On blocks with cyclic defect group and their head orders.

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ABSTRACT: It is shown that [Ple83, Theorem 8.5] describes blocks of cyclic defect group up to Morita equivalence. In particular such a block is determined by its planar embedded Brauer tree. Applying the radical idealiser process the head order of such blocks is calculated explicitly.

1 Introduction.

Blocks with cyclic defect group are very well understood. Despite of their very special structure these blocks are extensively used to study examples for the validity of various conjectures, since they are essentially described by combinatorial means. A detailed introduction to the theory of blocks with cyclic defect groups, that also deals with rationality questions of the involved characters, is given in [Fei82, Chapter 7]. Using the known character theoretic information and some new methods, essentially based on linear algebra, Plesken [Ple83, Chapter 8] gives a rather explicit description of blocks with cyclic defect group $B = \mathbb{Z}_p G\epsilon$ of p-adic group rings. The aim of the first part of this paper is to show, how far [Ple83, Theorem 8.5 and 8.10] determine blocks with cyclic defect group of group rings over discrete valuation rings. The fact that blocks of group rings are symmetric orders yields the additional information needed to get a complete description up to isomorphism (see Theorem 2.5). Even the Hasse invariants of the occurring skew fields can be read off from the action of the Galois group on the Brauer characters (see Theorem 2.8). In particular the planar embedded Brauer tree together with the character fields and the Galois action on the characters determine the Morita equivalence class of B.

Section 3 deals with the radical idealiser chain of B. This is a finite chain associated to an order Λ

$$\Lambda =: \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_N = \Lambda_{N+1}$$

where $\Lambda_{i+1} = \mathrm{Id}(J(\Lambda_i))$ $(i = 0, \ldots, N)$ is the 2-sided idealiser of the Jacobson radical of Λ_i that necessarily ends in a hereditary order Λ_N called the head order of Λ. In particular this radical idealiser process associates to a usually quite complicated object Λ two simple data: the length N of the chain and the head order Λ_N. It is an interesting question which information about Λ can be read off from these data. For instance [CPW87] shows that the length of the radical idealiser chain of a centre of a block over a splitting ring equals the valuation of the order of the defect group. Furthermore [Jac84] characterises blocks of defect 0 as those blocks where $\Lambda = \Lambda_N$ and shows that blocks with radical idealiser length 1 are Brauer tree orders. It seems to be desirable to have more complicated examples of radical idealiser chains for blocks of group rings.

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Section 3 investigates these chains and calculates the head order Λ_N of blocks B with cyclic defect group. In particular the B-composition factors of the simple Λ_N-modules can be easily read off from the planar embedded Brauer tree (see Remark 3.16).

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2 Blocks with cyclic defect group.

Throughout the paper let R be a (not necessarily commutative) discrete valuation ring with prime element π and residue class field $k = R/\pi R$ and let K be the skew-field of fractions of R.

A convenient language to describe certain R-orders are exponent matrices.

Definition 2.1. *(see [Rei75, Definition 39.2], [Ple83]) Let* $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$, $D := \sum_{i=1}^n d_i$ *and* $M \in \mathbb{Z}^{n\times n}$. *Then*

$$\Lambda(R, d, M) := \{X = (x_{ij}) \in K^{D\times D} \mid x_{ij} \in \pi^{m_{ij}} R^{d_i \times d_j}\}.$$

Example: (hereditary orders, see [Rei75, Section 39]) Let Λ be a hereditary order and P be a projective Λ-lattice. Then $\overline{P} := P/\pi P$ is uniserial. Let $d_1, \ldots, d_n$ be the dimensions of the simple Λ-modules in the order in which they occur in the radical series of $\overline{P}$. Then with respect to a suitable R-basis of P (adapted to this lattice chain),

$$\Lambda = \Lambda(R, d, H_n), \text{ where } H_n = \begin{pmatrix} 0 & 1 & \ldots & \ldots & 1 \\ 0 & 0 & 1 & \ldots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & \ldots & 0 & 0 \end{pmatrix}.$$

The description of blocks with cyclic defect given in [Ple83] will be repeated briefly. All the following results can be found in Chapter 8 of this lecture notes, so we omit the detailed citations. Let K be an **unramified** extension of $\mathbb{Q}_p$, R its ring of integers and $\mathcal{B}$ be a block of RG with cyclic defect group of order p^a. Assume that $k := R/pR$ is a splitting field of $k\mathcal{B}$. By [Fei82, Chapter 7], the minimal choice of such a field K is the character field of any of the non-exceptional characters in $\mathcal{B}$.

Let $\epsilon_1, \ldots, \epsilon_h$ be the central primitive idempotents in $\mathcal{A} := K\mathcal{B}$. Then $h = a + e$ where e is the number of simple $k\mathcal{B}$-modules and after a suitable permutation of the indices the centres $Z(\mathcal{A}\epsilon_s) \cong K$ for $s = a+1, \ldots, a+e$ and $Z(\mathcal{A}\epsilon_s) =: Z_s$ is a totally ramified extension of K of degree $\frac{p^s - p^{s-1}}{e}$ for $s = 1, \ldots, a$. The centre of $\mathcal{B}\epsilon_s$ is the maximal order R_s in Z_s.

The vertices in the Brauer tree are the exceptional vertex $\{1, \ldots, a\}$ and $a+1, \ldots, a+e$ corresponding to the other simple $\mathcal{A}$-modules. Let T_{odd} resp. T_{even} denote the set of vertices having an odd (resp. even) distance from the exceptional vertex.

For $s \in \{1, \ldots, h\}$ let $r_s \subset \{1, \ldots, e\}$ be the set of indices of the simple constituents of any $\mathcal{B}\epsilon_s$ lattice. Then for the exceptional vertex $r_1 = \ldots = r_a$ and the sets r_s are the orbits of certain permutations δ (if $s \in T_{even}$) resp. ρ (if $s \in T_{odd}$). Let $d_1, \ldots, d_e$ be the k-dimensions of the simple $k\mathcal{B}$-modules and $f_1, \ldots, f_e$ be orthogonal idempotents of $\mathcal{B}$ that lift the corresponding central primitive idempotents of $\mathcal{B}/J(\mathcal{B})$.

Theorem 2.2. *([Ple83, Theorem 8.3]) With the notation above let $i \in r_s$. Then*

$$(i) \qquad \mathcal{B}\epsilon_s \cong \Lambda(R_s, (d_i, d_{\delta(i)}, \ldots, d_{\delta^{|r_s|-1}(i)}), H_{|r_s|}) \quad \textit{for } s = 1, \ldots, a$$

and

$$(ii) \qquad \mathcal{B}\epsilon_s \cong \Lambda(R, (d_i, d_{\sigma(i)}, \ldots, d_{\sigma^{|r_s|-1}(i)}), aH_{|r_s|}) \quad \textit{for } s = a+1, \ldots, a+e$$

where $\sigma = \delta$ if $s \in T_{even}$ and $\sigma = \rho$ if $s \in T_{odd}$.

It remains to describe how $\mathcal{B}$ sits inside the direct sum of the $\mathcal{B}\epsilon_s$, that is to describe the amalgamations between the $\mathcal{B}\epsilon_s$.

Theorem 2.3. *([Ple83, Theorem 8.5])*

(i) For the exceptional vertex $\Gamma_a := (\epsilon_1 + \ldots + \epsilon_a)\mathcal{B}$ one gets an inductive description: For $s = 2, \ldots, a$ let

$$X_s := \mathcal{B}\epsilon_s / J(\mathcal{B}\epsilon_s)^{x_s} = \mathcal{B}\epsilon_s / \pi_s^{y_s}\mathcal{B}\epsilon_s$$

where $x_s = |r_1|y_s$ and $y_s = \frac{p^{s-1}-1}{e}$ and let $\nu_s : \mathcal{B}\epsilon_s \to X_s$ be the natural epimorphism. Define R-orders Γ_s $(s = 1, \ldots, a)$ inductively by

$$\Gamma_1 := \epsilon_1\mathcal{B} \textit{ and } \Gamma_s := \{(x, y) \in \Gamma_{s-1} \oplus \mathcal{B}\epsilon_s \mid \varphi_{s-1}(x) = \nu_s(y)\}$$

where φ_{s-1} is an epimorphism from Γ_{s-1} onto X_s.

