$(2,3)$-GENERATION OF EXCEPTIONAL GROUPS

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Abstract. We study two aspects of generation of large exceptional groups of Lie type. First we show that any finite exceptional group of Lie rank at least four is $(2,3)$-generated, i.e., a factor group of the modular group $\mathrm{PSL}_2(\mathbb{Z})$. This completes the study of $(2,3)$-generation of groups of Lie type. Secondly we complete the proof that groups of type E_7 and E_8 over fields of odd characteristic occur as Galois groups of geometric extensions of $\mathbb{Q}^{\mathrm{ab}}(t)$, where $\mathbb{Q}^{\mathrm{ab}}$ denotes the maximal abelian extension field of $\mathbb{Q}$. Finally, we show that all finite simple exceptional groups of Lie type have a pair of strongly orthogonal classes. The methods of proof in all three cases are very similar and require the Lusztig theory of characters of reductive groups over finite fields as well as the classification of finite simple groups.

1. Introduction

A group G is called a $(2,3)$-group if it can be generated by an involution and an element of order 3. This is equivalent to saying that G is a factor group of $\mathrm{PSL}_2(\mathbb{Z})$, which is the free product of two cyclic groups of order two and three. Thus every $(2,3)$-group corresponds to a normal (in general: non-congruence) subgroup of $\mathrm{SL}_2(\mathbb{Z})$. There has recently been some interest in determining the $(2,3)$-generated finite simple groups (see for example [**3**, **5**, **12**], and the references cited there). For the simple exceptional groups of Lie type, the question has been settled positively for ${}^2G_2(q)$ and $G_2(q)$ in [**11**], for ${}^3D_4(q)$ and ${}^2F_4(q)$ in [**12**]. In this paper we consider the five families of exceptional groups of large rank.

Theorem 1.1. *The simple groups of types $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, $E_8(q)$, $q = p^n$, p a prime, are $(2,3)$-generated.*

In [**11**, **12**] the $(2,3)$-generation property was proved for exceptional groups of small rank (apart from ${}^2B_2(2^{2n+1})$ and $G_2(2)'$). Thus all exceptional groups of Lie type (apart from the exceptions mentioned) are $(2,3)$-generated. Since the subgroup of $\langle \sigma,\tau \mid \sigma^2=\tau^3=1\rangle$ generated by $\{\sigma,\sigma^\tau,\sigma^{\tau^2}\}$ respectively $\{\tau,\tau^\sigma\}$ has index 3 respectively 2, we immediately obtain the following consequence, the first part of which strengthens [**21**, Theorem B]:

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Corollary 1.2. *Let G be an exceptional simple group of Lie type different from ${}^2B_2(2^{2n+1})$ and $G_2(2)'$.*
(a) *G is generated by three conjugate involutions.*
(b) *G is generated by two conjugate elements of order 3.*

Together with previous results of Liebeck and Shalev [**5**] for classical groups, the second author for exceptional groups [**11**, **12**] and of Miller [**15**] for alternating groups Theorem 1.1 completes the determination of finite simple $(2,3)$-groups, up to finitely many open cases:

Corollary 1.3. *Let G be a finite nonabelian simple group different from $Sp_4(2^n)$, $Sp_4(3^n)$, ${}^2B_2(2^{2n+1})$. Then, up to a finite number of possible exceptions, G is $(2,3)$-generated.*

With similar methods we show that certain exceptional groups of Lie type in bad characteristic occur as Galois groups over $\mathbb{Q}^{\mathrm{ab}}(t)$:

Theorem 1.4. *The groups $E_7(3^n)$, $E_8(3^n)$ and $E_8(5^n)$ occur as Galois groups of geometric field extensions of $\mathbb{Q}^{\mathrm{ab}}(t)$.*

Together with previous results of the second author [**10**] this implies:

Corollary 1.5. *Let G be an exceptional group of Lie type in odd characteristic. Then G occurs as the Galois group of a geometric field extension of $\mathbb{Q}^{\mathrm{ab}}(t)$.*

A pair (C_1, C_2) of conjugacy classes of a finite group G is called *strongly orthogonal* if there exist only two irreducible characters χ of G that such $\chi(C_1)\chi(C_2) \neq 0$. We show that pairs of strongly orthogonal classes exist for exceptional groups and use this to prove:

Corollary 1.6. *Let G be a finite simple exceptional group of Lie type. Then G has a conjugacy class C such that $G = C^2 \cup C^3$.*

2. Methods of the proof

Let G be a finite group and $\mathbf{C} := (C_1, C_2, C_3)$ a triple of conjugacy classes of G. Then

$$n(\mathbf{C}) := \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|}{\chi(1)} \prod_{i=1}^{3} \frac{\chi(\sigma_i)}{|C_G(\sigma_i)|}, \qquad \text{where } \sigma_i \in C_i, \tag{2.1}$$

where χ runs over the set $\mathrm{Irr}(G)$ of complex irreducible characters of G, is called the *normalized structure constant* of the class vector $\mathbf{C}$. If moreover no non-trivial element of G has centralizer intersecting all three classes of $\mathbf{C}$, then $n(\mathbf{C})$ equals the cardinality of the set

$$\bar{\Sigma}(\mathbf{C}) := \{(\sigma_1, \sigma_2, \sigma_3) \mid \sigma_i \in C_i,\ \sigma_1\sigma_2\sigma_3 = 1\}$$

modulo G-conjugation. In this case this quantity can hence be calculated from the ordinary character table of G.

We always choose C_2 to be a class of involutions and C_3 a class of elements of order 3. Then we show for suitable C_1 that

1. $n(\mathbf{C}) > 0$ and
2. not all triples from $\bar{\Sigma}(\mathbf{C})$ generate proper subgroups of G.

From this it follows that G is $(2,3)$-generated. For the existence of Galois realizations we show more precisely that $n(\mathbf{C}) = 1$, so that $\mathbf{C}$ is a rigid class vector.

To show 2. we will use results on maximal subgroups of G and sometimes compute structure constants inside those subgroups.

We now describe the computation of the structure constants $n(\mathbf{C})$ in some detail. Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{F}_q$, q a power of a prime, and $G = \mathbf{G}(q)$ the corresponding finite group of Lie type. We refer the reader to [**1**, **6**–**9**] for more information on Lusztig's theory.

In [**6**] and [**8**] Lusztig gives a parameterization of the complex irreducible characters of G in terms of semisimple classes of the dual group $G^* = \mathbf{G}^*(q)$. There is still no general method known to compute all character values of G, but we can compute those values we need for our applications, using the following ideas.

