The Linear Matrix Problems and the Determination of p-Groups

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Goal: to show one method of determination of p-groups up to isomorphisms by the applying of linear matrix problems.

Matrix problems

Definition 1.

- Let M be a matrix (or a set of matrices)
- T be a set of transformations applicable to M

$$M \mapsto T(M)$$

(often T is also given by a matrix or a set of matrices)

A **matrix problem** is a problem to find a canonical form of M with respect to T.

The method of linear matrix problems (L.M.P.)

was proposed by prof. A.Roiter,

was worked out by Kyiv School of Representation Theory

Examples.

- Gauß. $M \in \mathsf{Mat}_{\mathbb{k}}(n \times m)$ $S \in \mathsf{GL}(\mathbb{k}, n)$
  $R \in \mathsf{GL}(\mathbb{k}, m)$
  $$M \mapsto S^{-1}MR$$
- Jordan Norman Form. $M \in \mathsf{Mat}_{\mathbb{k}}(n \times n)$
  $S \in \mathsf{GL}(\mathbb{k}, n)$
  $$M \mapsto S^{-1}MS$$
- Congruence problem. $M \in \mathsf{Mat}_{\mathbb{k}}(n \times n)$
  $S \in \mathsf{GL}(\mathbb{k}, n)$
  $$M \mapsto S^{*}MS$$
- "Wild problem". $M_1, M_2 \in \mathsf{Mat}_{\mathbb{k}}(n \times n)$
  $S \in \mathsf{GL}(\mathbb{k}, n)$
  $$(M_1, M_2) \mapsto (S^{-1}M_1S,\ S^{-1}M_2S)$$
- $M_1, M_2 \in \mathsf{Mat}_{\mathbb{k}}(n \times n)$ $S, R, T \in \mathsf{GL}(\mathbb{k}, n)$
  $$(M_1, M_2) \mapsto (S^{-1}M_1R,\ S^{-1}M_2R + S^{-1}M_1T)$$

etc.

Extension

Let G be a central abelian extension group of an elementary abelian group H:

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 1.$$

where the sequence is exact.

To determine *a central abelian extension* we need

- a group H;
- an abelian N;
- a homomorphism $\varphi : H \rightarrow \mathsf{Aut}\, N$;
- a system of factors $\theta : (H \times H) \hookrightarrow N$.

The extension is the set of pairs

$$G = \langle (h, c) | h \in H, c \in N \rangle \tag{1}$$

with the multiplication low

$$(h_1, c_1)(h_2, c_2) = (h_1 h_2,\ c_1^{h_2} c_2\, \theta(h_1, h_2)) \tag{2}$$

- Let H be **elementary abelian** p-group (p - odd), $H = \langle h_1 \rangle \times \langle h_2 \rangle \times ... \times \langle h_n \rangle$;
- Let N be **abelian**, $|N| < p^{p-1}$.

To construct θ we need only

$$\theta(h_i^{p-1},\ h_i) = a_i \in N; \tag{3}$$

$$\theta(h_i^{-1} h_j^{-1},\ h_i h_j) = b_{ij} \in N \tag{4}$$

with restrictions

$$a_i^{\varphi(h_i)-1} = 1; \tag{5}$$

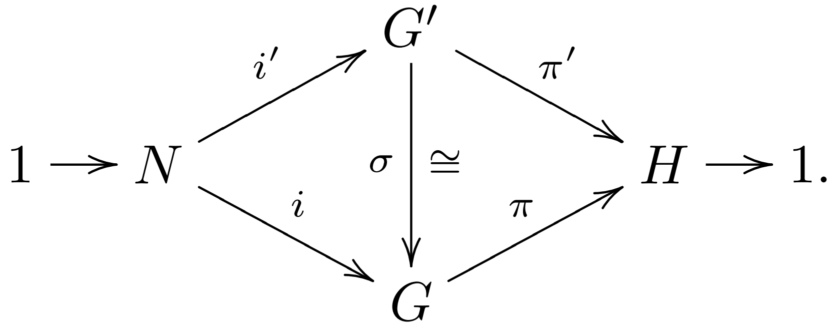
$$b_{ij} = b_{ji}^{-1}; \tag{6}$$

$$b_{ij}^{p^2} = 1. \tag{7}$$

An extension group G can be determined by the following setup:

$$G \leftrightsquigarrow (H, N, \varphi, \theta)$$

Definition 2. Two extensions $G = (H, N, \varphi, \theta)$ and $G = (H', N', \varphi', \theta')$ are called **equivalent** if the following diagram commutes:



Lemma 1. For $|N| = p^m$ with $m < p - 1$ two extensions (H, N, φ, θ) and $(H, N, \varphi', \theta')$ are equivalent iff

- $\varphi = \varphi'$
- There exist elements $\eta_1, \ldots \eta_n \in N$ such that $a_{ij}\eta_i^{h_j - 1}\eta_j^{-h_i + 1} = a'_{ij}$ and $b_i\eta_i^p = b'_i$.

Note. For each pair $(h_i, 1)' \in G' = (H', N', \varphi', \theta')$ the image

$$\sigma((h_i, 1)') = (h_i, \eta_i)$$

gives another representative of coset h_i by N.

Definition 3. Two extensions $G = (H, N, \varphi, \theta)$ and $G = (H', N', \varphi', \theta')$ are called **weakly equivalent** if the following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N' & \xrightarrow{i'} & G' & \xrightarrow{\pi'} & H' & \longrightarrow & 1 \\
 & & \rho \downarrow \cong & & \sigma \downarrow \cong & & \tau \downarrow \cong & & \\
1 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{\pi} & H & \longrightarrow & 1
\end{array}$$

Lemma 2. Let $G \leftrightsquigarrow (H, N, \varphi, \theta)$ and N be a characteristic subgroup of the group G.

G is isomorphic $G' \leftrightsquigarrow (H', N', \varphi', \theta')$ iff corresponding extensions are weakly equivalent.

Note. If G' is weakly equivalent G, then

$$\varphi'(h) = \rho^{-1}(\varphi(h^{\tau}))\rho \tag{8}$$

for each $h \in H$.

Assume additionally that N is an elementary abelian p-group too,

$$N = \langle c_1 \rangle \times \langle c_2 \rangle \times ... \times \langle c_m \rangle.$$

Thus

- $a_i = \Pi_{k=1}^{m} c_k^{\alpha_{ki}}$;
- $b_{ij} = \Pi_{k=1}^{m} c_k^{\beta_{kij}}$.

We obtain the matrices over the field F_p with p elements:

$$A = (\alpha_{ki})_{k=1..m;i=1..n} \tag{9}$$

$$B = (\beta_{kij})_{k=1..m;i,j=1..n}, \tag{10}$$

where $B = (B_1, B_2, ..., B_m)$,

B_k is antisymmetric $n \times n$ matrix for each $k = 1..m$.

Lemma 3. Let H, N are elementary abelian of order p^n, p^m respectively, $m < p - 1$.

Each set (H, N, φ, A, B), where A, B are matrices (9),(10) for which the restrictions (5)-(7) hold, determines an extension G.

Algorithm

Note.

$AutH \cong GL_n(F_p)$,

$AutN \cong GL_m(F_p)$

for the elementary abelian p-groups H, N.

1. **Find the homomorphism** $\varphi : H \to \mathsf{Aut}\, N$.
   According (8) we may determine φ up to conjugacy.
2. For fixed φ **find all pairs of transformations** $(\rho, \tau) \in Aut(N) \times Aut(H)$ saving φ:

   $$\rho^{-1}(\varphi(h^{\tau}))\rho = \varphi(h) \qquad (11)$$

   for all $h \in H$.
3. Establish, which elements a_i, b_{ij} can be made equal to the identity element by **re-choosing the representatives** of the cosets by N.

Algorithm (continued)

4. **Reduce** a set of matrices $M = (A, B_1, B_2, ..., B_m)$ **to the canonical form** W.R.T. transformations (ρ, τ):

$$A' = R^{-1}AT \tag{12}$$

$$\begin{pmatrix} B'_1 \\ B'_2 \\ \dots \\ B'_m \end{pmatrix} = R^{-1} \begin{pmatrix} T^* B_1 T \\ T^* B_2 T \\ \dots \\ T^* B_m T \end{pmatrix}, \tag{13}$$

where $R,\ T$ are matrices corresponding $\rho,\ \tau$ respectively,

T^* is the matrix transposed to T.