(ii) Let $\Gamma_0 := (\epsilon_{a+1} + \ldots + \epsilon_{a+e})\mathcal{B}$. Then for any $i, j \in \{1, \ldots, e\} - r_1$ with $i \neq j$ one gets

$$f_i\Gamma_0 f_j = \oplus_{s=a+1}^{a+e} f_i\mathcal{B}\epsilon_s f_j$$

and

$$f_i\Gamma_0 f_i \cong \{(x, y) \mid x, y \in R^{d_i \times d_i}, x \equiv y \pmod{p^a}\} \subset f_i\epsilon_s\mathcal{B}f_i \oplus f_i\epsilon_t\mathcal{B}f_i \cong R^{d_i \times d_i} \oplus R^{d_i \times d_i}$$

if $i \in r_s \cap r_t$.

(iii) Finally there are epimorphisms ν and μ of Γ_0 and Γ_a onto $\oplus_{i \in r_1}(R/p^aR)^{d_i \times d_i}$ such that

$$\mathcal{B} = \{(x, y) \in \Gamma_0 \oplus \Gamma_a \mid \nu(x) = \mu(y)\}.$$

Note that $X_s \cong \Gamma_{s-1}/J(\Gamma_{s-1})^{x_s}$.

The possible ambiguity in this description is the choice of the epimorphisms in (i) and in (iii). It is clear that one can always fix one of the two epimorphisms. The choices for the other one correspond to the automorphisms of the image. So the question is, whether these automorphisms lift to automorphisms of $\mathcal{B}$. This is clear for the maps in (iii). For (i) this is unfortunately not always the case.

To simplify notation, it is convenient to pass to the Morita-equivalent basic order. Let S be any discrete valuation ring with prime element π_S and $\Lambda := \Lambda(S, (1, \dots, 1), H_n)$ the basic hereditary S-order of degree n. Let $X := \Lambda/\pi_S\Lambda$. Then Λ is generated by the idempotents $e_i = \operatorname{diag}(0, \dots, 0, 1, 0, \dots 0)$ (the 1 is on the i-th place), the elements

$$e_{i+1,i} = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & 1 & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \in e_{i+1}\Lambda e_i \text{ and } g_{1,n} = \begin{pmatrix} 0 & \dots & 0 & \pi_S \\ 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \in e_1\Lambda e_n$$

where $i = 1, \dots, n-1$. These generators map onto generators $\overline{e_i}$, $\overline{e_{i+1,1}}$ and $\overline{g_{1,n}}$ of X corresponding to the Ext-quiver of X which is a directed n-gon. They satisfy the relation that

$$\overline{g_{1,n} e_{n,n-1}} \cdots \overline{e_{2,1}} = 0$$

and similarly for any cyclic permutation of this product.

Lemma 2.4. *Let Λ and $X = \Lambda/\pi_S\Lambda$ be as above and let φ be an automorphism of X that fixes all the idempotents $\overline{e_i}$.*

Then there are $0 \neq \overline{\lambda_i} \in S/\pi_S S =: k_S$ with $\varphi(\overline{e_{i+1,i}}) = \overline{\lambda_i}\overline{e_{i+1,i}}$ for $i = 1, \dots, n-1$ and $\varphi(\overline{g_{1,n}}) = \overline{\lambda_n}\overline{g_{1,n}}$.

There is an automorphism ϕ of Λ that lifts φ if and only if the product $\overline{\lambda_1} \cdots \overline{\lambda_n} = 1$.

In particular, there is always an automorphism ϕ of Λ with

$$\overline{\phi(e_{i+1,i})} = \varphi(\overline{e_{i+1,i}}), \text{ and } \phi(e_j) = e_j \text{ for all } i = 1, \dots, n-1, j = 1, \dots, n.$$

Proof. The automorphism φ maps the generator $\overline{e_{i+1,i}} \in \overline{e_{i+1}} X \overline{e_i} = k_S \overline{e_{i+1,i}}$ to some other generator of this module ($i = 1, \dots, n-1$) and similar for $\overline{g_{1,n}}$. Hence there are such units $\overline{\lambda_i} \in k_S^*$ as described in the lemma. Moreover any such tuple $(\overline{\lambda_1}, \dots, \overline{\lambda_n}) \in (k_S^*)^n$ determines a unique automorphism of X fixing all the idempotents $\overline{e_i}$.

Choose units $\lambda_i \in S^*$ that map to $\overline{\lambda_i}$ in k_S. Then the matrix

$$D := \operatorname{diag}(1, \lambda_1, \lambda_1\lambda_2, \dots, \lambda_1 \cdots \lambda_{n-1}) \in \Lambda^*$$

fixes all the e_i and conjugates $e_{i+1,i}$ to $\lambda_i e_{i+1,i}$ for all $i = 1, \dots, n-1$ and $g_{1,n}$ to $(\lambda_1 \cdots \lambda_{n-1})^{-1} g_{1,n}$. Hence if the product of the $\overline{\lambda_i}$ is 1, then conjugation by D is the desired automorphism ϕ.

On the other hand it is easy to see that all automorphisms of Λ that fix the idempotents e_i are given by conjugation with a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ mapping the matrix units e_{ij} to $\frac{d_i}{d_j} e_{ij}$. □

Therefore there is a tiny bit missing in Theorem 2.3 (i) to describe the exceptional vertex Γ_a up to isomorphism. Since blocks of group rings are symmetric orders, however, the missing information can easily be obtained from the trace bilinear form.

Theorem 2.3 gives the centre $Z := Z(\Gamma_a)$ up to isomorphism. Instead of continuing with Plesken's description, it seems to be easier to give generators for Γ_a over the centre

Z using the Ext-quiver of Γ_a. To this aim, we pass to the Morita equivalent basic order and assume that all simple Γ_a-modules are of dimension 1 over k. All projective Γ_a-lattices are uniserial when reduced modulo p, where the sequence of composition factors is given by the permutation δ. Therefore the Ext-quiver of Γ_a is again a directed n-gon, where $n = |r_1|$ is the number of simple Γ_a-modules. If $n = 1$, then $\Gamma_a = Z(\Gamma_a)$ is already described completely by Theorem 2.3. So we will assume that $n > 1$. Let $e_1, \ldots, e_n \in \Gamma_a$ be orthogonal lifts of the central primitive idempotents of $\Gamma_a/J(\Gamma_a)$ ordered in such a way that $\delta_{|r_1} = (1, \ldots, n)$. Denote the corresponding matrix units in $e_i K\Gamma_a \epsilon_s e_j$ by e^s_{ij} ($i, j \in \{1, \ldots, n\}, s \in \{1, \ldots, a\}$). Then according to Theorem 2.3 and Lemma 2.4, after a choice of a suitable basis, generators of Γ_a over its centre Z can be chosen as

$$e_1, \ldots, e_n, (e^1_{2,1}, \ldots, e^a_{2,1}) =: e_{2,1} \in e_2\Gamma_a e_1, \ldots, (e^1_{n,n-1}, \ldots, e^a_{n,n-1}) =: e_{n,n-1} \in e_n\Gamma_a e_{n-1}$$

and

$$(x_1\pi_1 e^1_{1,n}, \ldots, x_a\pi_a e^a_{1,n}) =: g_{1,n} \in e_1\Gamma_a e_n \text{ for certain units } x_i \in Z_i = Z(\Gamma_a\epsilon_i).$$

Theorem 2.5. *Let $Z := Z(\Gamma_a)$ and let $Z^\#$ be the dual of Z with respect to the sum of the usual trace bilinear forms. Then there are units $x_i \in Z_i = Z(\Gamma_a\epsilon_i)$ ($i = 1, \ldots, a$) such that*

$$p^a Z^\# = (x_1\pi_1, \ldots, x_a\pi_a)Z.$$

With the choice of these x_i, the order Γ_a is generated by

$$Z, e_1, \ldots, e_n, e_{i+1,i}\ (i = 1, \ldots, n-1), \text{ and } g_{1,n}$$

as defined above.