A class function on G is called *uniform* if it is a linear combination of Deligne-Lusztig characters $R_{T,\theta}$. We always choose classes C for our class triples which have the following property:

(*) The characteristic function $\chi_C : G \to \mathbb{C}$, $\chi_C(\sigma) = \begin{cases} 1 & \text{for } \sigma \in C, \\ 0 & \text{else,} \end{cases}$ is uniform.

Hence the class functions of G orthogonal to all $R_{T,\theta}$ take value zero on these classes. It follows that we can compute all values of the irreducible characters on these classes from the values of the Deligne-Lusztig characters, if we know the decomposition of the $R_{T,\theta}$. This decomposition is described explicitly in [**6**] for groups with connected center and refined to the general case in [**8**].

In fact the conjugacy classes in our class triples will always either be semisimple or unipotent. Semisimple classes always satisfy property (*), see [**1**, Prop. 7.5.5], and the values of $R_{T,\theta}$ on such classes can be computed by using [**1**, Prop. 7.5.3].

For unipotent classes the values of the $R_{T,\theta}$ are given by the Green functions Q_T^G, see [**1**, Cor. 7.2.9]. Up to one problem, which will be described presently, these Green functions can now be calculated by a general method: In [**7**, 24.] Lusztig describes an algorithm to compute Green functions associated to character sheaves of the algebraic group $\mathbf{G}$. Later it was shown by Lusztig [**9**] and Shoji [**20**] that the Green functions computed by this algorithm in fact coincide with the Green functions Q_T^G appearing in the formula for the Deligne-Lusztig characters. (The Q_T^G were previously already known for exceptional groups in good characteristic by work of Shoji and Beynon-Spaltenstein, and they had been computed by the second author for $F_4(2^n)$ and $E_6(2^n)$, and by Porsch for $E_6(3^n)$. But the results cited above cover the other bad characteristic cases, too, which is important for our applications.)

The problem with computing Green functions mentioned above is the following: for the classes $C = u^G$ considered in our applications, we can in general compute the values $Q_T^G(u)$ only up to a complex scalar of absolute value one (which is the same for all T). (In general, the functions Y_i appearing in [**7**, 24.2.3] are only known up to such a scalar.) But our class vectors will contain at most one unipotent class and so we can determine this unknown scalar from a structure constant $n(\mathbf{C}) \neq 0$ by the condition $n(\mathbf{C}) > 0$.

For the unipotent classes in our class triples the property (*) follows from two facts, satisfied in all cases: First, using the character formula [**1**, Thm. 7.2.8] for $R_{T,\theta}$, it is easy to see that

$$\frac{1}{|T|}\sum_{\theta\in\hat{T}} R_{T,\theta}(\sigma) = \begin{cases} Q_T^G(\sigma) & \text{for } \sigma \text{ unipotent,} \\ 0 & \text{else.} \end{cases}$$

for a maximal torus T of G with character group $\hat{T}$. Secondly, we check in each case that the characteristic function on the relevant class is a linear combination of Q_T^G's.

The actual computations of the structure constants were done using computer programs written by the first named author. These programs are written in the programming language of the computer algebra system GAP [**17**] and they will become part of the CHEVIE package [**4**].

All statements about semisimple classes which appear in the rest of the paper were checked (and sometimes found) during these computations.

In the sequel we specify semisimple classes by giving the Dynkin type of the centralizers of their elements. For unipotent classes we give references where explicit representatives can be found.

The orders of elements in unipotent classes from our class triples are always prime. This can be checked in each case by using explicit representatives. In many cases these are given as products of elements of pairwise commuting root subgroups with respect to some maximally split torus. The order of elements of such a root subgroup is equal to the characteristic of the ground field. In the remaining cases the order of the representatives can be calculated directly, using the commutator relations described in the cited papers.

For semisimple classes we get explicit representatives from our calculations. So the orders can easily be determined.

3. The groups $F_4(q)$

Theorem 3.1. *The groups $F_4(q)$, $q = p^n$, p a prime, are $(2,3)$-generated.*

Proof. We use the strategy of proof outlined in Section 2. For the first conjugacy class C_1 in $G := F_4(q)$ we choose the class of a generator of a cyclic Coxeter torus T of order $q^4 - q^2 + 1$ of G. Thus elements of C_1 are semisimple and moreover regular, since all non-identity elements of T are regular.

We will see that this choice of C_1 has two advantages: There are only a few maximal subgroups of G containing T (and they are known), and there are only a few Deligne-Lusztig characters having nonzero value on all three classes simultaneously.

If $p \neq 2$, G has two (semisimple) classes of involutions, one with centralizer of type $B_4(q)$, the other with centralizer of type $C_3(q) \circ A_1(q)$. We let C_2 be the class of an involution with centralizer $C_3(q) \circ A_1(q)$. If $p \neq 3$, G has a class of elements of order 3 with centralizer of type $A_2(q) \circ A_2(q)$, if $q \equiv 1 \pmod 3$ or ${}^2A_2(q) \circ {}^2A_2(q)$, if $q \equiv 2 \pmod 3$. Let C_3 be the class of such an element.

In even characteristic, we let C_2 be the (unipotent) class of an involution with centralizer of order $q^{20}(q^2-1)^2$, denoted x_4 in [**18**]. In characteristic 3, we define C_3 to be the unipotent class with centralizer of order $q^{14}(q^2-1)$, denoted x_{11} in [**19**].

Let $\mathbf{C}$ be the class vector consisting of the three conjugacy classes defined above. We show that only unipotent characters can contribute to the structure constant given by (2.1). First we see from the character formula [**1**, Thm. 7.2.8] that a Deligne-Lusztig character $R_{T',\theta}$ is zero on the class C_1 unless $T' \sim T$. In the present case for G the dual group G^* is isomorphic to G and T^* is isomorphic to T. Since all non-identity elements of T are regular this shows that also all non-identity elements of the dual torus T^* are regular. Hence for $\theta \neq 1$ either $R_{T,\theta}$ or $-R_{T,\theta}$ is an irreducible character of G. But if $p \neq 2$ the centralizer of an element from C_2 does not contain a conjugate of T, so all $R_{T,\theta}$ vanish on C_2. Similarly for $p \neq 3$ all $R_{T,\theta}$ vanish on C_3. Thus we are left to consider the unipotent characters.

Calculation now yields the structure constants given in Table 3.2.

