Γ-conjugate weight enumerators and invariant theory

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Abstract. Let K be a field, Γ a finite group of field automorphisms of K, F the Γ-fixed field in K and $G \leq \mathrm{GL}_v(K)$ a finite matrix group. Then the action of Γ defines a grading on the symmetric algebra of the F-space K^v which we use to introduce the notion of homogeneous Γ-conjugate invariants of G. We apply this new grading in invariant theory to broaden the connection between codes and invariant theory by introducing Γ-conjugate complete weight enumerators of codes. The main result of this paper applies the theory from Nebe, Rains, Sloane to show that under certain extra conditions these new weight enumerators generate the ring of Γ-conjugate invariants of the associated Clifford-Weil groups. As an immediate consequence we obtain a result by Bannai etal that the complex conjugate weight enumerators generate the ring of complex conjugate invariants of the complex Clifford group. Also the Schur-Weyl duality conjectured and partly shown by Gross etal can be derived from our main result.
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1. Introduction

A complex conjugate polynomial of degree (N, N) in v variables is a homogeneous polynomial $p \in \mathbb{C}[x_1, \ldots, x_v, \overline{x}_1, \ldots, \overline{x}_v]$ that is of degree N in the variables $x_1, \ldots, x_v$ and of degree N in their complex conjugates. An invariant theory for complex conjugate polynomials has been developed in [3]. Given a finite complex unitary matrix group $G \leq \mathrm{U}_v(\mathbb{C})$ and some $t \in \mathbb{N}$ such that for all $N = 1, \ldots, t$ all complex conjugate invariants of degree (N, N) of G are multiples of the Nth power of the invariant Hermitian form, then all G-orbits on $\mathbb{C}^v$ define projective t-designs. More generally if $x \in \mathbb{C}^v$ is a zero of all harmonic G-invariant polynomials up to degree (t,t), then the G-orbit xG gives rise to a projective t-design.

The textbook [9] gives a very general notion of a Type of a code. Associated to a Type ρ and an integer $m \geq 1$ there is a finite complex matrix group $\mathcal{C}_m(\rho)$, the associated Clifford-Weil group of genus m, such that the genus-m complete weight enumerator of any code of Type ρ and length N is an invariant polynomial of $\mathcal{C}_m(\rho)$, homogeneous of degree N. The Weight Enumerator Conjecture states that the space of homogeneous degree N invariants of $\mathcal{C}_m(\rho)$ is spanned by these weight enumerators. If v is the size of the alphabet of the codes of Type ρ, then $\mathcal{C}_m(\rho)$ consists of matrices of size v^m. Despite of this exponentially growing dimension, the space of invariants of a given degree N can be obtained by enumerating all codes of length N of Type ρ.

For the Type of doubly even binary codes, where $v = 2$, the associated Clifford-Weil groups are the complex Clifford groups $\mathcal{X}_m \leq \mathrm{GL}_{2^m}(\mathbb{C})$ which have a tight connection to quantum information theory (see [11]). The main result of [1] shows that the ring of complex conjugate invariants of $\mathcal{X}_m$ is spanned by the genus-m complex conjugate weight enumerators of self-dual doubly even binary codes. The paper also enumerates all such codes up to length $(5,5)$ therewith proving a conjecture from [11] that whenever an $\mathcal{X}_m$-orbit forms a projective 4-design then it is automatically a projective 5-design.

We extend the approach in [1] to the more general set-up in [9]. Any Type ρ also determines an abelian number field K such that the associated Clifford-Weil groups consist of matrices over K. Given a subgroup Γ of the automorphism group of K, we introduce the concept of Γ-conjugate weight enumerators of codes of Type ρ. These are elements of the ring of Γ-conjugate invariants (Definition 2.1) of $\mathcal{C}_m(\rho)$. Our main result, Theorem 6.3, shows that the space of Γ-conjugate homogeneous invariants of $\mathcal{C}_m(\rho)$ is spanned by these weight enumerators of codes of a given length. Therefore the enumeration of such codes of small length determines the Γ-conjugate invariants of small degree of the genus-m Clifford-Weil group associated to ρ for all $m \in \mathbb{N}$.

With a view to possible applications to measurement schemes for low rank matrix recovery from complex projective t-designs as in [6], Section 7 gives a few examples of self-dual codes of Type (N,N) for $N \leq 4$ for several small representations ρ. Section 8 contains, as another application of Theorem 6.3, a short proof of the Schur-Weyl duality conjectured and partly established in [4].

