Extremal Lattices

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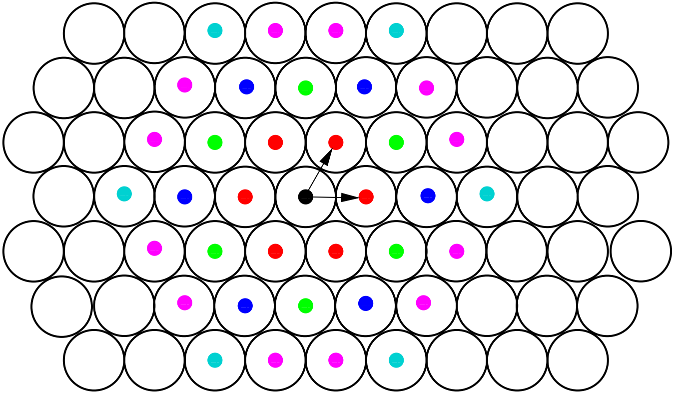
Lattices, modular forms and sphere packings



Hexagonal Circle Packing

$$\theta = 1 + 6q^2 + 6q^6 + 6q^8 + 12q^{14} + 6q^{18} + \dots .$$

Lattices $\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})/O_n(\mathbb{R})$

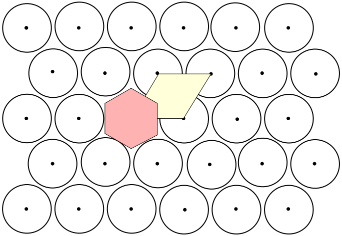
- $n \in \mathbb{N}$, $(\mathbb{R}^n, \cdot)$ Euclidean space
- $B = (b_1, \ldots, b_n)$ basis $\qquad \mathrm{GL}_n(\mathbb{R})$
- $L(B) = \{\sum_{i=1}^n a_i b_i \mid a_i \in \mathbb{Z}, 1 \leq i \leq n\}$ lattice $\qquad \mathrm{GL}_n(\mathbb{Z})$
- $G(B) = B^{tr} B$ Gram matrix $\qquad O_n(\mathbb{R})$
- $L \cong M$ isometric $\Leftrightarrow$ there is $f \in O_n(\mathbb{R})$ such that $f(L) = M$ $\Leftrightarrow$ there are lattice basis B_L and B_M such that $G(B_L) = G(B_M)$.
- $L \sim M$ similar $\Leftrightarrow \exists\ B_L, B_M, a > 0$ such that $G(B_L) = aG(B_M)$.

Density of associated lattice sphere packing

- measured by Hermite function γ.
- invariant under similarity

Voronoi around 1900

- Finitely many local extrema of density function $\qquad \gamma' = 0$
- local maxima: convexity condition $\qquad \gamma'' < 0$
- used to find all densest lattices (up to dimension 8).



The density of a lattice

- Squared covolume of L: $\det(B_L)^2 =: \det(L)$ determinant

  squared volumne of space needed for one sphere
- $\min(L) := \min\{\ell \cdot \ell \mid 0 \neq \ell \in L\}$ minimum

  squared diameter of one sphere
- $\gamma(L) := \frac{\min(L)}{\det(L)^{1/n}}$ Hermite function measures density
- $\gamma_n := \max\{\gamma(L) : L$ lattice in $\mathbb{R}^n\}$ Hermite constant

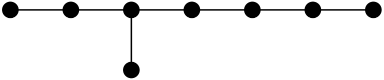
The densest lattices.

n	1	2	3	4	5	6	7	8	24
γ_n	1	1.15	1.26	1.41	1.52	1.67	1.81	2	4
L	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{D}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	Λ_{24}
loc. max.	1	1	1	2	3	6	30	2408	

The $\mathbb{E}_8$-lattice

n	1	2	3	4	5	6	7	8	24
L	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{D}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	Λ_{24}

In dimension $1, \ldots, 8$ all densest lattices are root lattices.



$$G(B) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{matrix} \det(\mathbb{E}_8) = 1 \\ \min(\mathbb{E}_8) = 2 \\ \gamma(\mathbb{E}_8) = 2 \end{matrix}$$

The $\mathbb{E}_8$-lattice

- $\mathbb{D}_8 = \{\mathbf{x} \in \mathbb{Z}^8 \mid \sum_{i=1}^8 x_i \text{ even } \}$
- $\mathbb{E}_8 = \mathbb{D}_8 \cup \{\frac{1}{2}^8 + \mathbf{x} \mid \mathbf{x} \in \mathbb{D}_8\}$

Even unimodular lattices

$\mathbb{E}_8$ and Λ_{24} are even unimodular lattices

Definition

- $L = L(B)$ lattice.
- $L^{\#} = L(B^*) = \{x \in \mathbb{R}^n \mid x \cdot \ell \in \mathbb{Z}$ for all $\ell \in L\}$ dual lattice.
- $\det(L)\det(L^{\#}) = 1$, $G(B)G(B^*) = 1$.
- L unimodular if $L = L^{\#}$.
- L even if $\ell \cdot \ell \in 2\mathbb{Z}$ for all $\ell \in L$.