Table 3.2. Normalized $(2,3)$-structure constants in $F_4(q)$

$q \pmod 6$	$n(\mathbf{C})$
1	$q^8+4q^6+2q^5+8q^4+2q^3+4q^2+1$
2	$q^2(q^6+4q^4-2q^3+7q^2-2q+2)$
3	$q^2(q^6+3q^4+5q^2-1)$
4	$q^2(q^6+4q^4+2q^3+7q^2+2q+2)$
5	$q^8+4q^6-2q^5+8q^4-2q^3+4q^2+1$

Clearly this shows $n(\mathbf{C}) > 0$ for all q.

By the remarks in Section 2 it remains to prove generation. Assume first that $q \geq 4$. Then by [**21**, 4(f)] the only proper subgroups of G containing the Coxeter torus T lie inside maximal subgroups isomorphic to $M := {}^3D_4(q).3$. There is one such class of maximal subgroups if q is odd and there are two if q is even. Assume that a triple $(\sigma_1, \sigma_2, \sigma_3) \in \bar{\Sigma}(\mathbf{C})$ generates a subgroup of (some conjugate of) M. Then there exists a class vector $\mathbf{C}' = (C_1', C_2', C_3')$ of M with $n(\mathbf{C}') > 0$. It hence suffices to show that the sum of all structure constants in M of class vectors $\mathbf{C}'$ is strictly smaller than that in G (respectively half of that in G), where the sum runs over all class triples such that C_1' is a fixed conjugacy class in the intersection of C_1 with M, C_2' is the unique class of involutions of M if p is odd, and runs through the 2 classes of involutions of M in even characteristic, and C_3' runs over the conjugacy classes of elements of order 3 in M. (Note that we need not consider classes outside the simple socle $M' := {}^3D_4(q)$ of M, since the other two classes in $\mathbf{C}$ are certainly contained in M'.) Table 3.3 contains the sum of the structure constants for those class vectors $\mathbf{C}' = (C_1', C_2', C_3')$ of M. These values are computed from the generic character table for the groups ${}^3D_4(q)$ contained in CHEVIE [**4**].

It is an easy exercise to check that the difference of the F_4-structure constant and the ${}^3D_4$-structure constant is always positive. This proves the claim for $q \geq 4$.

Table 3.3. Normalized $(2,3)$-structure constants in ${}^3D_4(q)$

	$\sum_{\mathbf{C}'} n(\mathbf{C}')$
$q \equiv 0 \pmod 3$	q^2+1
$q \not\equiv 0 \pmod 3$	$2q^2+2$

The maximal subgroups of $F_4(2)$ are explicitly known [**2**]. The only ones with order divisible by $|T| = 13$ are two classes of ${}^3D_4(2).3$, one class of ${}^2F_4(2)$ and one class $L_4(3).2_2$. From the Atlas [**2**] we find that the $(2,3,13)$-structure constants of ${}^2F_4(2)$ add up to 18, while those of $L_4(3).2_2$ give 14. The structure constants for ${}^3D_4(2).3$ are contained in Table 3.3. Since $n(\mathbf{C}) = 552$ by Table 3.1 the result follows in this case.

If $q = 3$, then by [**21**, 4(f)] the only other possibility for a maximal subgroup of G containing T would be an almost simple group with derived group isomorphic to $U_3(9)$. But the centralizer order in $U_3(9)$ of an involution is divisible by 5^2, while the elements in class C_2 of $F_4(3)$ have centralizer order only divisible by 5 to the first power. Thus the involutions of a possible subgroup of type $U_3(9)$ do not fuse into C_2 and a triple from $\mathbf{C}$ cannot lie inside such a subgroup. The rest of the proof is as in the case $q \geq 4$. $\square$

4. The groups $E_6(q)$

Theorem 4.1. *The groups $E_6(q)_{sc}$ and the simple groups $E_6(q)$, $q = p^n$, p a prime, are $(2,3)$-generated.*

Proof. It is sufficient to show that $G := E_6(q)_{sc}$ is $(2,3)$-generated (by non-central elements) since $E_6(q) = G/Z(G)$.

Let C_1 be the conjugacy class of a generator of a cyclic torus T of G of order q^6+q^3+1. If $p \neq 2$ let C_2 be the class of an involution with centralizer $A_5(q) \circ A_1(q)$. If $p \neq 3$, G has a class of elements of order 3 with centralizer of type $A_2(q)^3$ or $A_2(q^2) \circ {}^2A_2(q)$, depending on the congruence of $q \pmod 3$. Let C_3 be the class of such an element.

In even characteristic, we let C_2 be the class of an involution with centralizer of order $q^{31}(q^2-1)^2(q^3-1)$, denoted $3A_1$ in [**16**, Table 1]. In characteristic 3, we define C_3 to be the unipotent class with centralizer order $q^{22}(q^2-1)$, denoted $2A_2 + A_1$.

Let $\mathbf{C}$ be the class vector consisting of the three conjugacy classes defined above and $T^* \subset G^* = E_6(q)_{ad}$ the dual torus of T.

If $q \not\equiv 1 \pmod 3$ then $T^* \setminus Z(G^*)$ consists of regular elements. Similar considerations to those in the case of F_4 show that the only irreducible characters that can contribute to $n(\mathbf{C})$ are among the unipotent ones.

If $q \equiv 1 \pmod 3$ then T^* additionally contains elements from two classes with centralizer of type $A_2(q^3).3$. Let θ be a character of T corresponding to an element of T^* with centralizer H of type $A_2(q^3).3$. There is only one torus S of G, up to conjugacy, which contains representatives of C_3 and whose dual S^* is contained in H.

So, for this θ, the only Deligne-Lusztig character which can have nonzero value on C_3 is $R_{S,\theta}$. But the explicit computation shows that the value of $R_{S,\theta}$ on C_3 is zero, too, for non regular θ. We get again that only unipotent characters contribute to $n(\mathbf{C})$.

The normalized structure constants $n(\mathbf{C})$ are given in Table 4.2.