Proof. We may assume that $n > 1$. $\mathcal{B}$ is a symmetric order with respect to the associative symmetric bilinear form

$$(x, y) \mapsto \frac{1}{|G|}\mathrm{trace}_{reg}(xy) = \mathrm{trace}_{red}(xyz) =: \mathrm{Tr}_z(x, y)$$

where trace_{reg} and trace_{red} denote the regular respectively reduced trace of $K\mathcal{B}$ and $z = \sum_{s=1}^{a+e} \frac{\chi_s(1)}{|G|}\epsilon_s$, where $\epsilon_1, \ldots, \epsilon_{a+e}$ are the central primitive idempotents of $K\mathcal{B}$ and $\chi_1, \ldots, \chi_{a+e}$ some corresponding absolutely irreducible (complex) characters of G.

Let $f_1, \ldots, f_n$ denote orthogonal idempotents in $\mathcal{B}$ that map onto the central primitive idempotents of $\mathcal{B}/J(\mathcal{B})$ such that

$$e_i = f_i(\epsilon_1 + \ldots + \epsilon_a)\ (i = 1, \ldots, n).$$

Since $n > 1$

$$\langle g_{1,n}\rangle_Z = e_1\Gamma_a e_n = f_1\mathcal{B}f_n = (f_n\mathcal{B}f_1)^\# = (e_n\Gamma_a e_1)^\# = \langle e_{n,n-1}\cdots e_{2,1}\rangle^\#_Z$$

can be calculated via the symmetrising form above. Since the character degrees of the absolutely irreducible characters belonging to the exceptional vertex are all equal the dual with respect to Tr_z is as stated in the theorem, yielding the remaining generator $g_{1,n}$ for Γ_a. □

Note that the x_i do not depend on the degrees of the irreducible complex characters in $\mathcal{B}$, since all exceptional absolutely irreducible characters have the same degree. Therefore one gets

Corollary 2.6. *Let $\mathcal{B}_i$ ($i = 1, 2$) be two blocks with cyclic defect group $\cong C_{p^a}$ and assume that R is an unramified extension of $\mathbb{Z}_p$ that is large enough so that k is a splitting field for $k\mathcal{B}_i$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are Morita equivalent if and only if their Brauer trees (including the permutations δ and ρ) and the character fields $Z_1, \ldots, Z_a$ coincide.*

Also, symmetric orders remain symmetric orders, when one extends the ground ring. Therefore the explicit description in [Ple83, Theorem 8.5] shows that the Brauer tree determines a block of cyclic defect up to Morita equivalence (over an algebraically closed field). This is also shown in [Lin96, Theorem 2.7(ii)] with completely different methods.

Corollary 2.7. *Let $\mathcal{B}_i$ ($i = 1, 2$) be two blocks with isomorphic cyclic defect group and assume that R is large enough so that k and K are splitting fields for $k\mathcal{B}_i$ and $K\mathcal{B}_i$. (Here we drop the assumption that K is unramified over $\mathbb{Q}_p$.) Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are Morita equivalent if and only if their planar embedded Brauer trees coincide.*

2.1 Galois descent.

We now perform the Galois descent to obtain a description over $\mathbb{Z}_p$ (see [Ple83, Chapter 8]). So let B be a block of $\mathbb{Z}_pG$ such that $\mathcal{B}$ is a summand of $R \otimes B$. We assume that K is chosen to be minimal, i.e. $K = \mathbb{Q}_p[\chi_{a+1}] = \ldots = \mathbb{Q}_p[\chi_{a+e}]$ is the character field of any non-exceptional absolutely irreducible Frobenius character that belongs to $\mathcal{B}$. The maximal unramified subfield $\tilde{K}$ of the character field $\tilde{Z}_s := \mathbb{Q}_p[\chi_s]$ ($s = 1, \ldots, a$) of any exceptional absolutely irreducible Frobenius character in $\mathcal{B}$ does not depend on the character and is a subfield of K. Let $m := [K : \tilde{K}]$ denote its index.

If $\tilde{R}$ denotes the ring of integers in $\tilde{K}$, then $\tilde{R}$ embeds into the centre of B such that B can be viewed as an $\tilde{R}$-order and $R \otimes_{\tilde{R}} B \cong \mathcal{B}$.

The Galois group $\mathrm{Gal}(K/\tilde{K}) = \mathrm{Gal}(k/\tilde{k}) \cong C_m$ (where $\tilde{k} := \tilde{R}/p\tilde{R}$) acts on the simple $\mathcal{B}$-modules and the corresponding idempotents $f_1, \ldots, f_e$ with orbits of length m. Therefore orthogonal lifts of the central primitive idempotents of $B/J(B)$ can be chosen as $\tilde{f}_1, \ldots, \tilde{f}_{\tilde{e}} \in B$ where $\tilde{e} := \frac{e}{m}$ is the number of simple $\mathbb{F}_pB$-modules, each of which has character field $k = R/pR$.

The central primitive idempotents in $A := \mathbb{Q}_p \otimes B$ are $\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_a, \tilde{\epsilon}_{a+1}, \ldots, \tilde{\epsilon}_{a+\tilde{e}}$ indexed in such a way that $\tilde{Z}_s$ is a totally ramified extension of $\tilde{K}$ of degree $\frac{p^s - p^{s-1}}{e}$ for $s = 1, \ldots, a$.

For an appropriate ordering of the index set $\{1, \ldots, e\}$ the k-dimensions of the simple $\mathbb{F}_pB$-modules are $d_1, \ldots, d_{\tilde{e}}$ and the set of indices of the simple $\tilde{\epsilon}_sB$-modules is $\tilde{r}_s = r_s \cap \{1, \ldots, \tilde{e}\}$.

For $s = a + 1, \ldots, a + \tilde{e}$, the centre of $B\tilde{\epsilon}_s$ is isomorphic to R, $\tilde{r}_s = r_s$, and $B\tilde{\epsilon}_s$ is isomorphic to one of the R-orders in Theorem 2.2 (ii).

Let $n' := |\tilde{r}_1| = \frac{|r_1|}{m}$. For $s = 1, \ldots, a$ let D_s be a central $\tilde{Z}_s$-division algebra of index m and Ω_s be its maximal order with prime element $\wp_s$. Then

$$B\tilde{\epsilon}_s \cong \Lambda(\Omega_s, (d_i, d_{\delta(i)}, \ldots, d_{\delta^{n'-1}(i)}), H_{n'}).$$

Then the Hasse invariant of D_s (as defined in [Rei75, (31.7)]) is independent of s and can be read off from the planar embedded Brauer tree together with the Galois action of $\mathrm{Gal}(k/\tilde{k}) \cong \mathrm{Gal}(K/\tilde{K})$ on the modular constituents of any exceptional character in $\mathcal{B}$:

Theorem 2.8. *Let ψ be a p-modular constituent of any of the exceptional characters in $\mathcal{B}$. Let F denote the Frobenius automorphism of $k/\tilde{k}$. Then there is some $r \in \mathbb{Z}$ prime to m such that*

$$\delta^{n'}(\psi) = F^r(\psi) \text{ where } n' = |\tilde{r}_1| = \frac{|r_1|}{m}.$$

Let $t = r^{-1} \in \mathbb{Z}/m\mathbb{Z}$. Then for all $s \in \{1, \ldots, a\}$ the Hasse invariant of D_s is $\frac{t}{m}$.

Proof. To simplify notation we again assume that all the character degrees d_i are equal to 1. Then for $s \in \{1, \ldots, a\}$ the order $B\epsilon_s = \Lambda(\Omega_s, (1, \ldots, 1), H_{n'})$ and

$$P = \begin{pmatrix} 0 & \ldots & 0 & \wp_s \\ 1 & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{pmatrix}$$

is a generator of $J(B\tilde{\epsilon}_s)$. Then P also generates the Jacobson radical of $\mathcal{B}\epsilon_s = R \otimes_{\tilde{R}} B\epsilon_s$. Let L_ψ be a $\mathcal{B}\epsilon_s$-lattice whose head has character ψ. Then the head of $L_\psi P^{n'}$ has character $\delta^{n'}(\psi)$ which is Galois conjugate to ψ and hence of the form $F^r(\psi)$ for some r. Therefore conjugation by $P^{n'} = \mathrm{diag}(\wp_s, \ldots, \wp_s)$ induces the Galois automorphism F^r on the inertia subfield K of D_s. By the general theory of division algebras over local fields (see [Rei75]) r is prime to m and the Hasse invariant of D_s is $\frac{t}{m}$ as stated in the theorem. □

The amalgamations in B are described as in Theorem 2.3 (see [Ple83, Theorem 8.10]), where now the epimorphisms in (i) are only mappings between $\tilde{R}$-orders. For (iii) one should note that $R/pR \cong \Omega_s/\wp_s\Omega_s$ for all $s = 1, \ldots, a$.