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2. Conjugate polynomials

Let K be a field and let $\Gamma := \{\alpha_1, \ldots, \alpha_n\} \leq \mathrm{Aut}(K)$ be a group of field automorphisms of K that is finite of order n. Put

$$F := \{a \in K \mid \alpha_i(a) = a \text{ for all } i = 1, \ldots, n\}$$

to denote the fixed field of Γ. Then K/F is a Galois extension with Galois group Γ.

For a vector space U over K of finite dimension v we denote by $K[U]$ the ring of polynomials on U. We now restrict scalars and write U_F for U regarded as an nv-dimensional space over F. For a K-basis $(u_1, \ldots, u_v)$ of U and an F-basis $(b_1, \ldots, b_n)$ of K the tuple $B := (b_j u_i : 1 \leq j \leq n, 1 \leq i \leq v)$ is an F-basis of U_F. This yields an F-algebra isomorphism between the symmetric algebra $F[U_F]$ of the dual space U_F^* and $F[y_{ji} : 1 \leq j \leq n, 1 \leq i \leq v]$, where the functions $y_{ji} : U_F \to F$ form the dual basis to the previously chosen basis B. The latter is a polynomial ring in nv variables over F with well studied gradings given by multi-degrees. As we are mostly interested in invariant theory of finite groups over fields of characteristic 0 we assume that K is an infinite field and see elements of $K[U] \cong K[x_1, \ldots, x_v]$ as polynomial functions on $U \cong K^v$.

Definition 2.1. *For $v \in \mathbb{N}$ we denote by $\underline{v} := \{1, \ldots, v\}$ and by K^{K^v} the K-algebra of K-valued functions on K^v. Assume that K is an infinite field. The* ring of Γ-conjugate polynomials over K in v variables $\mathbf{x} := (x_1, \ldots, x_v)$ *is denoted by*

$$K[\mathbf{x} \circ \Gamma] := K[x_i \circ \alpha_j : 1 \leq i \leq v, 1 \leq j \leq n] \leq K^{K^v}$$

and defined as the K-subspace spanned by the monomial functions $M((m_{ij} \mid (i,j) \in \underline{v} \times \underline{n})) \in K^{K^v}$, where

$$M((m_{ij} \mid (i,j) \in \underline{v}\times\underline{n})) := \prod_{j=1}^{n}\prod_{i=1}^{v}(x_i \circ \alpha_j)^{m_{ij}} : (k_1, \ldots, k_v) \mapsto \prod_{j=1}^{n}\prod_{i=1}^{v}(\alpha_j(k_i))^{m_{ij}}.$$

The degree of such a monomial is

$$\deg(M((m_{ij} \mid (i,j) \in \underline{v} \times \underline{n})) := (d_1, \ldots, d_n)$$

with $d_j = \sum_{i=1}^{v} m_{ij}$ for all $j \in \underline{n}$.

The K-algebra structure of $K[\mathbf{x} \circ \Gamma]$ is inherited from the K-algebra structure of K^{K^v}. In particular the multiplication of two monomials is given by

$$M((\ell_{ij} \mid (i,j) \in \underline{v}\times\underline{n}))M((m_{ij} \mid (i,j) \in \underline{v}\times\underline{n})) = M((\ell_{ij}+m_{ij} \mid (i,j) \in \underline{v}\times\underline{n}))$$

and as usual the degree of the product is just the sum of the degrees of the two factors.

The Γ-action provides a finer notion of degree of a polynomial in $K[U_F] := K \otimes_F F[U_F]$. Note that also the functions $(x_i \circ \alpha_j \mid (i,j) \in \underline{v} \times \underline{n})$ form a basis of the dual space $K \otimes_F U_F^*$. So we obtain a K-algebra isomorphism

$$\varphi : K[\mathbf{x} \circ \Gamma] \to K[y_{ji} \mid (i,j) \in \underline{v} \times \underline{n}]. \tag{1}$$

This shows that the linear functions $(x_i \circ \alpha_j : (i,j) \in \underline{v} \times \underline{n})$ in K^{K^v} are algebraically independent over K.