Table 4.2. Normalized $(2,3)$-structure constants in $E_6(q)_{sc}$

$q \pmod 6$	$n(\mathbf{C})$
1	$(q^2+1)(q^8+3q^6+3q^5+5q^4+3q^3+3q^2+1)$
2	$q(q^2-q+1)(q^7+q^6+2q^5+q^3-2q^2-q-1)$
3	$q^2(q^2+1)(q^6+q^4+2q^2-q-1)$
4	$q(q^9+4q^7+3q^6+7q^5+5q^4+6q^3+2q^2+2q-1)$
5	$(q^2+1)(q^2-q+1)(q^6+q^5+q^4-q^3+q^2+q+1)$

By [**21**, 4(g)] a maximal subgroup of G containing a conjugate of the maximal torus T is isomorphic to $SL_3(q^3).3$, and there exists just one such class in G. In order to show that not all triples in $\bar{\Sigma}(\mathbf{C})$ lie inside such a subgroup we calculate the corresponding normalized structure constant in the derived group $SL_3(q^3)$. This can again be done with CHEVIE [**4**]. It turns out that the structure constant is equal to 1 in all cases. Thus there exist generating triples in $\bar{\Sigma}(\mathbf{C})$ for all q. □

5. The groups ${}^2E_6(q)$

Theorem 5.1. *The groups ${}^2E_6(q)_{sc}$ and the simple groups ${}^2E_6(q)$, $q = p^n$, p a prime, are $(2,3)$-generated.*

Proof. It is sufficient to prove that $G := {}^2E_6(q)_{sc}$ is $(2,3)$-generated.

Let C_1 be the conjugacy class of a generator of a cyclic torus T of order q^6-q^3+1 of G. If $p \neq 2$ let C_2 be the class of an involution with centralizer ${}^2A_5(q) \circ A_1(q)$. If $p \neq 3$, G has a class of elements of order 3 with centralizer of type ${}^2A_2(q)^3$ or $A_2(q^2) \circ A_2(q)$, depending on the congruence of $q \pmod 3$. Let C_3 be the class of such an element.

In even characteristic, we let C_2 be the class of an involution with centralizer of order $q^{31}(q^2-1)^2(q^3+1)$, denoted $3A_1$ in [**16**, Table 1]. In characteristic 3, we define C_3 to be the unipotent class with centralizer of order $q^{22}(q^2-1)$, denoted $2A_2+A_1$.

Let $\mathbf{C}$ be the class vector consisting of the three conjugacy classes defined above. By considerations very similar to those in case $E_6(q)$ we obtain the normalized structure constants $n(\mathbf{C})$ given in Table 5.2. (Here for $q \equiv -1 \pmod 3$ we have to exclude the possibility that elements in G^* with centralizer of type ${}^2A_2(q^3).3$ give a non-zero contribution.)

Again by [**21**, Table I and 4(h)] for $q \geq 4$ the subgroup lattice in G above the maximal torus T has $SU_3(q^3).3$ as unique maximal element. The structure constant inside $SU_3(q^3)$ for those class vectors which can possibly fuse into $\mathbf{C}$ is computed to be equal to 1, using CHEVIE.

Table 5.2. Normalized $(2,3)$-structure constants in ${}^2E_6(q)_{sc}$

$q \pmod 6$	$n(\mathbf{C})$
1	$(q^2+1)(q^2+q+1)(q^6-q^5+q^4-q^3+q^2-q+1)$
2	$q(q^9+4q^7-3q^6+7q^5-5q^4+6q^3-2q^2+2q+1)$
3	$q^2(q^2+1)(q^6+q^4+2q^2+q-1)$
4	$q(q^2+q+1)(q^7-q^6+2q^5+q^3+2q^2-q+1)$
5	$(q^2+1)(q^8+3q^6-3q^5+5q^4-3q^3+3q^2+1)$

Thus it remains to deal with the cases $q \in \{2,3\}$. By the arguments in [**10**, §7] it follows that a maximal subgroup containing the group M generated by a triple from $\bar{\Sigma}(\mathbf{C})$ is the normalizer of a non-abelian simple subgroup S of G. More precisely, either S is one of the simple groups listed in [**10**, Lemma 6.1], or it is of Lie type in characteristic p. The first possibility can be excluded since none of the relevant groups contains elements of order q^6-q^3+1 in its automorphism group. If on the other hand S is of Lie type in the same characteristic as G, then by the reasoning in [**10**] we conclude that $S = U_3(q^3)$. The argument in [**21**, 4(f)] now shows that there is a unique class of such subgroups in G. Thus we can conclude as above by comparing structure constants. □

It may be noticed that the structure constants for ${}^2E_6(q)$ in Table 5.2 are obtained from those for $E_6(q)$ in Table 4.2 by formally replacing q by $-q$. This is a consequence of the so-called Ennola-duality.

6. Galois realizations for $E_7(3^n)$

In [**10**, Thm. 8.2] it was shown that the simple groups $E_7(q)$, $q = p^n$ with $p \geq 5$, occur as the Galois groups of geometric field extensions of $\mathbb{Q}^{\mathrm{ab}}(t)$. (A field extension $N/\mathbb{Q}^{\mathrm{ab}}(t)$ is called regular if $\mathbb{Q}^{\mathrm{ab}}$ is algebraically closed in N.) Here we show that this remains true for $p = 3$. For this we verify that $E_7(q)$ has a rigid class vector $\mathbf{C}$ and then apply the rigidity criterion of Belyi, Matzat, and Thompson. We refer the reader to [**14**, **10**] for details on constructive Galois theory.

We start by defining the conjugacy classes for the class vector $\mathbf{C}$. Let p be an odd prime and $G := E_7(q)_{ad}$. The simple group $G' = E_7(q)$ then has index 2 in G. Let C_1 be the class of a generator of a maximal torus of order $(q+1)(q^6-q^3+1)$ of G. Choose C_2 so as to contain involutions from $G\backslash G'$. More precisely, in the case $q \equiv 1 \pmod 4$ an involution with (non-connected) centralizer ${}^2A_7(q).2$ is not contained in G', and in the case $q \equiv -1 \pmod 4$ the one with centralizer $A_7(q).2$ lies outside G'. For $q \equiv \epsilon \pmod 4$, $\epsilon \in \{1,-1\}$, let the class C_2^ϵ consist of such elements. Define C_p to be the unipotent class denoted $4A_1$ in [**16**, Table 2] with centralizer order $q^{51}(q^2-1)(q^4-1)(q^6-1)$.

Proposition 6.1. *The class vector $\mathbf{C} = (C_1, C_2^\epsilon, C_p)$ of $E_7(q)_{ad}$, $q = p^n$ for an odd prime p, satisfies $n(\mathbf{C}) = 1$.*

Proof. This structure constant was computed in [**10**, Prop. 8.1] in the case $p > 3$. In fact, the whole calculation remains the same for $p = 3$. For this it suffices to check that the values of the Green functions on C_p are given by the same polynomials in q for all odd p, which is the case.