L unimodular $\Rightarrow \det(L) = 1$ and $\gamma(L) = \min(L)$.

- Densest known lattices in dim. 8, 24, 48, 72: even unimodular
- Even unimodular lattices are regular integral quadratic forms.

Modular forms and theta series

Let L be an even unimodular lattice of dimension n

- $n \in 8\mathbb{Z}$.
- The theta series of L

$$\theta_L = \sum_{\ell \in L} q^{\ell \cdot \ell} = 1 + \sum_{k=\min(L)}^{\infty} a_k q^k$$

  where $a_k = |\{\ell \in L \mid \ell \cdot \ell = k\}|$.
- $q := \exp(\pi i z)$ yields holomorphic function on the upper half plane
- θ_L is a modular form of weight $\frac{n}{2}$

$$\theta_L \in \mathcal{M}_{\frac{n}{2}}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, \Delta]_{\frac{n}{2}}$$

  where $E_4 = \theta_{\mathbb{E}_8} = 1 + 240q^2 + \dots$ Eisenstein series of weight 4

$$\Delta = q^2 - 24q^4 + \dots \text{ cusp form of weight } 12$$

Lattices and modular forms

lattices		modular forms
$\bigcup_n \mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})/O_n(\mathbb{R})$		$\mathbb{C}[E_4, \Delta]$
union of homogenous spaces		finitely generated graded ring
dimension n		weight $n/2$
arithmetic properties	$\Rightarrow$	invariance properties
upper bounds on density	$\Leftarrow$	triagonal basis

Extremal Lattices

Extremal modular form

Basis of $\mathcal{M}_{4k}(\mathrm{SL}_2(\mathbb{Z}))$:

$$\begin{array}{llllll} E_4^k = & 1+ & 240kq^2+ & *q^4+ & \dots & \\ E_4^{k-3}\Delta = & & q^2+ & *q^4+ & \dots & \\ E_4^{k-6}\Delta^2 = & & & q^4+ & \dots & \\ \vdots & & & & & \\ E_4^{k-3m_k}\Delta^{m_k} = & & \dots & & q^{2m_k}+ & \dots \end{array}$$

where $m_k = \lfloor \frac{k}{3} \rfloor$.

$\mathcal{M}_{4k}(\mathrm{SL}_2(\mathbb{Z}))$ contains a unique

$$f^{(k)} := 1 + 0q^2 + 0q^4 + \ldots + 0q^{2m_k} + a(f^{(k)})q^{2m_k+2} + b(f^{(k)})q^{2m_k+4} + \ldots$$

the extremal modular form of weight $4k$.

$f^{(1)} = 1 + 240q^2 + \ldots = \theta_{\mathbb{E}_8}$, $f^{(2)} = 1 + 480q^2 + \ldots = \theta_{\mathbb{E}_8}^2$,
$f^{(3)} = 1 + 196,560q^4 + \ldots = \theta_{\Lambda_{24}}$, $f^{(6)} = 1 + 52,416,000q^6 + \ldots = \theta_{P_{48}}$,
$f^{(9)} = 1 + 6,218,175,600q^8 + \ldots = \theta_{\Gamma}$.

Extremal even unimodular lattices

$$f^{(k)} := 1 + 0q^2 + 0q^4 + \ldots + 0q^{2m_k} + a(f^{(k)})q^{2m_k+2} + b(f^{(k)})q^{2m_k+4} + \ldots$$

Theorem (Siegel, 1969)

$a(f^{(k)}) > 0$ for all k and $b(f^{(k)}) < 0$ for $k \geq 21000$.