Corollary 2.2. *The space of homogeneous polynomials of degree* $(d_1, \dots, d_n)$ *in* $K[\mathbf{x} \circ \Gamma]$ *is the span of all monomials* $M((m_{ij} \mid (i,j) \in \underline{v} \times \underline{n}))$ *with* $d_j = \sum_{i=1}^{v} m_{ij}$. *As these monomials form a basis its dimension is*

$$\dim(K[\mathbf{x} \circ \Gamma]_{d_1,\dots,d_n}) = \prod_{j=1}^{n} \dim(K[x_1, \dots, x_v]_{d_j}) = \prod_{j=1}^{n} \binom{d_j + v - 1}{d_j}.$$

3. Invariant Theory

We keep the assumptions of the previous section, in particular K is an infinite field and $\Gamma = \{\alpha_1, \dots, \alpha_n\}$ is a group of automorphisms of K of finite order n. Let $G \leq \mathrm{GL}_v(K)$ be a group. Then the right action of G on K^v defines a right action of G by K-algebra automorphisms on the K-algebra of K-valued functions on K^v by

$$f \cdot g : K^v \to K, k \mapsto f(kg^{-1}) \text{ for all } g \in G, f \in K^{K^v}.$$

This action preserves the subalgebra $K[\mathbf{x} \circ \Gamma]$ as well as its subspaces of homogeneous polynomials of a given degree.

Definition 3.1. *Let* $K[\mathbf{x} \circ \Gamma]^G$ *denote the* K*-algebra of* G*-invariant functions in* $K[\mathbf{x} \circ \Gamma]$.

Via the isomorphism φ from Equation (1) the K-algebra of G-invariant functions $K[\mathbf{x} \circ \Gamma]^G$ is isomorphic to the ring of G-invariant polynomials in $K[y_{ji} \mid (i,j) \in \underline{v} \times \underline{n}]$. In particular classical invariant theory gives us Molien's formula for the Hilbert series of this invariant ring.

The grading from Definition 2.1 refines the classical degree function thus giving a notion of Γ-conjugate Hilbert series of the invariant ring $K[\mathbf{x} \circ \Gamma]^G$, generalising the Forger series (where $K = \mathbb{C}$ and $F = \mathbb{R}$) [3] to our situation.

Definition 3.2. *For any* $\mathbf{d} := (d_1, \dots, d_n) \in \mathbb{N}_0^n$ *let*

$$a_{\mathbf{d}} := \dim(K[\mathbf{x} \circ \Gamma]_{\mathbf{d}} \cap K[\mathbf{x} \circ \Gamma]^G)$$

denote the dimension of the space of G*-invariant conjugate polynomials that are homogeneous of degree* $\mathbf{d}$. *Putting* $\mathbf{z}^{\mathbf{d}} := z_1^{d_1} \dots z_n^{d_n}$ *we define*

$$\mathcal{H}(K[\mathbf{x} \circ \Gamma]^G) := \sum_{\mathbf{d} \in \mathbb{N}_0^n} a_{\mathbf{d}} \mathbf{z}^{\mathbf{d}} \in \mathbb{Z}[[z_1, \dots, z_n]]$$

the Γ-conjugate Hilbert series *of the ring* $K[\mathbf{x} \circ \Gamma]^G$.

3.1. Molien's theorem

If G is finite and K has characteristic 0, then Molien's theorem [2, Theorem 2.5.2] gives a useful expression of the Hilbert series of the classical invariant ring of G. With a completely analogous proof (see also [3] for $K = \mathbb{C}$ and $F = \mathbb{R}$) we obtain the following theorem.

Theorem 3.3. *Let K be a field of characteristic 0 and let G be a finite subgroup of $\mathrm{GL}_v(K)$. Then*

$$\mathcal{H}(K[\mathbf{x} \circ \Gamma]^G) = \frac{1}{|G|} \sum_{g \in G} \prod_{j=1}^{n} \frac{1}{\det(I_v - z_j \alpha_j(g))}.$$

Proof. For a given degree $\mathbf{d} \in \mathbb{N}_0^n$ the Reynolds operator $\frac{1}{|G|} \sum_{g \in G} g \in K[G]$ induces a K-linear projection $P_{\mathbf{d}}$ from $K[\mathbf{x} \circ \Gamma]_{\mathbf{d}}$ onto the fixed space $K[\mathbf{x} \circ \Gamma]_{\mathbf{d}}^G$. Since $\mathrm{char}(K) = 0$ the dimension of this fixed subspace is equal to the trace of $P_{\mathbf{d}}$.
Therefore

$$\mathcal{H}(K[\mathbf{x} \circ \Gamma]^G) = \sum_{\mathbf{d} \in \mathbb{N}_0^n} \mathrm{trace}(P_{\mathbf{d}}) \mathbf{z}^{\mathbf{d}}.$$