(Of course, it would not be difficult to redo the whole calculation in the framework of this paper.) $\square$

Theorem 6.2. *The groups $E_7(q)_{ad}$ and the simple groups $E_7(q)$, $q = 3^n$, possess Galois realizations over $\mathbb{Q}^{\mathrm{ab}}(t)$.*

Proof. The corresponding statement was shown for $p > 3$ in [**10**, Ths. 8.1 and 8.2]. By the rigidity criterion [**14**, II, §4, Satz 2] it suffices to exhibit a rigid class vector of $G := E_7(q)_{ad}$. By Proposition 6.1 the class vector $\mathbf{C}$ satisfies $n(\mathbf{C}) = 1$. It remains to prove that any triple $(\sigma_1, \sigma_2, \sigma_3) \in \bar{\Sigma}(\mathbf{C})$ generates G. Let $H := \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ for such a triple. We follow the argument in [**10**], which can now be shortened considerably by using the information in [**21**]. The element $\sigma_1 \in C_1$ generates a maximal torus T of G of order $(q+1)(q^6 - q^3 + 1)$. By [**21**, 4(i)] for $q > 3$ the only maximal overgroup of the torus T in G is a subgroup $M \cong ((q+1).\,{}^2E_6(q)).2$. If $H \leq M$, then consider $\bar{H} := H/(H \cap M_1)$ with the normal subgroup $M_1 \cong {}^2E_6(q)$ of M. Since M/M_1 has order prime to p, $\bar{H}$ has a $(2, 2, 1)$ or a $(1, 1, 1)$-generating system. Thus H is contained in an extension of M_1 of degree 2. But such a group does not contain elements of order $(q+1)(q^6-q^3+1)$. This contradiction shows that $H = G$. Hence $\mathbf{C}$ is rigid and there exists a geometric Galois extension $N/\mathbb{Q}^{\mathrm{ab}}(t)$ with group G ramified at three points.

The simple group $G' := E_7(q)$ has index 2 in G. The standard descent argument now proves that the fixed field of N under G' is a rational function field $\mathbb{Q}^{\mathrm{ab}}(u)$, so that $N/\mathbb{Q}^{\mathrm{ab}}(u)$ yields a geometric Galois extension with group G'.

For the case $q = 3$ we may first apply [**10**, Lemma 1.2] to restrict the possible types of maximal subgroups M of G containing $\langle \sigma_1 \rangle$. The reasoning in [**10**, §8] then shows that any such M has a normal non-abelian simple subgroup S of order divisible by $q^6 - q^3 + 1$. Again as in loc.cit. we can only have $S = U_3(27)$ or $S = {}^2E_6(3)$. But in both cases $\mathrm{Aut}(S)$ does not contain elements of order $(q + 1)(q^6 - q^3 + 1)$, so if $H \leq \mathcal{N}_G(S)$ then $\mathcal{C}_G(S)$ is nontrivial and contains an element of order prime to 3. The centralizers of semisimple elements are known and it follows that any S as above is contained in a subgroup of type $((q+1).\,{}^2E_6(q)).2$. We may now proceed as before. $\square$

Since C_2 contains involutions and C_p contains elements of order p the preceding proof also shows that $E_7(3^n)_{ad}$ is $(2, 3)$-generated. But since $E_7(3^n)$ has index 2 in $E_7(3^n)_{ad}$, this cannot be used to yield $(2, 3)$-generation for the simple group.

7. (2, 3)-generation for $E_7(q)$

Theorem 7.1. *The groups $E_7(q)_{sc}$ and the simple groups $E_7(q)$, $q = p^n > 2$, p a prime, are $(2, 3)$-generated.*

Proof. Let $\delta = \delta(q)$ be defined to be -1 if $q \equiv -1 \pmod 3$, and to be 1 if $q \equiv 0, 1 \pmod 3$. Let C_1 be the conjugacy class of a generator of a cyclic torus T of order $(q + \delta)(q^6 - \delta q^3 + 1)$ of $G := E_7(q)_{sc}$. If $p \neq 2$ let C_2 be the class of an involution with centralizer of type $A_7(q)$ if $q \equiv 1 \pmod 4$, respectively of type ${}^2A_7(q)$ if $q \equiv -1 \pmod 4$. If $p \neq 3$ let C_3 be the class of an element of order 3 with centralizer of type $A_5(q) \circ A_2(q)$ if $q \equiv 1 \pmod 3$, respectively of type ${}^2A_5(q) \circ {}^2A_2(q)$ if $q \equiv 2 \pmod 3$.

In even characteristic, we let C_2 be the class of an involution with centralizer of order $q^{51}(q^2-1)(q^4-1)(q^6-1)$, denoted $4A_1$ in [**16**, Table 2]. In characteristic 3, we define C_3 to be the unipotent class with centralizer of order $q^{39}(q^2-1)^2$, denoted $2A_2 + A_1$.

Let $\mathbf{C}$ be the class vector consisting of the three conjugacy classes defined above.

In Table 7.2 we list the types of centralizers of semisimple elements s in the torus T^* dual to T. Here $\mathcal{E}(H,1)$ denotes the set of unipotent characters of H, and ${}^{-\delta}E_6(q)$ is to be interpreted as $E_6(q)$ if $\delta = -1$ and ${}^2E_6(q)$ if $\delta = 1$.

TABLE 7.2. Centralizers of elements in T^*

$\mathcal{C}_{G^*}(s)$	# classes	$\lvert\mathcal{E}(\mathcal{C}_{G^*}(s),1)\rvert$
$E_7(q)_{ad}$	1	76
$((q+\delta).{}^{-\delta}E_6(q)).2$	$\begin{cases} 1 & q \text{ odd} \\ 0 & q \text{ even} \end{cases}$	60
$(q+\delta).{}^{-\delta}E_6(q)$	$\begin{cases} (q-2+\delta)/2 & q \text{ odd} \\ (q-1+\delta)/2 & q \text{ even} \end{cases}$	30
T^*	$(q^7+\delta\, q^6-\delta\, q^4-q^3)/18$	1

The characters of G corresponding to the regular elements (those with centralizer T^*) do not contribute to the structure constant $n(\mathbf{C})$, because no conjugate of T lies in the centralizer of semisimple elements in C_2 and C_3.

In the cases $q \equiv 1, 9, 11 \pmod{12}$ a maximal torus in the centralizer of an element in C_2 is never conjugate to a torus which is dual to some torus in a centralizer with connected component of type E_6. So, in these cases only unipotent characters contribute to the structure constant.