Similarly as in Theorem 2.5 one shows:

Theorem 2.9. *The description above (see [Ple83, p. 140ff]) determines B up to isomorphism.*
More precisely let $\tilde{\Gamma}_a := (\tilde{\epsilon}_1 + \ldots + \tilde{\epsilon}_a)B$. and let $\tilde{e}_1, \ldots, \tilde{e}_{n'}$ ($n' = |\tilde{r}_1| = \frac{|r_1|}{m}$) be lifts of the central primitive idempotents in $\tilde{\Gamma}_a/J(\tilde{\Gamma}_a)$. Again we assume that the k-dimensions of the simple $\tilde{\Gamma}_a$-modules are 1. Then $\tilde{e}_i\tilde{\Gamma}_a\tilde{e}_i$ is generated as a $Z(\tilde{\Gamma}_a)$-order by $(\zeta_1, \ldots, \zeta_a)$ and $(\wp_1, \ldots, \wp_a)$, where $\zeta_s \in \Omega_s$ is a primitive (q^m-1)st root of unity ($q := |\tilde{k}| = |\tilde{R}/p\tilde{R}|$) and the prime elements $\wp_s \in \Omega_s$ are chosen such that $\zeta_s^{\wp_s} = F^r(\zeta_s) = \zeta_s^{q^r}$ where r is as in Theorem 2.8 (i.e. $\frac{t}{m}$ is the Hasse invariant of D_s where $rt \equiv 1 \pmod{m}$). The remaining generators of $\tilde{\Gamma}_a$ are $\tilde{e}_{i+1,i} \in \tilde{e}_{i+1}\tilde{\Gamma}_a\tilde{e}_i$ ($i = 1, \ldots, n'-1$) and $\tilde{g}_{1,n'} \in \tilde{e}_1\tilde{\Gamma}_a\tilde{e}_{n'}$ defined analogously to the ones in Theorem 2.5.

Proof. Let $L_i = \tilde{e}_i\tilde{\Gamma}_a$ be any projective indecomposable $\tilde{\Gamma}_a$-lattice ($i = 1, \ldots, n'$). Then, by the above, the endomorphism ring of L_i is a successive amalgam of the orders Ω_s, $s = 1, \ldots, a$. Since $\tilde{k}$-automorphisms of $\Omega_s/\wp_s\Omega_s$ lift to (inner) $\tilde{R}$-automorphisms of Ω_s, this ring is uniquely determined by [Ple83, Theorem 8.10] up to isomorphism and

$$\tilde{e}_i\tilde{\Gamma}_a\tilde{e}_i = \mathrm{End}_{\tilde{\Gamma}_a}(L_i) = \langle(\zeta_1, \ldots, \zeta_a), (\wp_1, \ldots, \wp_a), Z(\tilde{\Gamma}_a)\rangle.$$

To generate $\tilde{\Gamma}_a$, by Nakayama's lemma, it is enough to choose additional elements of $\tilde{e}_i\tilde{\Gamma}_a\tilde{e}_j$ ($i \neq j \in \{1, \ldots n'\}$) that generate

$$\tilde{e}_i\tilde{\Gamma}_a\tilde{e}_j/(\tilde{e}_iJ(\tilde{\Gamma}_a)^2\tilde{e}_j + p\tilde{e}_i\tilde{\Gamma}_a\tilde{e}_j)$$

as an $\tilde{e}_i\tilde{\Gamma}_a\tilde{e}_i$-module. The same arguments as in the proof of Theorem 2.5 now imply the theorem. ☐

Corollary 2.10. *The planar embedded Brauer tree together with the character fields K, $\tilde{Z}_1$, ..., $\tilde{Z}_a$ and the Galois action on the modular constituents of the exceptional characters determine the block B of $\mathbb{Z}_pG$ up to Morita equivalence.*

3 The radical idealiser chain for blocks with cyclic defect groups

In this section we will investigate the radical idealiser chain for blocks with cyclic defect group, where we mainly concentrate on describing the head order. Head orders are hereditary orders and hence they are the maximal elements for the "radically covering" relation, where an order Γ radically covers and order Λ, $\Gamma \succ \Lambda$, if $\Gamma \supseteq \Lambda$ and $J(\Gamma) \supseteq J(\Lambda)$. Then for all orders in the idealiser chain $\Lambda_i \succ \Lambda_{i-1}$ (see [Rei75, Section 39]). Moreover it is easy to see that if $\Gamma \succ \Lambda$ then every simple Γ module is semi-simple as a Λ-module (see [Neb04, Lemma 2.2]). In particular the simple Λ_N-modules are semi-simple Λ-modules.

We will use the notation introduced in the last section and perform the calculations for the block B of $\mathbb{Z}_pG$. The results for the block $\mathcal{B}$ of RG then follow easily (see Corollary 3.17). However, it is crucial for the whole process that R is an unramified extension of $\mathbb{Z}_p$.

For the radical idealiser process we treat the exceptional vertex $\tilde{\Gamma}_a$ and $\tilde{\Gamma}_0$ separately always keeping track of the amalgamations between them, which are controlled by the following lemma.

Lemma 3.1. *Let S be a discrete valuation ring with prime element π and let Λ_i ($i = 1, 2$) be S-orders. Given epimorphisms $\varphi_i : \Lambda_i \to X := S^{s\times s}/\pi^t S^{s\times s}$ let*

$$\Lambda := \{(x_1, x_2) \in \Lambda_1 \oplus \Lambda_2 \mid \varphi_1(x_1) = \varphi_2(x_2)\}.$$

Then

$$\mathrm{Id}(J(\Lambda)) \supseteq \{(x_1, x_2) \in \Lambda_1 \oplus \Lambda_2 \mid \overline{\varphi_1(x_1)} = \overline{\varphi_2(x_2)}\} =: \Gamma$$

where $\overline{} : X \to S^{s\times s}/\pi^{t-1}S^{s\times s}$ *is the natural epimorphism.*

Proof. Clearly $J(\Lambda) = \{(y_1, y_2) \in J(\Lambda_1) \oplus J(\Lambda_2) \mid \varphi_1(y_1) = \varphi_2(y_2)\}$ and $\varphi_i(J(\Lambda_i)) = J(X) = \pi X$ for $i = 1, 2$. Let $(x_1, x_2) \in \Gamma$ and $(y_1, y_2) \in J(\Lambda)$. Then clearly $x_i y_i$ and $y_i x_i$ are in $J(\Lambda_i)$ $(i = 1, 2)$. Since φ_1 is surjective, there is $z_1 \in \Lambda_1$ with $\pi\varphi_1(z_1) = \varphi_1(y_1)$. Choose $z_2 \in \Lambda_2$ with $\varphi_2(z_2) = \varphi_1(z_1)$. Then

$$\varphi_1(y_1 x_1) = \varphi_1(z_1)\pi\varphi_1(x_1) = \varphi_2(z_2)\pi\varphi_2(x_2) = \varphi_2(y_2 x_2)$$

and similarly $\varphi_1(x_1 y_1) = \varphi_2(x_2 y_2)$. Hence $(x_1, x_2) \in \mathrm{Id}(J(\Lambda))$. □

The following trivial lemma suffices to deduce the head order of $\tilde{\Gamma}_a$.

Lemma 3.2. *Let Λ be an order in $\mathcal{A}$ and ϵ a central idempotent of $\mathcal{A}$. Then*

$$\Lambda\epsilon \subseteq \mathrm{Id}(J(\Lambda))\epsilon \subseteq \mathrm{Id}(J(\Lambda\epsilon)).$$

Corollary 3.3. *The head order of $\tilde{\Gamma}_a$ is $\oplus_{s=1}^a B\tilde{\epsilon}_s$.*
Similarly the head order of Γ_a is $\oplus_{s=1}^a \mathcal{B}\epsilon_s$.

Proof. The orders $\mathcal{B}\epsilon_s = \mathrm{Id}(J(\mathcal{B}\epsilon_s))$ and $B\tilde{\epsilon}_s$ are already hereditary for $s = 1, \ldots, a$. □

Note that this corollary is not true, when R is replaced by a ramified extension of $\mathbb{Z}_p$.