As $\mathrm{trace}(P_{\mathbf{d}}) = \frac{1}{|G|} \sum_{g \in G} t_{\mathbf{d}}(g)$ it suffices to compute the trace $t_{\mathbf{d}}(g)$ of the action of g on $K[\mathbf{x} \circ \Gamma]_{\mathbf{d}}$ for all $g \in G$.
To do so we may and will assume that K contains a $|G|$th primitive root of unity. Then each $g \in G$ is diagonalizable over K so after a suitable choice of basis we assume that $g^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_v)$. Then for $\alpha_j \in \Gamma$,

$$\det(I_v - z_j \alpha_j(g)) = \prod_{i=1}^{v} (1 - \alpha_j(\lambda_i) z_j).$$

The monomials from Definition 2.1 form an eigenvector basis for the action of g on $K[\mathbf{x} \circ \Gamma]_{\mathbf{d}}$, where $M((m_{ij} \mid (i,j) \in \underline{v} \times \underline{n}))$ has eigenvalue $\prod_{j=1}^{n} \prod_{i=1}^{v} \alpha_j(\lambda_i)^{m_{ij}}$.
So we get

$$\sum_{\mathbf{d} \in \mathbb{N}_0^n} t_{\mathbf{d}}(g) \mathbf{z}^{\mathbf{d}} = \prod_{j=1}^{n} \prod_{i=1}^{v} \sum_{m \in \mathbb{N}_0} (\alpha_j(\lambda_i) z_j)^m =$$

$$\prod_{j=1}^{n} \prod_{i=1}^{v} \frac{1}{1 - \alpha_j(\lambda_i) z_j} = \prod_{j=1}^{n} \frac{1}{\det(I_v - z_j \alpha_j(g))}.$$

□

4. The Type of a self-dual code

This section briefly recalls the relevant notions from [9]. Classically a self-dual code C of length N over a finite field $\mathbb{F}_q$ is a linear subspace $C \leq \mathbb{F}_q^N$ that is self-dual (i.e. $C = C^\perp$) with respect to the standard inner product. Loosely speaking, to define self-dual codes in a more general sense we need a ring R, a left R-module V and a non-singular form β on V such that β can be used to define the orthogonal code $C^\perp$ of an R-submodule $C \leq V^N$. Extra conditions (such as being doubly even) can be imposed by means of isotropy conditions with respect to some set Q of quadratic maps on V. We call such an admissible quadruple $\rho := (R, V, Q, \beta)$ a Type, and the self-dual isotropic codes that arise from ρ are called codes of Type ρ. More precisely, a Type is a representation of an abstract form ring.

4.1. Form rings

A form ring (R, M, ψ, Φ) is a quadruple, where R is a (unital and associative) ring, M a right $R \otimes R$-module, $\psi : R_R \to M_{1\otimes R}$ an isomorphism of right R-modules. It comes with an involution $\tau : M \to M$ such that $\tau(m)(s \otimes r) = \tau(m(r \otimes s))$ for all $m \in M$, $r, s \in R$ and such that $\epsilon := \psi^{-1}(\tau(\psi(1)))$ is a unit in R. The isomorphism ψ defines an anti automorphism $\,^J : R \to R$ by $r^J := \psi^{-1}(\psi(1)(r \otimes 1))$. The last ingredient is an R-qmodule Φ, i.e. an abelian group Φ together with a (pointed quadratic) map $[] : R \to \mathrm{End}_{\mathbb{Z}}(\Phi)$ such that $[1] = 1$ and $[rs] = [r][s]$ for all $r, s \in R$. There are qmodule homomorphisms $\{\!\{\ \}\!\} : M \to \Phi$ and $\lambda : \Phi \to M$ such that for all $m \in M$, $\phi \in \Phi$,

$$\{\!\{\tau(m)\}\!\} = \{\!\{m\}\!\},\ \tau(\lambda(\phi)) = \lambda(\phi),\ \lambda(\{\!\{m\}\!\}) = m + \tau(m)$$

and

$$\{\!\{\lambda(\phi)(r \otimes s)\}\!\} = \phi[r + s] - \phi[r] - \phi[s] \text{ for all } r, s \in R, \phi \in \Phi.$$

Taking $M = R$ and ψ the identity we abbreviate $(R, \Phi) := (R, M, \psi, \Phi)$.