For all other congruences we must explicitly compute the values of the Deligne-Lusztig characters and their constituents corresponding to the centralizers of type E_6 on the classes in $\mathbf{C}$. For the computation of the structure constants the possibilities for the congruence of q modulo 24 had to be distinguished. In all cases these characters give non-zero contributions to the structure constant. The results turn out to depend only on the congruence of q modulo 12. The normalized structure constants $n(\mathbf{C})$ are given in Table 7.3.

In [**21**, 4(i)] the maximal overgroups M of T are classified for all $q \geq 4$. The only possibilities are either a maximal parabolic subgroup of type E_6 or the normalizer of a Levi factor of type E_6 or ${}^2E_6$. Let H denote the group generated by a triple of elements from $\bar{\Sigma}(\mathbf{C})$. By our choice of δ the order of elements from C_1 is not divisible by 3. Thus the commutator factor group H/H' of H has order at most 2, so H' contains elements of order $|T|/2$ (respectively $|T|$ if $p = 2$). On the other hand, the derived group of a maximal parabolic subgroup of G of type E_6 does not contain

TABLE 7.3. Normalized (2, 3)-structure constants in $E_7(q)_{sc}$

q (mod 12)	$n(\mathbf{C})$
1	$q^{20} - q^{19} + 3q^{18} + q^{17} + 4q^{16} + 5q^{15} + 11q^{14} + 9q^{13} + 19q^{12} + 18q^{11}$ $+24q^{10} + 18q^9 + 19q^8 + 9q^7 + 11q^6 + 5q^5 + 4q^4 + q^3 + 3q^2 - q + 1$
2, 8	$(q^{19} + q^{18} + 3q^{17} + 5q^{15} - q^{14} + 11q^{13} - 3q^{12} + 13q^{11} - 11q^{10}$ $+10q^9 - 9q^8 + 6q^7 - q^4 - 2q^3 - 2q^2 - 2q - 1)q$
3	$q^4(q^2 - q + 1)(q^{12} + q^{10} - q^9 + 3q^8 - 3q^7 + 3q^6 - 4q^5 + 3q^4 - q^3$ $-5q^2 + 2q - 2)(q^2 + 1)$
4	$q(q^{19} - q^{18} + 3q^{17} + 5q^{15} + q^{14} + 11q^{13} + 3q^{12} + 13q^{11} + 11q^{10}$ $+10q^9 + 9q^8 + 6q^7 + q^4 - 2q^3 + 2q^2 - 2q + 1)$
5	$(q^2 + 1)(q^2 + q + 1)(q^{16} + q^{14} - q^{13} + 4q^{12} + 5q^{10} - 3q^9 + 6q^8$ $-3q^7 + 5q^6 + 4q^4 - q^3 + q^2 + 1)$
7	$(q^2 + 1)(q^2 - q + 1)(q^{16} + q^{14} + q^{13} + 4q^{12} + 5q^{10} + 3q^9 + 6q^8$ $+3q^7 + 5q^6 + 4q^4 + q^3 + q^2 + 1)$
9	$q^6(q^{14} - q^{13} + 3q^{12} - q^{11} + 5q^{10} - q^9 + 8q^8 + q^7 + 7q^6 + 6q^5 + 3q^4$ $+8q^3 - 4q^2 + 2q - 3)$
11	$q^{20} + q^{19} + 3q^{18} - q^{17} + 4q^{16} - 5q^{15} + 11q^{14} - 9q^{13} + 19q^{12} - 18q^{11}$ $+24q^{10} - 18q^9 + 19q^8 - 9q^7 + 11q^6 - 5q^5 + 4q^4 - q^3 + 3q^2 + q + 1$

elements of that order unless $q = 3$. Also, the normalizer of a Levi subgroup of type E_6 or ${}^2E_6$ in G does not contain such elements for $q \neq 3$.

For the remaining case $G = E_7(3)_{sc}$ we may argue as in the proof of Theorem 6.2 to show generation. □

Proposition 7.4. *The group $E_7(2)$ is (2, 3)-generated.*

Proof. For this proof let $q := 2$. We choose C_1 to be the class of a generator of the cyclic torus T of order q^7+1. Further, as above we let C_2 be the class of an involution with centralizer of order $q^{51}(q^2-1)(q^4-1)(q^6-1)$, denoted $4A_1$ in [**16**, Table 2], and C_3 the class of an element of order 3 with centralizer of type ${}^2A_5(q) \circ {}^2A_2(q)$. The structure constant of the corresponding class vector $\mathbf{C} = (C_1, C_2, C_3)$ turns out to be $n(\mathbf{C}) = 1605916$.

We next determine the possible maximal subgroups M of $G = E_7(2)$ which might contain a triple from $\bar{\Sigma}(\mathbf{C})$. Using [**10**, Lemma 1.2] and arguments similar to those in [**10**, proof of Thm. 8.1] we find that either M is local, or M is contained in the normalizer of a simple subgroup of G of order divisible by $43 = (2^7+1)/3$.

If M is a simple subgroup of G of order divisible by 43 then by [**10**, Lemma. 8.1] it is contained in the following list:

$$\{U_3(7), U_7(2), U_8(2)\}.$$

(Note that the orders of J_4 and $O_{14}^-(2)$ do not divide the order of G.) By considering element orders and centralizer orders it can be seen that only the unipotent class of

$U_8(2)$ having four Jordan blocks of size 2 in the natural representation can fuse into C_2. But all $(2,3,3.43)$-structure constants of $U_8(2)$ involving this class of involutions vanish. Further, neither $U_3(7)$ nor $U_7(2)$ contain elements of order 3.43 in their automorphism group. Hence if they are involved in M then their centralizer is non-trivial, i.e., they lie inside the centralizer of a 3-element. Thus we are left to consider the local subgroups. If M is local then either $M = \mathcal{N}_G(T)$ or M is the normalizer in G of a subgroup $3 \times U_7(2)$. There exists one class of each in G. Clearly no subgroup of $\mathcal{N}_G(T)$ is $(2,3,3.43)$-generated. It is known that G contains at least one class of subgroups $U_8(2)$ and this in turn contains a $3 \times U_7(2)$. But we already saw that $U_8(2)$ does not contribute, and since there exists just one class of $3 \times U_7(2)$, it also gives no contribution. So G is $(2,3)$-generated. □

8. Galois realizations for $E_8(3^n)$ and $E_8(5^n)$

Let $G := E_8(q)$, $q = p^n$ with $p \neq 2$. In [**10**, Thm. 9.1] it was shown that G occurs as the Galois group of a geometric field extension of $\mathbb{Q}^{\mathrm{ab}}(t)$ unless $p \leq 5$. Here we extend this result to cover the cases $p = 3, 5$ as well by exhibiting a rigid class vector $\mathbf{C}$ of G.

