^3\phi_2^2\phi_6^2$	9	1	${}^3D_4(q)$	$\phi_{2,1}$
$1/2\, q^3\phi_2^2\phi_{12}$	9	1	${}^3D_4(q)$	$\phi_{2,2}$
$\phi_1\phi_3\phi_6\phi_{12}$	9	$1/2\, q$	$A_1(q^3) \times \phi_2$	$[2]$

with the following exception for small q:
$q = 2$:

Degree	Mult.	$C(s)$	Char.
26	1	${}^3D_4(q)$	$\phi'_{1,3}$
52	1	${}^3D_4(q)$	${}^3D_4[1]$
196	1	${}^3D_4(q)$	${}^3D_4[-1]$
273	1	$A_1(q^3) \times \phi_2$	$[2]$
324	1	${}^3D_4(q)$	$\phi_{2,1}$
351	3	$A_2(q) \times \phi_3$	$[3]$

Case $q \equiv 1 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$q\phi_{12}$	5	1	${}^3D_4(q)$	$\phi'_{1,3}$
$\phi_3\phi_6\phi_{12}$	8	1	$A_1(q) \times A_1(q^3)$	$[2], [2]$
$1/2\, q^3\phi_1^2\phi_{12}$	9	1	${}^3D_4(q)$	${}^3D_4[1]$
$1/2\, q^3\phi_1^2\phi_3^2$	9	1	${}^3D_4(q)$	${}^3D_4[-1]$
$1/2\, q^3\phi_2^2\phi_6^2$	9	1	${}^3D_4(q)$	$\phi_{2,1}$
$1/2\, q^3\phi_2^2\phi_{12}$	9	1	${}^3D_4(q)$	$\phi_{2,2}$

with the following exception for small q:
$q = 3$:

Degree	Mult.	$C(s)$	Char.
219	1	${}^3D_4(q)$	$\phi'_{1,3}$
3942	1	${}^3D_4(q)$	${}^3D_4[1]$
6643	1	$A_1(q) \times A_1(q^3)$	$[2], [2]$
9126	1	${}^3D_4(q)$	${}^3D_4[-1]$
10584	1	${}^3D_4(q)$	$\phi_{2,1}$
13286	1	$A_1(q^3) \times \phi_2$	$[2]$

5.5 The Ree groups ${}^2F_4(q^2)$, $q = \sqrt{2^{2m+1}}$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$1/2\,\sqrt{2}q\phi_1\phi_2\phi_4^2\phi_{12}$	11	1	${}^2F_4(q^2)$	${}^2B_2[b][1]$
$1/2\,\sqrt{2}q\phi_1\phi_2\phi_4^2\phi_{12}$	11	1	${}^2F_4(q^2)$	${}^2B_2[a][1]$
$q^2\phi_{12}\phi_{24}$	14	1	${}^2F_4(q^2)$	ϵ'
$\phi_1\phi_2\phi_8^2\phi_{24}$	18	1	${}^2A_2(q^2)$	$[3]$
$1/12\, q^4\phi_1^2\phi_2^2{\phi''_8}^2\phi_{12}\phi''_{24}$	20	1	${}^2F_4(q^2)$	χ_9
$1/12\, q^4\phi_1^2\phi_2^2{\phi'_8}^2\phi_{12}\phi'_{24}$	20	1	${}^2F_4(q^2)$	χ_8

(where $\phi'_{24} = q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1$ and $\phi''_{24} = q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1$,
ϕ'_8 and ϕ''_8 are defined under the table for the Suzuki groups)
with the following exception for small q:

$q = \sqrt{2}$:

Degree	Mult.	$C(s)$	Char.
1	1	${}^2F_4(q^2)$	χ_9
27	1	${}^2F_4(q^2)$	${}^2B_2[b][1]$
27	1	${}^2F_4(q^2)$	${}^2B_2[a][1]$
27	1	${}^2F_4(q^2)$	χ_{11}
27	1	${}^2F_4(q^2)$	χ_{12}
52	1	${}^2F_4(q^2)$	χ_{17}

(Recall that ${}^2F_4(2)$ is not a simple group, see Section 3.)