4.2. Finite representations of form rings

A finite representation of a form ring (R, Φ) is a quadruple $\rho = (V, \rho_M, \rho_\Phi, \beta)$, where V is a left R-module of finite cardinality v, $\rho_M : M \to \mathrm{Bil}(V)$ is an $R \otimes R$-module homomorphism into the group $\mathrm{Bil}(V)$ of bi-additive $\mathbb{Q}/\mathbb{Z}$-valued maps on V compatible with the involution τ, i.e. $\rho_M(\tau(m))(x, y) = \rho_M(m)(y, x)$ for all $m \in M, x, y \in V$, and such that $\beta := \rho_M(\psi(1))$ is non-singular. Also ρ_Φ is an R-qmodule homomorphism from Φ into the group of $\mathbb{Q}/\mathbb{Z}$-valued quadratic maps on V satisfying

$$\rho_\Phi(\{\!\{m\}\!\})(x) = \rho_M(m)(x, x) \text{ and}$$
$$\rho_M(\lambda(\phi)) = \rho_\Phi(\phi)(x + y) - \rho_\Phi(\phi)(x) - \rho_\Phi(\phi)(y)$$

for all $x, y \in V, m \in M, \phi \in \Phi$.

Definition 4.1. *Let $\rho = (V, \rho_M, \rho_\Phi, \beta)$ be a finite representation of a form ring (R, Φ). Any R-submodule $C \leq V$ is called a* code in *ρ. For a code C in ρ the orthogonal module is*

$$C^\perp = C^{\perp,\beta} := \{x \in V \mid \beta(x, c) = 0 \textit{ for all } c \in C\}.$$

The code C is called self-dual *if $C = C^\perp$ and* self-orthogonal *if $C \subseteq C^\perp$. A self-orthogonal code C in ρ is called* isotropic *if $\rho_\Phi(\phi)(C) = \{0\}$ for all $\phi \in \Phi$.*

A code of Type *ρ is a self-dual isotropic code in ρ.*

MacWilliams transformations ([9, Section 2.2]) map the weight enumerator of a code to the one of its dual and fix weight enumerators of self-dual codes. To obtain a generating set of the associated Clifford-Weil group we need to include MacWilliams transformations for representatives of the conjugacy classes of primitive symmetric idempotents.

Definition 4.2. *An idempotent $\iota^2 = \iota \in R$ is called* symmetric *if $\iota R \cong \iota^J R$ as right R-modules. Such an isomorphism is given by left multiplication with some $v_\iota \in \iota^J R \iota$ with inverse $u_\iota \in \iota R \iota^J$ such that $u_\iota v_\iota = \iota$ and $v_\iota u_\iota = \iota^J$ (see* [9, Section 3.5.3]*).*

Remark 4.3. (see [9, Theorem 3.5.9]) Let $\iota \in R$ be a symmetric idempotent and $C = C^\perp \leq V$ a self-dual code. Then ιC is a self-dual code in ιV and in particular $|\iota C|^2 = |\iota V|$.

Definition 4.4. *The* value group *of the representation $\rho = (V, \rho_M, \rho_\Phi, \beta)$ is the subgroup $\nu(\rho)$ of $\mathbb{Q}/\mathbb{Z}$ generated by*

$$\{\beta(x,y) \mid x,y \in V\} \cup \{\rho_\Phi(\phi)(x) \mid x \in V, \phi \in \Phi\}.$$

As $\nu(\rho)$ is a finitely generated (and hence finite) subgroup of $\mathbb{Q}/\mathbb{Z}$ it is cyclic, so $\nu(\rho) = \langle \frac{1}{f} + \mathbb{Z} \rangle$, where $f = f(\rho) = |\nu(\rho)|$ is called the conductor *of ρ.*

Remark 4.5. Put $f := f(\rho)$ and let $a_1, \ldots, a_N \in \mathbb{Z}$ be prime to f and put $\mathbf{a} := (a_1, \ldots, a_N)$. Then the orthogonal sum

$$\rho^{\mathbf{a}} := (V^N, a_1\rho_M \perp \ldots \perp a_N\rho_M, a_1\rho_\Phi \perp \ldots \perp a_N\rho_\Phi, a_1\beta \perp \ldots \perp a_N\beta)$$

is a finite representation of the form ring (R, Φ).