For this let C_1 be the conjugacy class of a generator of a cyclic torus T of order $\Phi_{30}(q) = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$ of G. Denote by C_2 the class of involutions of G with centralizer of type D_8 and by C_p the class of unipotent elements with centralizer order $q^{100}(q^2-1)(q^4-1)(q^6-1)(q^8-1)$, denoted $4A_1$ in [**16**, Table 3].

Proposition 8.1. *The class vector* $\mathbf{C} = (C_1, C_2, C_p)$ *of* $E_8(q)$, $q = p^n$, $p \geq 3$ *a prime, satisfies* $n(\mathbf{C}) = 1$.

Proof. In [**10**, Prop. 9.1] this was proved under the restriction $p \geq 7$ since at that time only the Green functions in good characteristic were known. By the recent results of Lusztig and Shoji cited above the same calculation holds in characteristic 3 and 5. The only thing we have to check is that the values of the Green functions on C_p are given by the same polynomials in q for all odd p, which is the case. □

Theorem 8.2. *The groups* $E_8(q)$, $q = p^n$, $p \neq 2$, *possess Galois realizations over* $\mathbb{Q}^{\mathrm{ab}}(t)$.

Proof. By the rigidity criterion ([**14**, II, §4, Satz 2]) and Proposition 8.1 above it suffices to prove that any triple $(\sigma_1, \sigma_2, \sigma_3) \in \bar{\Sigma}(\mathbf{C})$ generates $G := E_8(q)$. By [**21**, 4(j)] the only maximal overgroup of the torus $T = \langle \sigma_1 \rangle$ in G is the normalizer N of T in G. But N is solvable of order $30|T|$, while the group generated by a triple from $\bar{\Sigma}(\mathbf{C})$ is perfect, since the element orders in C_1, C_2, C_p are pairwise coprime (see [**10**, Lemma 1.6] for example). This proves the theorem. □

Note that in the case $p = 3$ this shows that $E_8(3^n)$ is $(2,3)$-generated.

9. $(2,3)$-generation for $E_8(q)$

Theorem 9.1. *The groups* $E_8(q)$, $q = p^n$, p *a prime, are* $(2,3)$*-generated.*

Proof. By the result of Theorem 8.2 above we may assume that $p \neq 3$. Let C_1 be the conjugacy class of a generator of the cyclic torus T of order $\Phi_{30}(q) = q^8 + q^7 -$

TABLE 9.2. Normalized (2, 3)-structure constants in $E_8(q)$

$q(6)$	$n(\mathbf{C})$
1	$q^{40} - q^{39} + 2\,q^{38} + 3\,q^{36} + q^{35} + 6\,q^{34} + 3\,q^{33} + 12\,q^{32} + 7\,q^{31} + 19\,q^{30} + 16\,q^{29}$
	$+32\,q^{28} + 26\,q^{27} + 48\,q^{26} + 42\,q^{25} + 69\,q^{24} + 56\,q^{23} + 87\,q^{22} + 69\,q^{21} + 98\,q^{20}$
	$+69\,q^{19} + 87\,q^{18} + 56\,q^{17} + 69\,q^{16} + 42\,q^{15} + 48\,q^{14} + 26\,q^{13} + 32\,q^{12}$
	$+16\,q^{11} + 19\,q^{10} + 7\,q^{9} + 12\,q^{8} + 3\,q^{7} + 6\,q^{6} + q^{5} + 3\,q^{4} + 2\,q^{2} - q + 1$
2	$q^4(q^2 - q + 1)(q^{34} + q^{32} - q^{31} + 3\,q^{30} - q^{29} + 6\,q^{28} - 4\,q^{27} + 11\,q^{26} - 9\,q^{25}$
	$+20\,q^{24} - 16\,q^{23} + 35\,q^{22} - 29\,q^{21} + 49\,q^{20} - 45\,q^{19} + 68\,q^{18} - 48\,q^{17}$
	$+75\,q^{16} - 52\,q^{15} + 62\,q^{14} - 40\,q^{13} + 43\,q^{12} - 25\,q^{11} + 14\,q^{10} - 9\,q^{9}$
	$-4\,q^{8} + q^{7} - 10\,q^{6} + 5\,q^{5} - 9\,q^{4} + 4\,q^{3} - 5\,q^{2} + q - 2)$
3	$q^8(q^{32} - q^{31} + 2\,q^{30} - q^{29} + 4\,q^{28} - 2\,q^{27} + 6\,q^{26} - 2\,q^{25} + 11\,q^{24} - 5\,q^{23}$
	$+17\,q^{22} - 6\,q^{21} + 25\,q^{20} - 8\,q^{19} + 29\,q^{18} - 4\,q^{17} + 31\,q^{16} + 4\,q^{15}$
	$+19\,q^{14} + 20\,q^{13} + 5\,q^{12} + 27\,q^{11} - 12\,q^{10} + 19\,q^{9} - 15\,q^{8} + 11\,q^{7}$
	$-10\,q^{6} + q^{5} - 2\,q^{4} - 3\,q^{3} + 2\,q^{2} - 2\,q + 1)$
4	$q^5(q^{35} - q^{34} + 2\,q^{33} + 3\,q^{31} + q^{30} + 6\,q^{29} + 3\,q^{28} + 11\,q^{27} + 8\,q^{26} + 16\,q^{25}$
	$+17\,q^{24} + 27\,q^{23} + 26\,q^{22} + 39\,q^{21} + 41\,q^{20} + 54\,q^{19} + 53\,q^{18} + 65\,q^{17}$
	$+63\,q^{16} + 69\,q^{15} + 60\,q^{14} + 55\,q^{13} + 46\,q^{12} + 34\,q^{11} + 32\,q^{10} + 17\,q^{9}$
	$+16\,q^{8} + 5\,q^{7} + 8\,q^{6} + 2\,q^{4} - 1)$
5	$q^{40} - q^{39} + 2\,q^{38} - 2\,q^{37} + 5\,q^{36} - 5\,q^{35} + 10\,q^{34} - 11\,q^{33} + 22\,q^{32} - 25\,q^{31}$
	$+43\,q^{30} - 48\,q^{29} + 80\,q^{28} - 90\,q^{27} + 134\,q^{26} - 146\,q^{25} + 205\,q^{24} - 206\,q^{23}$
	$+261\,q^{22} - 241\,q^{21} + 286\,q^{20} - 241\,q^{19} + 261\,q^{18} - 206\,q^{17} + 205\,q^{16}$
	$-146\,q^{15} + 134\,q^{14} - 90\,q^{13} + 80\,q^{12} - 48\,q^{11} + 43\,q^{10} - 25\,q^{9} + 22\,q^{8}$
	$-11\,q^{7} + 10\,q^{6} - 5\,q^{5} + 5\,q^{4} - 2\,q^{3} + 2\,q^{2} - q + 1$