$q = \sqrt{8}$:

Degree	Mult.	$C(s)$	Char.
64638	1	${}^2F_4(q^2)$	${}^2B_2[b][1]$
64638	1	${}^2F_4(q^2)$	${}^2B_2[a][1]$
1839048	1	${}^2F_4(q^2)$	ϵ'
13778800	1	${}^2F_4(q^2)$	χ_9
119275975	1	${}^2A_2(q^2)$	$[3]$
133929936	1	${}^2F_4(q^2)$	χ_{11}
133929936	1	${}^2F_4(q^2)$	χ_{12}

5.6 The groups $F_4(q)$

For the computation the congruence classes of q modulo 12 must be distinguished. The results only depend on q modulo 2.

Case $q \equiv 0 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$1/2\, q\phi_1^2\phi_3^2\phi_8$	11	1	$F_4(q)$	$[[2],[]]$
$1/2\, q\phi_4\phi_8\phi_{12}$	11	1	$F_4(q)$	$\phi''_{2,4}$
$1/2\, q\phi_4\phi_8\phi_{12}$	11	1	$F_4(q)$	$\phi'_{2,4}$
$1/2\, q\phi_2^2\phi_6^2\phi_8$	11	1	$F_4(q)$	$\phi_{4,1}$
$q^2\phi_3^2\phi_6^2\phi_{12}$	14	1	$F_4(q)$	$\phi_{9,2}$
$\phi_1\phi_3\phi_4\phi_6\phi_8\phi_{12}$	15	$1/2\, q$	$C_3(q) \times \phi_2$	$[[3],[]]$
$\phi_1\phi_3\phi_4\phi_6\phi_8\phi_{12}$	15	$1/2\, q$	$B_3(q) \times \phi_2$	$[[3],[]]$

with the following exception for small q:
$q = 2$:

Degree	Mult.	$C(s)$	Char.
833	1	$F_4(q)$	$[[2], []]$
1105	1	$F_4(q)$	$\phi''_{2,4}$
1105	1	$F_4(q)$	$\phi'_{2,4}$
1326	1	$F_4(q)$	$F_4^{II}[1]$
1377	1	$F_4(q)$	$\phi_{4,1}$
21658	1	$F_4(q)$	$F_4^{I}[1]$

Case $q \equiv 1 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$\phi_3\phi_6\phi_{12}$	8	1	$B_4(q)$	$[[4], []]$
$1/2\, q\phi_1^2\phi_3^2\phi_8$	11	1	$F_4(q)$	$[[2], []]$
$1/2\, q\phi_4\phi_8\phi_{12}$	11	1	$F_4(q)$	$\phi''_{2,4}$
$1/2\, q\phi_4\phi_8\phi_{12}$	11	1	$F_4(q)$	$\phi'_{2,4}$
$1/2\, q\phi_2^2\phi_6^2\phi_8$	11	1	$F_4(q)$	$\phi_{4,1}$
$q^2\phi_3^2\phi_6^2\phi_{12}$	14	1	$F_4(q)$	$\phi_{9,2}$

with the following exception for small q:
$q = 3$:

Degree	Mult.	$C(s)$	Char.
6643	1	$B_4(q)$	$[[4], []]$
83148	1	$F_4(q)$	$[[2], []]$
89790	1	$F_4(q)$	$\phi''_{2,4}$
89790	1	$F_4(q)$	$\phi'_{2,4}$
96432	1	$F_4(q)$	$\phi_{4,1}$
5181540	1	$B_4(q)$	$[[0, 1, 4], []]$

5.7 The groups ${}^2E_6(q)_{sc}$

For the computation the congruence classes of q modulo 6 must be distinguished. The results only depend on q modulo 2.

Case $q \equiv 0 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$q\phi_8\phi_{18}$	11	1	${}^2E_6(q)$	$\phi'_{2,4}$
$\phi_3\phi_6^2\phi_{12}\phi_{18}$	16	q	${}^2D_5(q) \times \phi_2$	$[[0,5],[]]$
$q^2\phi_4\phi_8\phi_{10}\phi_{12}$	16	1	${}^2E_6(q)$	$\phi_{4,1}$
$1/2\,q^3\phi_8\phi_{10}\phi_{12}\phi_{18}$	21	1	${}^2E_6(q)$	$\phi'_{1,12}$
$1/2\,q^3\phi_4^2\phi_{10}\phi_{12}\phi_{18}$	21	1	${}^2E_6(q)$	$\phi''_{2,4}$
$1/2\,q^3\phi_2^4\phi_6^2\phi_{10}\phi_{18}$	21	1	${}^2E_6(q)$	$\phi'_{8,3}$

with the following exception for small q:
$q = 2$:

Degree	Mult.	$C(s)$	Char.
1938	1	${}^2E_6(q)$	$\phi'_{2,4}$
46683	2	${}^2D_5(q) \times \phi_2$	$[[0,5],[]]$
48620	1	${}^2E_6(q)$	$\phi_{4,1}$
554268	1	${}^2E_6(q)$	$\phi'_{1,12}$
815100	1	${}^2E_6(q)$	$\phi''_{2,4}$
1322685	1	${}^2A_5(q) \times \phi_2$	$[6]$

Case $q \equiv 1 \pmod 2$

Generically smallest non-trivial degrees:

Degree	A	Mult.	$C(s)$	Char.
$q\phi_8\phi_{18}$	11	1	${}^2E_6(q)$	$\phi'_{2,4}$
$\phi_3\phi_6^2\phi_{12}\phi_{18}$	16	q	${}^2D_5(q) \times \phi_2$	$[[0,5],[]]$
$q^2\phi_4\phi_8\phi_{10}\phi_{12}$	16	1	${}^2E_6(q)$	$\phi_{4,1}$
$\phi_3\phi_4\phi_6\phi_8\phi_{12}\phi_{18}$	20	1	${}^2A_5(q) \times A_1(q)$	$[6],[2]$
$1/2\,q^3\phi_8\phi_{10}\phi_{12}\phi_{18}$	21	1	${}^2E_6(q)$	$\phi'_{1,12}$
$1/2\,q^3\phi_4^2\phi_{10}\phi_{12}\phi_{18}$	21	1	${}^2E_6(q)$	$\phi''_{2,4}$

with the following exception for small q:
$q = 3$:

Degree	Mult.	$C(s)$	Char.
172938	1	${}^2E_6(q)$	$\phi'_{2,4}$
32690203	3	${}^2D_5(q) \times \phi_2$	$[[0,5],[]]$
32863140	1	${}^2E_6(q)$	$\phi_{4,1}$
3465418113	1	${}^2E_6(q)$	$\phi'_{1,12}$
3829423780	1	${}^2A_5(q) \times A_1(q)$	$[6],[2]$
4226119650	1	${}^2E_6(q)$	$\phi''_{2,4}$

5.8 The groups $E_6(q)_{sc}$

For the computation the congruence classes of q modulo 6 must be distinguished. The results only depend on q modulo 2.

Case $q \equiv 0 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$q\phi_8\phi_9$	11	1	$E_6(q)$	$\phi_{6,1}$
$q^2\phi_4\phi_5\phi_8\phi_{12}$	16	1	$E_6(q)$	$\phi_{20,2}$
$\phi_3^2\phi_6\phi_9\phi_{12}$	16	$(q-2)$	$D_5(q) \times \phi_1$	$[[0],[5]]$
$1/2\, q^3\phi_1^4\phi_3^2\phi_5\phi_9$	21	1	$E_6(q)$	$D_4[1]$
$1/2\, q^3\phi_5\phi_6^2\phi_8\phi_9$	21	1	$E_6(q)$	$\phi_{15,5}$
$1/2\, q^3\phi_5\phi_8\phi_9\phi_{12}$	21	1	$E_6(q)$	$\phi_{15,4}$

with the following exception for small q:
$q = 2$:

Degree	Mult.	$C(s)$	Char.
2482	1	$E_6(q)$	$\phi_{6,1}$
137020	1	$E_6(q)$	$\phi_{20,2}$
443548	1	$E_6(q)$	$D_4[1]$
1384956	1	$E_6(q)$	$\phi_{15,5}$
1693965	1	$A_5(q) \times \phi_2$	$[6]$
2000492	1	$E_6(q)$	$\phi_{15,4}$

Case $q \equiv 1 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$q\phi_8\phi_9$	11	1	$E_6(q)$	$\phi_{6,1}$
$q^2\phi_4\phi_5\phi_8\phi_{12}$	16	1	$E_6(q)$	$\phi_{20,2}$
$\phi_3^2\phi_6\phi_9\phi_{12}$	16	$(q-2)$	$D_5(q) \times \phi_1$	[[0], [5]]
$\phi_3\phi_4\phi_6\phi_8\phi_9\phi_{12}$	20	1	$A_5(q) \times A_1(q)$	[6], [2]
$1/2\, q^3\phi_1^4\phi_3^2\phi_5\phi_9$	21	1	$E_6(q)$	$D_4[1]$
$1/2\, q^3\phi_5\phi_6^2\phi_8\phi_9$	21	1	$E_6(q)$	$\phi_{15,5}$

with the following exception for small q:
$q = 3$:

Degree	Mult.	$C(s)$	Char.
186222	1	$E_6(q)$	$\phi_{6,1}$
65187540	1	$E_6(q)$	$\phi_{20,2}$
65373763	1	$D_5(q) \times \phi_1$	[[0], [5]]
3343656888	1	$E_6(q)$	$D_4[1]$
4123575820	1	$A_5(q) \times A_1(q)$	[6], [2]
4968496071	1	$E_6(q)$	$\phi_{15,5}$

5.9 The groups $E_7(q)_{sc}$

For the computation the congruence classes of q modulo 12 must be distinguished. The results only depend on q modulo 2.

Case $q \equiv 0 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$q\phi_7\phi_{12}\phi_{14}$	17	1	$E_7(q)$	$\phi_{7,1}$
$q^2\phi_3^2\phi_6^2\phi_9\phi_{12}\phi_{18}$	26	1	$E_7(q)$	$\phi_{27,2}$
$\phi_1^3\phi_3\phi_5\phi_7\phi_9\phi_{14}$	27	$1/2\, q$	${}^2E_6(q) \times \phi_2$	$\phi_{1,0}$
$q^3\phi_7\phi_9\phi_{14}\phi_{18}$	27	1	$E_7(q)$	$\phi_{21,3}$
$\phi_2^3\phi_6\phi_7\phi_{10}\phi_{14}\phi_{18}$	27	$1/2\,(q-2)$	$E_6(q) \times \phi_1$	$\phi_{1,0}$
$1/2\, q^3\phi_1^4\phi_3^2\phi_5\phi_7\phi_9\phi_{14}$	33	1	$E_7(q)$	D_4[[3], []]

with the following exception for small q:
$q = 2$:

Degree	Mult.	$C(s)$	Char.
141986	1	$E_7(q)$	$\phi_{7,1}$
86507701	1	${}^2E_6(q) \times \phi_2$	$\phi_{1,0}$
95420052	1	$E_7(q)$	$\phi_{27,2}$
181785768	1	$E_7(q)$	$\phi_{21,3}$
2422215628	1	$E_7(q)$	$D_4[[3],[]]$
3876501772	1	$E_7(q)$	$\phi_{21,6}$

Case $q \equiv 1 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$q\phi_7\phi_{12}\phi_{14}$	17	1	$E_7(q)$	$\phi_{7,1}$
$q^2\phi_3^2\phi_6^2\phi_9\phi_{12}\phi_{18}$	26	1	$E_7(q)$	$\phi_{27,2}$
$1/2\,\phi_1^3\phi_3\phi_5\phi_7\phi_9\phi_{14}$	27	1	${}^2E_6(q).2 \times \phi_2$	$\phi_{1,0}$
$1/2\,\phi_1^3\phi_3\phi_5\phi_7\phi_9\phi_{14}$	27	1	${}^2E_6(q).2 \times \phi_2$	$\phi_{1,0}$
$1/2\,\phi_2^3\phi_6\phi_7\phi_{10}\phi_{14}\phi_{18}$	27	1	$E_6(q).2 \times \phi_1$	$\phi_{1,0}$
$1/2\,\phi_2^3\phi_6\phi_7\phi_{10}\phi_{14}\phi_{18}$	27	1	$E_6(q).2 \times \phi_1$	$\phi_{1,0}$

(In the cases of non-connected centralizers of s there are two characters in the Lusztig series corresponding to the trivial representation of the connected centralizer. We don't distinguish them with the given labels.)
with the following exception for small q:
$q = 3$:

Degree	Mult.	$C(s)$	Char.
130933749	1	$E_7(q)$	$\phi_{7,1}$
2847685879324	1	${}^2E_6(q).2 \times \phi_2$	$\phi_{1,0}$
2847685879324	1	${}^2E_6(q).2 \times \phi_2$	$\phi_{1,0}$
2895338589507	1	$E_7(q)$	$\phi_{27,2}$
5695371758648	1	${}^2E_6(q) \times \phi_2$	$\phi_{1,0}$
5743024468832	1	$E_6(q).2 \times \phi_1$	$\phi_{1,0}$
5743024468832	1	$E_6(q).2 \times \phi_1$	$\phi_{1,0}$

5.10 The groups $E_8(q)$

For the computation the congruence classes of q modulo 60 must be distinguished. The results only depend on q modulo 2.