5. Clifford-Weil groups and full weight enumerators

Let $\rho = (V, \rho_M, \rho_\Phi, \beta)$ be a finite representation of a form ring and put $v := |V|$. The group algebra $\mathbb{C}V$ is a v-dimensional complex vector space with basis $(b_w : w \in V)$. The full weight enumerator of a code in ρ is defined as

$$\mathrm{fwe}(C) := \sum_{c \in C} b_c \in \mathbb{C}V.$$

The associated Clifford-Weil group $\mathcal{C}(\rho)$ is a group of linear operators on $\mathbb{C}V$ whose generators are explicitly given in [9, Definition 5.3.1]; these generators are the obvious transformations that stabilise $\mathrm{fwe}(C)$ for any code of Type ρ. In particular the full weight enumerators of codes of Type ρ are invariant under $\mathcal{C}(\rho)$ (see [9, Theorem 5.5.1]).

Remark 5.1. The Weight Enumerator Conjecture [9, Conjecture 5.5.2] states that in this general situation the fixed space of $\mathcal{C}(\rho)$ is spanned by the full weight enumerators of codes of Type ρ.

In fact we do not know a counterexample and [9, Theorem 5.5.5 and Theorem 5.5.7] assert the truth of the Weight Enumerator Conjecture for fairly large classes of finite form rings including matrix rings over finite fields.

We recall the action of the associated Clifford-Weil group for the representation $\rho^{\mathbf{a}}$ from Remark 4.5 with respect to the $\mathbb{C}$-basis $(b_w : w = (w_1, \ldots, w_N) \in V^N)$ of the group algebra of V^N:

$$\mathcal{C}(\rho^{\mathbf{a}}) = \langle m_r, d_\phi, h_{\iota,u_\iota,v_\iota} : r \in R^\times, \phi \in \Phi, \iota = u_\iota v_\iota \text{ sym. idem. } \rangle$$

where

$$m_r : b_w \mapsto b_{rw}, \qquad d_\phi : b_w \mapsto \prod_{j=1}^{N} \exp(2\pi i \rho_\Phi(\phi)(w_j))^{a_j} b_w$$

and

$$h_{\iota,u_\iota,v_\iota} : b_w \mapsto \frac{1}{|\iota V|^{N/2}} \sum_{u \in \iota V} \prod_{j=1}^{N} \exp(2\pi i \beta(u_j, v_\iota w_j))^{a_j} b_{u+(1-\iota)w}.$$

Here the ι runs through the set of all $R^\times$-conjugacy classes of symmetric idempotents in R and $u_\iota, v_\iota \in R$ are as in Definition 4.2.

So the transformations m_r are represented as permutation matrices on the chosen basis and the transformations d_ϕ as diagonal matrices. It is clear that fwe(C) is invariant under all d_ϕ and all m_r, if C is an isotropic code in $\rho^{\mathbf{a}}$. For self-dual codes the invariance under $h_{\iota,u_\iota,v_\iota}$ follows from a general MacWilliams theorem, see [9, Example 2.2.6].

Definition 5.2. *Let* $f := f(\rho)$ *and let* $\zeta_f := \exp(\frac{2\pi i}{f}) \in \mathbb{C}$. *Put* $F_1 := \mathbb{Q}(\{\sqrt{|\iota V|} : \iota \in R \text{ sym. idem.}\})$ *and* $K(\rho) := \mathbb{Q}(\zeta_f)F_1$ *the abelian number field containing all entries of* $\mathcal{C}(\rho)$. *We choose a complement* F *of* $F_1 \cap \mathbb{Q}(\zeta_f)$ *in* F_1. *For* $a \in (\mathbb{Z}/f\mathbb{Z})^\times$ *put* γ_a *to denote the Galois automorphism of* $K(\rho)$ *that is the identity on* F *and raises* ζ_f *to the ath power. Then* $\gamma : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathrm{Gal}(K(\rho)/F), a \mapsto \gamma_a$ *is a group isomorphism.*

Remark 5.3. Let $a \in (\mathbb{Z}/f\mathbb{Z})^\times$. For a symmetric idempotent $\iota \in R$ we put $\epsilon(\iota, a) \in \{1, -1\}$ such that $\gamma_a(\sqrt{|\iota V|}) = \epsilon(\iota, a)\sqrt{|\iota V|}$. Then

$$\mathcal{C}(\rho^{(a)}) = \langle m_r, \gamma_a(d_\phi), \epsilon(\iota, a)\gamma_a(h_{\iota,u_\iota,v_\iota}) \mid r \in R^\times, \phi \in \Phi, \iota \text{ sym. idem. } \rangle.$$

By [9, Theorem 5.5.3] the order of the scalar subgroup in $\mathcal{C}(\rho)$ is the greatest common divisor of the lengths of self-dual isotropic codes in ρ. Combining this with Remark 4.3 we obtain the following lemma.