$q^5 - q^4 - q^3 + q + 1$ of $G := E_8(q)$. If $p \neq 2$ let C_2 be the class of an involution with centralizer of type $D_8(q)$. Finally, let C_3 be the class of an element of order 3 with centralizer of type $A_8(q)$ if $q \equiv 1 \pmod 3$, respectively of type ${}^2A_8(q)$ if $q \equiv 2 \pmod 3$.

In even characteristic, we let C_2 be the class of an involution with centralizer of order $q^{100}(q^2-1)(q^4-1)(q^6-1)(q^8-1)$, denoted $4A_1$ in [**16**, Table 3].

Let $\mathbf{C}$ be the class vector consisting of the three conjugacy classes defined above. The normalized structure constants $n(\mathbf{C})$ are given in Table 9.2. As in the case $F_4(q)$ only unipotent characters contribute to $n(\mathbf{C})$, so this can be computed from the values of the $R_{T,1}$.

Thus $n(\mathbf{C}) > 0$ for all q. To show generation we can invoke [**21**, 4(j)] as in the proof of Theorem 8.2 since elements from the class C_1 again generate a maximal torus of order $\Phi_{30}(q)$. □

10. Strongly orthogonal classes for exceptional groups

A pair (C_1, C_2) of conjugacy classes of a finite group G is called *strongly orthogonal* if there exist only two irreducible characters χ of G that such $\chi(C_1)\chi(C_2) \neq 0$.

It was shown in [**13**] that pairs of strongly orthogonal classes exist for all simple classical groups of Lie type with the possible exception of groups $O_{4n}^{+}(q)$. Here we extend this result to exceptional groups:

Theorem 10.1. *Let G be a finite simple exceptional group of Lie type. Then G has a pair of strongly orthogonal classes, and at least one of the classes is real.*

Proof. Let $G = G(q)$ be an exceptional simple group of Lie type and let T_1, T_2 be the two maximal tori of G of the orders indicated in Table 10.2 (where Φ_i stands for the i-th cyclotomic polynomial evaluated at q). Then it can be checked that both tori contain regular elements for all q. Let C_1, C_2 be classes of regular elements of T_1, T_2. Since both classes contain semisimple elements it is easy to verify that only the trivial and the Steinberg character simultaneously take non-zero values on both classes, using the Deligne-Lusztig theory. (Candidates for the maximal tori were found by considering the values of unipotent characters of G on regular semisimple classes. Then it was checked that no centralizer of a non-central semisimple element in the dual group G^* contains T_1^* and T_2^* simultaneously.) Thus C_1, C_2 are strongly orthogonal in the sense introduced above. For the cases missing in Table 10.2, i.e., $G = {}^2G_2(3)'$, $G = G_2(2)'$ and $G = {}^2F_4(2)'$, we take the pairs of classes $(3A, 7A)$, $(7A, 8A)$, and $(12A, 13A)$ respectively, in Atlas notation [**2**].

The Weyl groups of type G_2, F_4, E_7, E_8 contain -1, so any semisimple element in groups of those types is real. Furthermore, for ${}^2B_2(q^2)$, ${}^2G_2(q^2)$, ${}^2F_4(q^2)$ and ${}^3D_4(q)$ it is easy to see that the elements from C_2 are real. Finally, for E_6 and ${}^2E_6$ we choose elements in T_2 of order $\Phi_8 = q^4 + 1$. These are contained in the subgroup $F_4(q)$, hence they are real by the above argument. Moreover, elements of this order in T_2 are also regular since no unipotent element has centralizer order divisible by Φ_8. This completes the proof. □

As in [**13**, Thm. 3.2] the existence of a a pair of strongly orthogonal classes, one of which is real, allows to conclude that there exists a conjugacy class C such that G is covered by $C^2 \cup C^3$ (Corollary 1.6):

Proof of Corollary 1.6. Let (C_1, C_2) be the pair of strongly orthogonal classes for G from Theorem 10.1, with C_2 real. Then only two characters $\chi_1 = 1, \chi_2$ contribute to the normalized structure constant $n(C_1, C_2, C_3)$ for any class C_3 of G. But $1 = \chi_1(C_1)\chi_1(C_2) = -\chi_2(C_1)\chi_2(C_2)$, and $|\chi_2(C_3)| < \chi_2(1)$ for $1 \notin C_3$ since G is simple, so $n(C_1, C_2, C_3) > 0$ for all non-trivial classes C_3. In particular, choosing $C_3 = C_2$ we see that any element in C_1 lies in C_2^2. Since moreover C_2 is real, the identity is also contained in C_2^2. Together we hence obtain $G \subset C_2^2 \cup C_2^3$. □

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TABLE 10.2. Pairs of strongly orthogonal classes

G	$\|T_1\|$	$\|T_2\|$
${}^2B_2(q^2), q^2 > 2$	$q^2 - 1$	$q^2 + \sqrt{2}q + 1$
${}^2G_2(q^2), q^2 > 3$	$q^2 - 1$	$q^2 + \sqrt{3}q + 1$
${}^2F_4(q^2), q^2 > 2$	$q^4 - 1$	$q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1$
$G_2(q), q > 2$	$\Phi_1\Phi_2$	Φ_3
${}^3D_4(q)$	$\Phi_1\Phi_2\Phi_3$	Φ_{12}
$F_4(q)$	$\Phi_1\Phi_2\Phi_3$	Φ_8
$E_6(q)$	Φ_9	$\Phi_1\Phi_2\Phi_8$
${}^2E_6(q)$	Φ_{18}	$\Phi_1\Phi_2\Phi_8$
$E_7(q)$	$\Phi_1\Phi_7$	$\Phi_2\Phi_{18}$
$E_8(q)$	$\Phi_1\Phi_2\Phi_4\Phi_8$	Φ_{15}

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