5. For each canonical form **construct the group** G:

$$G = \langle \overline{h}_1, \overline{h}_2, ...\overline{h}_n, \overline{c}_1, \overline{c}_2, ...\overline{c}_m | \, [\overline{h}_i, \overline{h}_j] = \prod_{k=1}^{m} \overline{c}_k^{\ \alpha_{kij}},$$

$$[\overline{c}_k, \overline{h}_i] = \overline{c}^{\ \varphi(h_i)-1}, [\overline{c}_k, \overline{c}_l] = 1, \overline{c}_k^p = 1,$$

$$\overline{h}_i^p = \prod_{k=1}^{m} \overline{c}_k^{\ \beta_{ki}}, (i, j = 1..n, k, l = 1..m) \rangle$$

Example. Let H and N be elementary abelian of orders p^2 and p^3 respectively ($p > 3$),

$H = \langle h_1 \rangle \times \langle h_2 \rangle$ and $N = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle$.

Determine all extensions of H by N.

There are 6 morphisms

$\varphi : H \to S_p(\mathsf{Aut}(N)) = UT_3(\mathbb{F}_p)$

up to conjugacy :

1. $h_1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $h_2 \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
2. $h_1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $h_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3. $h_1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $h_2 \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
4. $h_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $h_2 \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
5. $h_1 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $h_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
6. $h_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $h_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Consider **the homomorphism**
$\varphi : H \to S_p(\mathsf{Aut}(N)) = UT_3(\mathbb{F}_p)$.
$h_1 \mapsto \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $h_2 \mapsto \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

The system of factors:

$$\theta(h_1^{p-1},\ h_1) = a_1 \in N$$

$$\theta(h_2^{p-1},\ h_2) = a_2 \in N$$

$$\theta(h_1^{-1}h_2^{-1},\ h_1h_2) = b_{12} \in N$$

where a_1, a_2, b_{12} satisfy the restrictions (5)-(7).

Corresponding matrices:

$$A = \left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ 0 & 0 \end{smallmatrix}\right)$$

$$B_1 = \begin{pmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{pmatrix}\ B_2 = \begin{pmatrix} 0 & \beta_2 \\ -\beta_2 & 0 \end{pmatrix}\ B_3 = \begin{pmatrix} 0 & \beta_3 \\ -\beta_3 & 0 \end{pmatrix}$$

The re-choosing of the representatives gives:

$$A = \left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ 0 & 0 \end{smallmatrix}\right)$$

$$B_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\ B_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\ B_3 = \begin{pmatrix} 0 & \beta_3 \\ -\beta_3 & 0 \end{pmatrix}$$

$\beta_3 \neq 0$ gives groups **from the isoclinism family** Φ_6.

$\beta_3 = 0$ gives groups **from the isoclinism family** Φ_4.

Consider the case $\beta \neq 0$.

Transformations ρ and τ satisfying (11) have a matrices

$$R = \delta \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{33} \\ 0 & 0 & 1 \end{pmatrix} \text{ where } \delta = r_{11}r_{22} - r_{12}r_{21};$$

$$T = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

Thus

$$A' = R^{-1}AT \tag{14}$$

$$\begin{pmatrix} B'_1 \\ B'_2 \\ B'_3 \end{pmatrix} = R^{-1} \begin{pmatrix} T^*B_1T \\ T^*B_2T \\ T^*B_3T \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \tag{15}$$

For $p > 3$ there are $p + 7$ canonical forms for A.

N **is a characteristic subgroup,** $N = \Phi(G)$, for each extension $G = (H, N, \varphi, A, B)$,

thus

we obtain $p + 7$ **non-isomorphic groups** of order p^5 from family Φ_6.

Applications

Easy modification of this method give possibility to obtain:

- **The determination of finite p-groups with abelian subgroup of index p.**

  L.A.Nazarova, A.V.Roiter, V.V.Sergeichuk, and V.N.Bondarenko, 1972.

- **The determination of finite p-groups which are an extension of cyclic by the abelian subgroup of index bigger than p.**

  V.V.Sergeichuk, 1974.

- **The investigation of p-groups which are an extension of elementary abelian of order p^2 by elementary abelian.**

  O.Pyliavska (O.S.Pilyavskaya), 1993

- **The determination of p-groups of order p^6.**

  O.Pyliavska (O.S.Pilyavskaya) 1983