Corollary

L even unimodular of dimension $n = 8k$ then

$$\min(L) \leq 2 + 2\lfloor \frac{n}{24} \rfloor = 2 + 2m_k.$$

Lattices achieving equality are called extremal.

extremal even unimodular lattices

n	8	16	24	32	40	48	72	80
min(L)	2	2	4	4	4	6	8	8
number	1	2	1	$\geq 10^7$	$\geq 10^{51}$	≥ 4	≥ 1	≥ 4

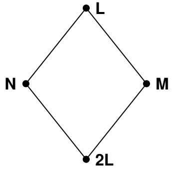
Extremal lattices in jump dimensions

$f^{(3)} = 1 + 196,560q^4 + \ldots = \theta_{\Lambda_{24}}$.
$f^{(6)} = 1 + 52,416,000q^6 + \ldots = \theta_{P_{48}}$.
$f^{(9)} = 1 + 6,218,175,600q^8 + \ldots = \theta_{\Gamma}$.

$\dim(L) = 24k$, $\min(L) = 2k + 2$

- Layers $\{\ell \in L \mid \ell \cdot \ell = const.\}$ form spherical 11-designs.
- L is a local maximum of the density function γ.
- $k = 1$: unique extremal lattice: Leech lattice Λ_{24}.
- Λ_{24} densest lattice of dimension 24 (Cohn, Kumar (2004)).
- Λ_{24} densest sphere packing (Viazowska et al (2016)).
- P_{48} and Γ are densest known lattices.

- $\mathbb{E}_8$ densest lattice (Blichfeldt (1935))
- $\mathbb{E}_8$ densest sphere packing (Viazowska (2016))



Turyn's construction

- Let L be an even unimodular lattice of dimension n.
- $Q(\ell) := \frac{1}{2}\ell \cdot \ell$ regular quadratic form.
- $M, N \leq L$ such that $M + N = L$, $M \cap N = 2L$,
- and $(M, \frac{1}{2}Q)$, $(N, \frac{1}{2}Q)$ even unimodular.
- Such a pair (M, N) is called a polarisation of L.

$$\mathcal{L}(M, N) = \{(m + a, m + b, m + c) \mid m \in M, a, b, c \in N, a + b + c \in 2L\}$$

- Define $\tilde{Q} : \mathcal{L}(M, N) \to \mathbb{Z}$,

$$\tilde{Q}(y_1, y_2, y_3) := \frac{1}{2}(Q(y_1) + Q(y_2) + Q(y_3)).$$

- $(\mathcal{L}(M, N), \tilde{Q})$ is an even unimodular lattice of dimension $3n$.

(m+a,m+b,m+c) in

- L ⊥ L ⊥ L
- L(M,N)
- 2L⊥2L⊥2L

m in M

a,b,c in N

a+b+c in 2L

Let $2d := \min(L, Q) = \min(M, \frac{1}{2}Q) = \min(N, \frac{1}{2}Q)$

Then $2\lceil \frac{3d}{2} \rceil \leq \min(\mathcal{L}(M, N)) \leq 4d$.

Proof:

- $v = (a, 0, 0)$ $a = 2\ell \in 2L$ with $\frac{1}{2}Q(2\ell) = 2Q(\ell) \geq 2d \Rightarrow v \cdot v \geq 4d$.
- $v = (a, b, 0)$ $a, b \in N$ with $\frac{1}{2}Q(a) + \frac{1}{2}Q(b) \geq 2d \Rightarrow v \cdot v \geq 4d$.
- $v = (a, b, c)$ then $\frac{1}{2}(Q(a) + Q(b) + Q(c)) \geq \frac{3}{2}d \Rightarrow v \cdot v \geq 2\lceil \frac{3d}{2} \rceil$.

$(a, b, c) \in \mathcal{L}(M, N)$ with $\tilde{Q}((a, b, c)) < 4d \Rightarrow a, b, c \neq 0$.

(m+a,m+b,m+c) in

- L ⊥ L ⊥ L
- L(M,N)
- 2L⊥2L⊥2L

m in M

a,b,c in N

a+b+c in 2L

$2d := \min(L, Q) = \min(M, \frac{1}{2}Q) = \min(N, \frac{1}{2}Q)$

Then $2\lceil \frac{3d}{2} \rceil \leq \min(\mathcal{L}(M, N)) \leq 4d$.

Theorem (Lepowsky, Meurman; Elkies, Gross)

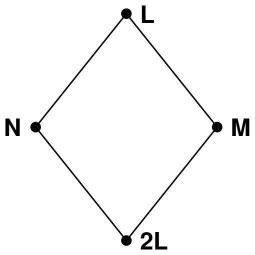
Let $(L, Q) \cong \mathbb{E}_8$ (so $d = 1$). Then for any polarisation $\min(\mathcal{L}(M, N)) \geq 4$, so $\mathcal{L}(M, N) \cong \Lambda_{24}$ is the Leech lattice.