3.1 The first steps.

The main task to calculate the idealiser chain for $\tilde{\Gamma}_0$ is to calculate the one of $\tilde{\epsilon}_s\tilde{\Gamma}_0$ for $s = a+1, \ldots, a+\tilde{e}$. These orders have a certain symmetry with respect to a cyclic permutation of their simple modules, and therefore can be encoded in a simple way. All orders in this radical idealiser chain share this symmetry.

Definition 3.4. *For $v = (v_0, \ldots, v_{n-1}) \in \mathbb{Z}^n$ and $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$ define*

$$\Lambda(d, v) := \Lambda(R, d, M) := \{X = (x_{ij}) \in K^{D\times D} \mid x_{ij} \in \pi^{m_{ij}} R^{d_i \times d_j}\}$$

where

$$m_{ij} = \begin{cases} v_{j-i} & \text{if } j \geq i \\ v_{n+j-i} - v_{n-1} & \text{if } j < i \end{cases}$$

and $D = \sum_{i=1}^n d_i$.

Remark 3.5. *Since the dimension vector d will be fixed most of the time, we will omit it and let $\Lambda(v_0, \ldots, v_{n-1}) := \Lambda(d, v)$.*

The order $\tilde{\Gamma}_0$ is an amalgam of the orders $\tilde{\epsilon}_s B = \tilde{\epsilon}_s\tilde{\Gamma}_0$ $(s = a+1, \ldots, a+\tilde{e})$ of the form $\Lambda(R, d, aH_n) \cong \Lambda(0, \underbrace{a, \ldots, a}_{n-1})$ for some dimension vector d and $n = |r_s|$. The amalgamations in $\tilde{\Gamma}_0$ are only on the diagonal, more precisely, the part of B belonging to $\tilde{\epsilon}_s B$ is of the form

$$\Lambda(R, d, \begin{pmatrix} \underline{0}_a & a & \dots & \dots & a \\ 0 & \underline{0}_a & a & \dots & a \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \underline{0}_a & a \\ 0 & \dots & \dots & 0 & \underline{0}_a \end{pmatrix}) = \Lambda(\underline{0}_a, a^{n-1})$$

where all the underlined entries obey a certain congruence modulo p^a to a diagonal entry in some other $\tilde{\epsilon}_t B$ ($t \neq s$) which is indicated by underlining the 0 and the index a. By Lemma 3.1 these amalgamations will decrease by 1 in each step until after a steps the order B_a contains the central primitive idempotents $\tilde{\epsilon}_{a+1}, \dots, \tilde{\epsilon}_{a+\tilde{e}}$.

In the following we fix some $s \in \{a+1, \dots, a+\tilde{e}\}$, put $n := |r_s|$, and let

$$\Lambda := \Lambda_0 := B\tilde{\epsilon}_s \subseteq \Lambda_1 := B_1\tilde{\epsilon}_s \subseteq \dots \subseteq \Lambda_N := B_N\tilde{\epsilon}_s$$

where

$$B =: B_0 \subset B_1 \subset \dots \subset B_N = B_{N+1}$$

is the radical idealiser chain of B. Together with the structure of Λ_i we keep track of the additional information, how B_i is embedded into the direct sum of the $B_i\tilde{\epsilon}_s$ using the notation above.

Lemma 3.6. *If $a \geq n$ then*

$$\Lambda_n \cong \Lambda(\underline{0}_{a-n}, (a-n)^{n-1}).$$

Proof. An easy induction on j shows that for $j = 1, \dots, n$

$$\Lambda_j = \Lambda(\underline{0}_{a-j}, (a-j+1), (a-j+2), \dots, a-1, a^{n-j}).$$

Then

$$\Lambda_n = \Lambda(\underline{0}_{a-n}, (a-n+1), (a-n+2), \dots, a-1) \cong \Lambda(\underline{0}_{a-n}, (a-n)^{n-1})$$

by conjugation with the diagonal matrix $\mathrm{diag}(1, \pi, \pi^2, \dots, \pi^{n-1})$. □

Inductively we get

Corollary 3.7. *Let $a = z_s n + b$ with $0 \leq b < n$ and $m_0 := z_s n + 1$. Then*

$$\Lambda_{m_0-1} = \Lambda(\underline{0}_b, b^{n-1}).$$

If $b = 0$ then Λ_{m_0-1} is already a maximal order and we are done.

Lemma 3.8. *Assume that $b > 0$ and define l_0, x_0 by $n = l_0 b + x_0$ with $0 < x_0 \leq b$. Then*

$$\Lambda_{m_0} = \Lambda(\underline{0}_{b-1}, b^{n-1}).$$

If $0 \leq m < n - l_0$ then

(a) *Λ_{m+m_0} is of the form $\Lambda_{m+m_0} = \Lambda(\underline{0}_{f(m)}, v_1, \ldots, v_{n-1})$ with $0 < v_1 \leq v_2 \leq \ldots \leq v_{n-1} = b$, where $f(m) = \max\{0, b-m-1\}$.*

(b) *If $m = l(b-1) + y$ with $0 \leq y < b-1$ then*

$$\Lambda_{m+m_0} = \Lambda(\underline{0}_{f(m)}, 1^l, 2^l, \ldots, (b-y-1)^l, (b-y)^{l+1}, \ldots, (b-1)^{l+1}, b^{n-m-1})$$

i.e. $\Lambda_{m+m_0} = \Lambda(\underline{0}_{f(m)}, v_1, \ldots, v_{n-1})$ with

$$v_j = \begin{cases} \lfloor \frac{j-1}{l} \rfloor + 1 & \text{if } 1 \leq j \leq (b-y-1)l \\ b - y + \lfloor \frac{j-1-(b-y-1)l}{l+1} \rfloor & \text{if } (b-y-1)l < j \leq (b-1)l + y \\ b & \text{if } j > (b-1)l + y \end{cases}$$

(c) *The radical $J(\Lambda_{m+m_0}) = \Lambda(\underline{1}_{f(m)}, v_1, \ldots, v_{n-1})$.*

Proof. The form of Λ_{m_0} is clear. For the other statements we argue by induction on m, where the case $m = 0$ is trivial. Assume that $m < n - l_0 - 1$ and that $\Lambda_{m+m_0} = \Lambda(\underline{0}_{f(m)}, v_1, \ldots, v_{n-1})$ has the properties $(a), (b), (c)$. Then Λ_{m+m_0+1} is of the form

$$\Lambda_{m+m_0+1} = \Lambda(\underline{0}_{f(m+1)}, \tilde{v}_1, \ldots \tilde{v}_{n-1}),$$

since the inequalities on the entries of the exponent matrix preserve the symmetry conditions in (a). The form of the amalgamations follows from Lemma 3.1. Clearly $\tilde{v}_i \leq v_i$ for all i and $\tilde{v}_0 = 0$. The remaining conditions in (a) and the property (c) follow once we have shown (b). Let $v'_i := v_i$ for $i > 0$ and $v'_0 := 1 = v_0 + 1$. Then the conditions on $m_{1j} = \tilde{v}_{j-1}$ $(j > 1)$ that Λ_{m+m_0+1} lies in the left idealiser of $J(\Lambda_{m+m_0})$ read as

$$\tilde{v}_{j-1} \geq \max\{v_{k-1} - v'_{k-j} \mid k = j, \ldots, n\} =: \max{}_1$$

and

$$\tilde{v}_{j-1} \geq \max\{b + v'_{k-1} - v_{k+n-j} \mid k = 1, \ldots, j-1\} =: \max{}_2.$$

The inequalities for the right idealiser of $J(\Lambda_{m+m_0})$ read as

$$\tilde{v}_{j-1} \geq \max\{v_{j-1} - 1, v'_{j-k_1} - v_{n+1-k_1} + b, v_{n+j-k_2} - v_{n+1-k_2} \mid k_1 = 2, \ldots, j, k_2 = j+1, \ldots, n\}$$

and agree with the conditions above after an easy variable transformation. Hence right and left idealiser of $J(\Lambda_{m+m_0})$ coincide and are equal to $\mathrm{Id}(J(\Lambda_{m+m_0}))$.