Case $q \equiv 0 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$q\phi_4^2\phi_8\phi_{12}\phi_{20}\phi_{24}$	29	1	$E_8(q)$	$\phi_{8,1}$
$q^2\phi_5\phi_7\phi_{10}\phi_{14}\phi_{15}\phi_{20}\phi_{30}$	46	1	$E_8(q)$	$\phi_{35,2}$
$1/2\, q^3\phi_1^4\phi_3^2\phi_5^2\phi_7\phi_8\phi_9\phi_{14}\phi_{15}\phi_{24}$	57	1	$E_8(q)$	$D_4[\phi_{1,0}]$
$1/2\, q^3\phi_4^2\phi_7\phi_8\phi_{12}\phi_{14}\phi_{15}\phi_{18}\phi_{20}\phi_{24}$	57	1	$E_8(q)$	$\phi_{28,8}$
$1/2\, q^3\phi_4^2\phi_7\phi_8\phi_9\phi_{12}\phi_{14}\phi_{20}\phi_{24}\phi_{30}$	57	1	$E_8(q)$	$\phi_{84,4}$
$1/2\, q^3\phi_2^4\phi_6^2\phi_7\phi_8\phi_{10}^2\phi_{14}\phi_{18}\phi_{24}\phi_{30}$	57	1	$E_8(q)$	$\phi_{112,3}$

with the following exception for small q:
$q = 2$:

Degree	Mult.	$C(s)$	Char.
545925250	1	$E_8(q)$	$\phi_{8,1}$
76321227908420	1	$E_8(q)$	$\phi_{35,2}$
46453389380074796	1	$E_8(q)$	$D_4[\phi_{1,0}]$
51320060161363500	1	$E_8(q)$	$\phi_{28,8}$
97697128859455125	1	$E_7(q) \times \phi_2$	$\phi_{1,0}$
144074197011621500	1	$E_8(q)$	$\phi_{84,4}$

Case $q \equiv 1 \pmod 2$

Generically smallest non-trivial degrees:

Degree	d	Mult.	$C(s)$	Char.
$q\phi_4^2\phi_8\phi_{12}\phi_{20}\phi_{24}$	29	1	$E_8(q)$	$\phi_{8,1}$
$q^2\phi_5\phi_7\phi_{10}\phi_{14}\phi_{15}\phi_{20}\phi_{30}$	46	1	$E_8(q)$	$\phi_{35,2}$
$\phi_3\phi_4^2\phi_5\phi_6\phi_8\phi_{10}\phi_{12}\phi_{15}\phi_{20}\phi_{24}\phi_{30}$	56	1	$E_7(q) \times A_1(q)$	$\phi_{1,0}, [2]$
$1/2\, q^3\phi_1^4\phi_3^2\phi_5^2\phi_7\phi_8\phi_9\phi_{14}\phi_{15}\phi_{24}$	57	1	$E_8(q)$	$D_4[\phi_{1,0}]$
$1/2\, q^3\phi_4^2\phi_7\phi_8\phi_{12}\phi_{14}\phi_{15}\phi_{18}\phi_{20}\phi_{24}$	57	1	$E_8(q)$	$\phi_{28,8}$
$1/2\, q^3\phi_4^2\phi_7\phi_8\phi_9\phi_{12}\phi_{14}\phi_{20}\phi_{24}\phi_{30}$	57	1	$E_8(q)$	$\phi_{84,4}$

with the following exception for small q:
$q = 3$:

Degree	Mult.	$C(s)$	Char.
68725813719000	1	$E_8(q)$	$\phi_{8,1}$
8986201692043878710595	1	$E_8(q)$	$\phi_{35,2}$
5863143319345635265071660 96	1	$E_8(q)$	$D_4[\phi_{1,0}]$
5895848162804500302031630 00	1	$E_7(q) \times A_1(q)$	$\phi_{1,0}, [2]$
5928642868279598519641515 00	1	$E_8(q)$	$\phi_{28,8}$
1175890162013321512831618500	1	$E_8(q)$	$\phi_{84,4}$

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