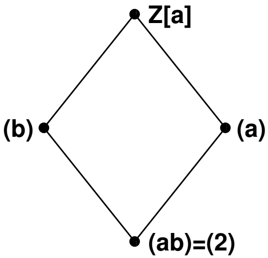
72-dimensional lattices from Leech (Griess, 2010)

If $(L, Q) \cong (M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q) \cong \Lambda_{24}$ then $6 \leq \min(\mathcal{L}(M, N)) \leq 8$.

Enumerate $A_6 := \{\ell \in \mathcal{L}(M, N) \mid \ell \cdot \ell = 6\}$: computation in Λ_{24}.

How to find polarisations





Hermitian polarisations

- Take $\alpha \in \mathrm{End}(L)$ such that $\alpha^2 - \alpha + 2 = 0$
- so $\mathbb{Z}[\alpha] =$ ring of integers in $\mathbb{Q}[\sqrt{-7}]$
- and $\alpha\overline{\alpha} = 2$.
- Then $M := \alpha L$, $N := \overline{\alpha} L$ defines a polarisation of L such that $(L, Q) \cong (M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q)$.

Hermitian structures of the Leech lattice

Theorem (M. Hentschel, 2009)

There are exactly nine $\mathbb{Z}[\alpha]$-structures P_i of the Leech lattice Λ_{24}.

i	$\mathrm{Aut}_{\mathbb{Z}[\alpha]}(P_i)$	$\lvert A_6\rvert$
1	$\mathrm{SL}_2(25)$	0
2	$2.A_6 \times D_8$	$2 \cdot 20,160$
3	$\mathrm{SL}_2(13).2$	$2 \cdot 52,416$
4	$(\mathrm{SL}_2(5) \times A_5).2$	$2 \cdot 100,800$
5	$(\mathrm{SL}_2(5) \times A_5).2$	$2 \cdot 100,800$
6	$2^9 3^3$	$2 \cdot 177,408$
7	$\pm\mathrm{PSL}_2(7) \times (C_7 : C_3)$	$2 \cdot 306,432$
8	$\mathrm{PSL}_2(7) \times 2.A_7$	$2 \cdot 504,000$
9	$2.J_2.2$	$2 \cdot 1,209,600$

Theorem (N. 2010) $\Gamma := \mathcal{L}(\alpha P_1, \overline{\alpha} P_1)$ is extremal.

Stehlé, Watkins proof of extremality

Theorem (Stehlé, Watkins (2010))

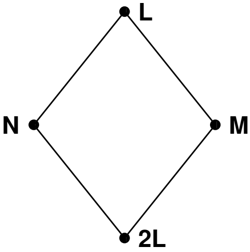
Let L be an even unimodular lattice of dimension 72 with $\min(L) \geq 6$. Then L is extremal, if and only if it contains at least $6,218,175,600$ vectors v with $v \cdot v = 8$.

Proof: L is an even unimodular lattice of minimum ≥ 6, so its theta series is

$$\theta_L = 1 + a_6 q^6 + a_8 q^8 + \ldots = f^{(9)} + a_6 \Delta^3.$$

$$\begin{array}{ccccccc} f^{(9)} & = & 1 & + & 6,218,175,600q^8 & + \ldots \\ \Delta^3 & = & & q^6 & -72q^8 & + \ldots \end{array}$$

So $a_8 = 6,218,175,600 - 72a_6 \geq 6,218,175,600$ if and only if $a_6 = 0$.



How to obtain all polarisations

A rough estimate shows that there are about 10^{10} orbits of $\mathrm{Aut}(\Lambda_{24})$ on the set of polarisations (M, N) such that $(M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q) \cong \Lambda_{24}$.

Theorem (Richard Parker, N.) (2014)

Unique orbit of polarisations (M, N) for which $\mathcal{L}(M, N)$ is extremal.

The extremal 72-dimensional lattice Γ

- Γ is an extremal even unimodular lattice of dimension 72.
- Have at least 3 independent proofs of extremality:
  Using Turyn's construction (N.),
  enumerating all norm 8 vectors (Stehlé, Watkins),
  using the Hermitian tensor product (Coulangeon, N.).
- Γ realises the densest known sphere packing
- and maximal known kissing number in dimension 72.
- Γ is the only extremal lattice obtained from the Leech lattice using Turyn's construction. (Parker, N.)