By the induction assumption for all $1 \leq i \leq n$

$$v_{i-1} = \begin{cases} \lfloor \frac{i-2}{l} \rfloor + 1 & \text{if } 1 \leq i-1 \leq (b-y-1)l \\ b - y + \lfloor \frac{i-2-(b-y-1)l}{l+1} \rfloor & \text{if } (b-y-1)l < i-1 \leq (b-1)l + y \\ b & \text{if } i-1 > (b-1)l + y. \end{cases}$$

Since the 'slope' of v is decreasing $v_{k-1} - v'_{k-j}$ is maximal if v'_{k-j} is the last 1 in v', hence if $k - j = l$ i.e. $k = l + j$. If $k := \min(l+j, n)$ then

$$\max{}_1 = v_{k-1} - 1 = \begin{cases} \lfloor \frac{j-2}{l} \rfloor + 1 & \text{if } 1 < j-1 \leq (b-y-2)l \\ b - y + \lfloor \frac{j-2-(b-y-2)l}{l+1} \rfloor - 1 & \text{if } (b-y-2)l < j-1 \leq (b-2)l + y \\ b - 1 & \text{if } (b-2)l + y < j-1. \end{cases}$$

This implies that $\max_1 = v_{j-1}$ if $j-1 \leq (b-y-2)l$. If $(b-y-2)l < j-1 \leq (b-y-1)l$ then

$$v_{j-1} = \lfloor \frac{j-2}{l} \rfloor + 1 = b-y-1 = \max_1 = b-y+\lfloor \frac{j-2-(b-y-2)l}{l+1} \rfloor - 1.$$

If $(b-y-1)l < j-1 \leq (b-2)l+y$ then $\max_1 = b-y-1+\lfloor \frac{j-2-(b-y-2)l}{l+1} \rfloor$ and $v_{j-1} = b-y+\lfloor \frac{j-2-(b-y-1)l}{l+1} \rfloor$. Therefore $\max_1 < v_{j-1}$ if and only if $j-3-y+b$ is divisible by $l+1$, i.e.

$$j-1 = (b-2)l+y-x(l+1), \; x = 0,1,\ldots,y-2$$

when $v_{j-1} = b-x-1$ is the first occurrence of $b-x-1$ in v.

If $(b-2)l+y < j-1 \leq (b-1)l+y$ then $v_{j-1} = b-1 = \max_1$ and if $j-1 > (b-1)l+y$ then $\max_1 = b-1 < v_{j-1} = b$.

For $\max_2$ one finds that $b+v'_{k-1} - v_{k+(n-j)}$ is maximal if $k = j-1$ since the 'slope' of v is decreasing. Hence

$$\max_2 = b + v'_{j-2} - v_{n-1} = v'_{j-2}.$$

Combining these conditions one finds that

$$\tilde{v}_{j-1} = \begin{cases} v_{j-1} - 1 & \text{if } j-1 = (b-2)l+y-x(l+1) \text{ for } x = 0,\ldots,y-1 \\ v_{j-1} & \text{otherwise} \end{cases}.$$

With these $\tilde{v}_j$ the multiplication by $\Lambda_{m+m_0+1} = \Lambda(\underline{0}_{f(m+1)}, \tilde{v}_1, \ldots, \tilde{v}_{n-1})$ preserves the congruences in $J(\Lambda_{m+m_0})$ given in (c), since

$$b-m-1 \leq f(m) \leq \tilde{v}_{j-1} + v_{n+1-j} - b \text{ and } b-m-1 \leq f(m) \leq \tilde{v}_{n+1-j} - b + v_{j-1}$$

for all j. This implies part (b) of the lemma. □

Corollary 3.9. *Let $m_1 := n - l_0 - 1 + m_0$ and $y = x_0 - 1$. Then*

$$\Lambda_{m_1} = \Lambda(0, 1^{l_0}, \ldots, (b-y-1)^{l_0}, (b-y)^{l_0+1}, \ldots, (b-1)^{l_0+1}, b^{l_0}) =: \Lambda(v^{(1)}).$$

Corollary 3.10. *For all $s \in \{1, \ldots, a+\tilde{e}\}$ the s-th component of the head order of B is equal to the head order of the projection $\tilde{\epsilon}_s B$.*

3.2 The head order.

If $n = (l_0+1)b$ is divisible by b, then the order Λ_{m_1} as defined in Corollary 3.9 is already hereditary. More precisely we have the following

Lemma 3.11. *Let b be a factor of $n = lb$. Then $\Lambda_{m_1} = \Lambda(0, 1^l, \ldots, (b-1)^l, b^{l-1})$ is hereditary, $\Lambda_{m_1} \sim \Lambda(R, (D_1, \ldots, D_l), H_l)$ where $D_i = \sum_{j=0}^{b-1} d_{jl+i}$.*

Proof. $\Lambda_{m_1} = \Lambda(R, d, M)$ where

$$m_{ij} = \lfloor \frac{j-i-1}{l} \rfloor + 1.$$

Let $t_i := m_{i1} = \lfloor \frac{-i}{l} \rfloor - 1$. Conjugating by the diagonal matrix $T := \mathrm{diag}(\pi^{t_i})$ one obtain the conjugate order $\Lambda_{m_1}^T = \Lambda(R, d, \tilde{M})$, where

$$\tilde{m}_{ij} = m_{ij} - t_i + t_j = \lfloor \frac{j-i-1}{l} \rfloor + 1 - \lfloor \frac{-i}{l} \rfloor + \lfloor \frac{-j}{l} \rfloor.$$

Writing $j = j_1 l + j_2$ and $i = i_1 l + i_2$ with $0 < j_2, i_2 \leq l$ one gets

$$\tilde{m}_{ij} \lfloor \frac{j_2 - i_2 - 1}{l} \rfloor + 1 - \lfloor \frac{-i_2}{l} \rfloor + \lfloor \frac{-j_2}{l} \rfloor = \lfloor \frac{j_2 - i_2 - 1}{l} \rfloor + 1 = \begin{cases} 0 & \text{if } j_2 \leq i_2 \\ 1 & \text{if } j_2 > i_2 \end{cases}.$$

Hence after reordering the constituents Λ_{m_1} has the form as claimed in the lemma.
□

We now assume that $0 < x_0 < b$. Continuing to trace down the radical idealiser process like in Lemma 3.8 seems to be a rather tedious work. If $x_0 \geq \frac{b}{2}$ then after $m_2 = b - x_0 - 1$ steps one arrives at an order

$$\Lambda_{m_1+m_2} = \Lambda(0, 1^{l_0}, 2^{l_0+1}, 3^{l_0}, 4^{l_0+1}, \ldots, z^{l_0}, (z+1)^{l_0+1}, (z+2)^{l_0+1}, \ldots, (b-1)^{l_0+1}, b^{l_0})$$

where $z = 2b - 2x_0 - 1$. If $x_0 \leq \frac{b}{2}$ then after $m_2 = x_0 - 1$ steps one arrives at an order

$$\Lambda_{m_1+m_2} = \Lambda(0, 1^{l_0}, 2^{l_0}, \ldots, z^{l_0}, (z+1)^{l_0+1}, (z+2)^{l_0}, \ldots, (b-2)^{l_0+1}, (b-1)^{l_0}, b^{l_0})$$

where $z = b - 2x_0 + 1$. If $x = \frac{b}{2} = \gcd(n, b) = d$ then $\Lambda_{m_1+m_2}$ is again hereditary

$$\Lambda_{m_1+m_2} \sim \Lambda(R, (D_1, \ldots, D_{2l_0+1}), H_{2l_0+1})$$

where $D_j = \sum_{i \equiv -l_0 j} d_i$, where the congruence is modulo $2l_0 + 1 = \frac{n}{d}$.

Instead of continuing like this, we prefer to calculate the head order $\Lambda_N = \tilde{\epsilon}_s B_N$, which is also the head order of Λ_{m_1}, directly where we need the following trivial lemma:

Lemma 3.12. *Let $\Lambda \subset \Gamma$ be two orders with $J(\Lambda) \subset J(\Gamma)$. If $e \in \Lambda$ is an idempotent then $J(e\Lambda e) \subset J(e\Gamma e)$.*

Proof.

$$J(e\Lambda e) = eJ(\Lambda)e \subseteq eJ(\Gamma)e = J(e\Gamma e).$$

□

The head order Λ_N of Λ_{m_1} has the following properties:

Properties 3.13. 0) Λ_N is of the form $\Lambda(w)$ for some $w \in \mathbb{Z}^n_{\geq 0}$.