Lemma 5.4. *Assume that there is a symmetric idempotent* $\iota \in R$ *for which* $|\iota V|$ *is not a square in* $\mathbb{Q}$. *Then* $-I_v \in \mathcal{C}(\rho)$.

So for all $a \in \mathbb{Z}$ prime to f the groups $\mathcal{C}(\rho^{(a)})$ and $\gamma_a(\mathcal{C}(\rho))$ are equal.

6. Conjugate weight enumerators and the main theorem

We keep the notation of the previous section, in particular ρ is a finite representation of a form ring (R, Φ) and $K := K(\rho)$ is an abelian extension of the field F from Definition 5.2. Put $f := f(\rho)$ and

$$\Gamma := \mathrm{Gal}(K/F) =: \{\alpha_1, \ldots, \alpha_n\},$$

a finite group of order n isomorphic to $(\mathbb{Z}/f\mathbb{Z})^\times$. We always assume that there is some length N_0 such that there is a self-dual isotropic code in ρ^{N_0}. Then $\mathcal{C}(\rho)$ is a finite subgroup of $\mathrm{GL}_v(K)$, where $v = |V|$. We also assume that the Weight Enumerator Conjecture 5.1 holds for the form ring (R, Φ). For $\mathbf{a} = (a_1, \ldots, a_N)$ as in Remark 4.5 we consider the Clifford-Weil groups $\mathcal{C}(\rho^{\mathbf{a}})$ introduced in Section 5.

Definition 6.1. *We say that* $\mathbf{a}$ *satisfies the* sign condition *if*

$$\prod_{i=1}^{N} \epsilon(\iota, a_i) = 1$$

for all symmetric idempotents $\iota \in R$.

For $1 \leq j \leq n$ we put

$$D_j := \{i \in \{1, \ldots, N\} \mid \gamma_{a_i} = \alpha_j\} \text{ and } d_j := |D_j| \in \mathbb{N}_0.$$

Put $\mathbf{d} := (d_1, \ldots, d_n)$.

Definition 6.2. *The* Γ-conjugate complete weight enumerator

$$\mathrm{ccwe}(C) := \mathrm{ccwe}^{\mathbf{a}}(C) \in K[\mathbf{x} \circ \Gamma]_{\mathbf{d}}$$

of a code C *in* $\rho^{\mathbf{a}}$ *is defined as*

$$\mathrm{ccwe}(C) := \sum_{c \in C} \prod_{j=1}^{n} \prod_{i \in D_j} x_{c_i} \circ \alpha_j.$$

Two codes C, D *in* $\rho^{\mathbf{a}}$ *are called* equivalent *if there are permutations* π_j *of* D_j *(*$1 \leq j \leq n$*) such that* $D = C \circ \pi$ *where* $\pi = \pi_1 \times \ldots \times \pi_n$.

We are now in the position to state and prove the main result of this paper.

Theorem 6.3. *Assume that* ρ *satisfies the Weight Enumerator Conjecture* 5.1 *and that* $\mathbf{a}$ *satisfies the sign condition. Then the space of invariants of* $\mathcal{C}(\rho)$ *in* $K[\mathbf{x} \circ \Gamma]_{\mathbf{d}}$ *is spanned by the* Γ*-conjugate complete weight enumerators of self-dual isotropic codes in* $\rho^{\mathbf{a}}$.