1) Λ_N is an order, i.e. for all $i < j < k$ one has

   (i) $b - w_{n+j-k} \leq w_{k-i} - w_{j-i} \leq w_{k-j}$

   (ii) $b - w_{n+i-k} \leq w_{k-j} - w_{n+i-j} + b \leq w_{k-i}$

(iii) $b - w_{n+i-j} \leq w_{n+j-k} - w_{n+i-k} \leq w_{j-i}$

which just expresses the fact that the entries m_{ij} in the exponent matrix of Λ_N satisfy $m_{ik} + m_{kj} \geq m_{ij}$ for all $i, j, k \in \{1, \ldots, n\}$.

2) Λ_N is hereditary, i.e.

$$w_{j-1} + w_{n+1-j} - b \in \{0, 1\} \text{ for all } j > 1.$$

3) Λ_N radically covers the order $\Lambda_{m_1} = \Lambda(v^{(1)})$ defined in Corollary 3.9. This property implies with Lemma 3.12 that $w_{j-1} = v^{(1)}_{j-1}$ and $w_{n-j+1} = v^{(1)}_{n-j+1}$ whenever $v^{(1)}_{j-1} + v^{(1)}_{n+1-j} - b = 1$. In particular

3') $w_1 = \ldots = w_{l_0} = 1$, $w_{n-1} = \ldots = w_{n-l_0} = b$, and $w_{n-l_0-1} = b - 1$.

Lemma 3.14. *Λ_N is uniquely determined by Properties 3.13 0), 1), 2), and 3'). More precisely let $n = l_0 b + x_0$ be as above and assume that $1 \leq x_0 \leq b - 1$. Then*

(i) $\Lambda_N = \Lambda(w)$ where $w = (0, 1^{l_1}, 2^{l_2} \ldots, b^{l_b})$ with $l_1 = l_0 = l_b$ and $l_j \in \{l_0, l_0 + 1\}$ for all $j = 1, \ldots, b$.

(ii) Let $e := (e_1, \ldots, e_b)$, where $e_k = l_k - l_0 \in \{0, 1\}$ for $k = 1, \ldots b - 1$ and $e_b := 1$. For all j let $a_j := \sum_{k=1}^{j} e_k$. Let $d := \gcd(n, b) = \frac{b}{i}$. Then $x_0 = \frac{a_i}{i} b$, $e_i = 1$ and

$$e = (e_1, \ldots, e_i)^d = (e_1, \ldots, e_i, e_1, \ldots, e_i, \ldots, e_1, \ldots, e_i).$$

The entries of w are uniquely determined by

$$a_j = \lfloor \frac{x_0 \cdot j}{b} \rfloor \text{ for all } j = 1, \ldots, b.$$

Proof. (i) Property 3.13 3') together with Property 3.13 1) (i) (for $i = 1$) show that for $0 < k - j \leq l_0$

$$0 \leq w_{k-1} - w_{j-1} \leq 1$$

and if $k - j \geq l_0 + 1$, then $w_{k-1} - w_{j-1} \geq 1$. This implies (i).
(ii) Put $d = \gcd(b, x_0) = \frac{b}{i}$. Then $i = \min\{j \in \{1, \ldots, b\} \mid \frac{b}{j} \text{ divides } x_0\}$.
We now show by induction on j that $a_j = \lfloor \frac{x_0 \cdot j}{b} \rfloor$ and $l_j = l_{b-j+1}$ for $j = 1, \ldots, i - 1$. This is clear for $j = 1$ since $l_1 = l_0 = l_b$ and $a_1 = 0 = \lfloor \frac{x_0}{b} \rfloor$. Assume that $1 < j \leq i - 1$ and that $a_k = \lfloor \frac{x_0 \cdot k}{b} \rfloor$ and $l_k = l_{b-k+1}$ for $k = 1, \ldots, j - 1$. Let

$$X_1 := ((t - 1), t^{l_t}, \ldots, (t + j - 1)^{l_{t+j-1}}, (t + j))$$

be a subsequence of w. Then the difference between the first and the last entry of X_1 is $j + 1$ and the distance between these entries is $\sum_{q=t}^{t+j-1} l_q + 1$. Since $w_{l_0 j + a_j} = j$, Property 3.13 1) (i) implies that

$$\sum_{q=t}^{t+j-1} l_q = l_0 j + \sum_{q=t}^{t+j-1} e_q \geq l_0 j + a_j \text{ for all } 1 \leq t < b + 1 - j.$$

Similarly for subsequences of (w,w) of the form

$$X_2 := ((b-t-1),(b-t)^{l_{b-t}},\dots,b^{l_b},0,1^{l_1},\dots,(j-t-1)^{l_{j-t-1}},(j-t))$$

Property 3.13 1) (ii) implies that

$$\sum_{q=b-t}^{b} l_q + 1 + \sum_{q=1}^{j-t-1} l_q = l_0 j + \sum_{q=b-t}^{b} e_q + \sum_{q=1}^{j-t-1} e_q \geq l_0 j + a_j \text{ for all } 0 \leq t \leq j.$$

This implies that for every subsequence of length j of the sequence (e,e), the sum over the entries in this subsequence is $\geq a_j$ and therefore

$$x_0 = \sum_{t=1}^{b} e_t \geq \frac{b}{j} a_j.$$

Let $b_j := \sum_{t=b-j+1}^{b} e_t$. Similar arguments as above, using the second and second last entries of the sequences X_1 and X_2 above and the fact that $w_{n-jl_0-b_j} = b-j$, show that

$$\sum_{q=t}^{t+j-1} l_q = l_0 j + \sum_{q=t}^{t+j-1} e_q \leq l_0 j + b_j \text{ for all } 1 \leq t \leq b+1-j$$

and

$$\sum_{q=b-t}^{b} l_q + 1 + \sum_{q=1}^{j-t-1} l_q = l_0 j + \sum_{q=b-t}^{b} e_q + \sum_{q=1}^{j-t-1} e_q \leq l_0 j + b_j \text{ for all } 0 \leq t \leq j$$

which yields

$$\frac{b}{j} a_j \leq x_0 \leq \frac{b}{j} b_j.$$

By induction hypothesis, we have $b_j = a_j + 1$ (if $l_{b-j+1} = l_j$) or $b_j = a_j$ (if $l_{b-j+1} = l_0$ and $l_j = l_0 + 1$). Note that the case $l_{b-j+1} = l_0 + 1$ and $l_j = l_0$ is not possible since then $w_{n-jl_0-b_j} + w_{jl_0+b_j} = b - j + 1 + j + 1 = 2$ contradicting Property 3.13 2). If $b_j = a_j$ then $x_0 = \frac{b}{j} a_j$ and $\frac{x_0 j}{b}$ is an integer showing that $j \geq i$. If $b_j = a_j + 1$ then $l_{b-j+1} = l_j$ and

$$\frac{jx_0}{b} - 1 \leq a_j \leq \frac{jx_0}{b}$$

which give $a_j = \lfloor \frac{jx_0}{b} \rfloor$ as claimed, since $j \leq i-1$ and hence $\frac{jx_0}{b}$ is not an integer.

It remains to show that if $j = i$, i.e. $\frac{b}{j} = \gcd(b, x_0) = \gcd(b, n)$, then $a_j = a_i = \frac{ix_0}{b}$ and e and Λ_N are as claimed. For this it is enough to show that $a_i = b_i$, since then every subsequence of e of length i contains exactly a_i times 1. Applying this to $(e_1,\dots,e_i)$ and $(e_2,\dots,e_{i+1})$ this shows that $e_{i+1} = e_1$. Repeating it follows that $e = (e_1,\dots,e_i,e_1,\dots,e_i,\dots e_1,\dots,e_i)$ as claimed.

Assume that $a_i \neq b_i$. Then $b_i = a_i + 1$ and either $a_i = \frac{x_0}{d}$ and $b_i = \frac{x_0}{d} + 1$ or $a_i = \frac{x_0}{d} - 1$ and $b_i = \frac{x_0}{d}$ (where $d := \frac{b}{i} = \gcd(n,b)$). Assume the latter, then

$$x_0 = \sum_{j=1}^{b} e_j = \sum_{k=0}^{d-1} \sum_{j=ki+1}^{ki+i} e_j \leq d b_i = x_0.$$

Hence for all k the sum $\sum_{j=ki+1}^{ki+i} e_j = b_i$, in particular $a_i = \sum_{j=1}^{i} e_j = b_i$. In the other case one argues similarly using a_i instead of b_i. $\square$

Theorem 3.15. *The head order of B is*

$$B_N = \bigoplus_{s=1}^{a+\tilde{e}} \Delta_s$$

where $\Delta_s = B\tilde{\epsilon}_s$ for $s = 1, \ldots, a$.
If $s \in \{a+1, \ldots, a+\tilde{e}\}$ then $\tilde{r}_s = r_s$ and δ and ρ induce permutations on $\tilde{r}_s$. As in Theorem 2.2 let $\sigma := \delta_{|\tilde{r}_s}$ if $s \in T_{even}$ and $\sigma := \rho_{|\tilde{r}_s}$ if $s \in T_{odd}$. Define $d := \gcd(|r_s|, a)$, $t := \frac{|r_s|}{d}$ and $c := (\frac{a}{d})^{-1} \in (\mathbb{Z}/t\mathbb{Z})^*$. Then the order of σ is $|r_s|$ and we define $\tau := \sigma^t$ and $\gamma := \sigma^c$ and choose $i \in \tilde{r}_s$ arbitrarily. Then

$$\Delta_s \cong \Lambda(R, (D_i, D_{\gamma(i)}, \ldots, D_{\gamma^{t-1}(i)}), H_t)$$

where $D_j = \sum_{l=0}^{d-1} d_{\tau^l(j)}$.

Proof. For $1 \le s \le a$ the theorem follows from Corollary 3.3. For $a+1 \le s \le a+\tilde{e}$ let $n := |r_s|$, $a = \mu n + b$ with $0 \le b < n$. If $b = 0$, then $\Delta_s \cong \Lambda_{m_1}$ as defined in Corollary 3.9 is already a maximal order and the theorem follows from Lemma 3.11.

So assume that $1 \le b \le n-1$. Then $d = \gcd(a, n) = \gcd(b, n)$ and we write $n = lb + x$ with $0 \le x < b$ and put $n = n'd,\ b = b'd,\ x = x'd$. Then there is $k \in \mathbb{Z}$ with $cb' = 1 + n'k$ where c is as defined in the theorem. For $j \in \mathbb{Z}$ put

$$f(j) := 1 + \lfloor \frac{j - 1 - \lfloor \frac{xj}{n} \rfloor}{l} \rfloor = 1 + \lfloor \frac{-1 - \lfloor \frac{(x'-n')j}{n'} \rfloor}{l} \rfloor.$$

Since $x' - n' = -b'l$ is divisible by l, one finds that

$$f(j + n') = f(j) + b' \text{ for all } j \in \mathbb{Z}.$$

Let

$$\Lambda := \Lambda(f(0), \ldots, f(n-1)) = \Lambda(R, d, M) \text{ where } m_{ij} = f(j-i).$$

We claim that $\Lambda = \Delta_s$. By Lemma 3.14 it is enough to show that Λ is a hereditary order that has property 3.13 3'). The latter is checked by a straightforward calculation. We show that Λ is hereditary, by establishing an isomorphism with the hereditary order in the theorem.

Put $t_i := m_{i1} = f(1-i)$. Conjugating by the diagonal matrix $T := \mathrm{diag}(\pi^{t_i})$ one obtains the conjugate order

$$\Lambda^T = \Lambda(R, d, \tilde{M}), \text{ where } \tilde{m}_{ij} = m_{ij} - t_i + t_j = f(j-i) - f(i) + f(j).$$

Writing $j = 1 + cj_2 + n'j_1$ and $i = 1 + ci_2 + n'i_1$ with $0 \le j_2, i_2 < n'$ one gets $\tilde{m}_{ij} = f(c(j_2 - i_2)) - f(-ci_2) + f(-cj_2)$

$$= 1 + \lfloor \frac{-1 - \lfloor \frac{-cb'l(j_2-i_2)}{n'} \rfloor}{l} \rfloor - \lfloor \frac{-1 - \lfloor \frac{cb'li_2}{n'} \rfloor}{l} \rfloor + \lfloor \frac{-1 - \lfloor \frac{cb'lj_2}{n'} \rfloor}{l} \rfloor.$$

Since $cb' = 1 - kn'$ one gets

$$\tilde{m}_{ij} = 1 + \lfloor \frac{-1 - \lfloor \frac{l(i_2-j_2)}{n'} \rfloor}{l} \rfloor - \lfloor \frac{-1 - \lfloor \frac{li_2}{n'} \rfloor}{l} \rfloor + \lfloor \frac{-1 - \lfloor \frac{lj_2}{n'} \rfloor}{l} \rfloor.$$

Now $0 \leq i_2 < n'$ implies that $0 \leq \lfloor \frac{li_2}{n'} \rfloor \leq l-1$ and therefore $\lfloor \frac{-1-\lfloor \frac{li_2}{n'} \rfloor}{l} \rfloor = -1$. Similarly $\lfloor \frac{-1-\lfloor \frac{lj_2}{n'} \rfloor}{l} \rfloor = -1$. For the first term we have $1 - n' \leq i_2 - j_2 \leq n' - 1$ implying that

$$\lfloor \frac{-1 - \lfloor \frac{l(i_2-j_2)}{n'} \rfloor}{l} \rfloor \in \{0, -1\}.$$

More precisely this yields

$$\tilde{m_{ij}} = \begin{cases} 0 & \text{if } i_2 \geq j_2 \\ 1 & \text{if } i_2 < j_2. \end{cases}$$

In particular Λ is a hereditary order and hence $\Lambda = \Delta_s$. After a suitable reordering of the constituents the order $\Lambda^T \cong \Delta_s$ has the form as claimed in the theorem. □

Remark 3.16. *Let $s \in \{a+1, \ldots, a+\tilde{e}\}$ and $\Delta_s := \tilde{\epsilon}_s B_N$. Let $n := |\tilde{r}_s| = n'd$, $d = \gcd(a, n)$, $a = a'd$, and $ca' \equiv 1 \pmod{n'}$. Let $\nu : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n'\mathbb{Z}$ be the natural epimorphism. Assume that the simple $B\tilde{\epsilon}_s$-modules are labelled S_i with $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\sigma(S_i) = S_{i+1}$, where σ is as in Theorem 3.15.*

Then the simple Δ_s-modules are T_j with $j \in \mathbb{Z}/n'\mathbb{Z}$ and can be labelled such that

$$(T_j)_{|B\tilde{\epsilon}_s} = \bigoplus_{i \in \nu^{-1}(cj)} S_i$$

The Δ_s-lattices in the simple $\mathcal{A}\tilde{\epsilon}_s$-module form a chain

$$\ldots \supset L_1 \supset L_2 \supset \ldots \supset L_{n'} \supset pL_1 =: L_{n'+1} \supset \ldots$$

where $L_j/L_{j+1} \cong T_j$ for $j = 1, \ldots, n'$.

It is a general and well known fact that if Λ is an S-order for some discrete valuation ring S and S' is an unramified extension of S then $J(S' \otimes \Lambda) = S' \otimes J(\Lambda)$ and hence also $\mathrm{Id}(J(S' \otimes \Lambda)) = S' \otimes \mathrm{Id}(J(\Lambda))$. Therefore the radical idealiser chain of the S'-order $S' \otimes \Lambda$ is obtained by extension of scalars of all orders in the chain. This immediately implies the following corollary:

Corollary 3.17. *Theorem 3.15 also holds when the block B of $\mathbb{Z}_pG$ is replaced by the block $\mathcal{B}$ of RG from Theorem 2.5.*

This is not true for ramified extensions. However, the calculation of the head order of $\tilde{\Gamma}_0$ above only depends on the special structure of this order. Replacing $\tilde{\Gamma}_0$ by $R \otimes \tilde{\Gamma}_0$ for some ramified extension R of $\mathbb{Z}_p$ yields an R-order with the same structure, where a has to be replaced by the π-adic valuation of p^a where π is a prime element in R.

Remark 3.18. *Replacing R by a ramified extension of $\mathbb{Z}_p$ in Remark 3.16 and a by the π-adic valuation of p^a still yields a description of the head order of the non exceptional vertex Γ_0.*

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