Proof. Define a K-linear map

$$\sigma : KV^N \to K[\mathbf{x} \circ \Gamma]_{\mathbf{d}}, (v_1, \ldots, v_N) \mapsto \prod_{j=1}^{n} \prod_{i \in D_j} x_{v_i} \circ \alpha_j.$$

If $\prod_{i=1}^{N} \epsilon(\iota, a_i) = 1$ for all symmetric idempotents $\iota \in R$ then σ is a $\mathcal{C}(\rho)$-module epimorphism. As $\mathcal{C}(\rho)$ is a finite group and $\mathrm{char}(K) = 0$, both modules are semisimple. In particular the space of invariants of $\mathcal{C}(\rho)$ in $K[\mathbf{x} \circ \Gamma]_{\mathbf{d}}$

is the image under σ of the $\mathcal{C}(\rho)$-fixed space in KV^N. By the Weight Enumerator Conjecture 5.1 this fixed space is spanned by the full weight enumerators of self-dual isotropic codes C in $\rho^{\mathbf{a}}$. Clearly $\sigma(\mathrm{fwe}(C)) = \mathrm{ccwe}(C)$, so the Γ-conjugate complete weight enumerators of these codes span the space $K[\mathbf{x}\circ\Gamma]_{\mathbf{d}}^{\mathcal{C}(\rho)}$ of homogeneous Γ-conjugate $\mathcal{C}(\rho)$-invariant polynomials of degree $\mathbf{d}$. □

Remark 6.4. The sign condition in Theorem 6.3 is necessary, as the following easy example shows: Take $R = \mathbb{F}_5$, the field with 5 elements. Then $J = \mathrm{id}$. We take $\Phi = \{\!\{M\}\!\}$. We specify the representation ρ by putting $V := \mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$ and $\beta(x, y) := xy/5 \in \mathbb{Q}/\mathbb{Z}$. Then $K(\rho) = \mathbb{Q}(\zeta_5)$ and $F = \mathbb{Q}$. Consider the representation $\rho^{(1,2,2,2)}$. As $\epsilon(1, 1) = 1$ and $\epsilon(1, 2) = -1$, the sign condition is not satisfied. Put $\Sigma := 4b_{(0,0,0,0)} - \sum_{v\in I} b_v$, where

$$I := \{0 \neq (v_1, v_2, v_3, v_4) \in \mathbb{F}_5^4 \mid v_1^2 + 2(v_2^2 + v_3^2 + v_4^2) = 0\}$$

is the set of isotropic vectors in $\rho^{(1,2,2,2)}$. Then $m_r(\Sigma) = d_\phi(\Sigma) = \Sigma$ for all $r \in \mathbb{F}_5^\times$ and $\phi \in \Phi$ but $h_{1,1,1}(\Sigma) = -\Sigma$. The map σ from the proof of Theorem 6.3 hence maps Σ to $0 \neq \sigma(\Sigma) \in K[\mathbf{x} \circ \Gamma]^{\mathcal{C}(\rho)}_{1,3,0,0}$. This invariant space is of dimension 1. However, as the discriminant of the quadratic form $x_1^2 + 2(x_2^2 + x_3^2 + x_4^2)$ is not a square in $\mathbb{F}_5$, there are no self-dual isotropic codes in $\rho^{(1,2,2,2)}$.

The same method gives $h_{1,1,1}$-anti-invariants for primes $p \equiv_4 1$.

6.1. Higher genus Clifford-Weil groups

The Clifford-Weil groups associated to a representation ρ form an infinite sequence of matrix groups $\mathcal{C}_m(\rho)$ in exponentially growing dimension v^m for $m \in \mathbb{N}$ that comes with ring epimorphisms $\Phi_m : \mathrm{Inv}(\mathcal{C}_m(\rho)) \to \mathrm{Inv}(\mathcal{C}_{m-1}(\rho))$ that are isomorphisms on the spaces of invariants of small degree. A similar phenomenon occurs for the space of Γ-conjugate invariants.

The full genus-m weight enumerator $\mathrm{fwe}_m(C)$ of C in ρ is the sum over all m-tuples of code words of C. For $C \leq V^N$ we can think of such an m-tuple $c \in C^m$ as a matrix $c \in V^{m\times N}$, where the rows correspond to code words. For $1 \leq i \leq N$ the ith column c_i of this matrix is an element of $V^m = R^m \otimes V$. In the notation of Definition 6.2 we denote by

$$\mathrm{ccwe}_m(C) := \sum_{c\in C^m} \prod_{j=1}^{n} \prod_{i\in D_j} x_{c_i} \circ \alpha_j$$

the Γ-conjugate complete weight enumerator of genus m of the code C in $\rho^{\mathbf{a}}$. Now the alphabet of C^m is V^m and hence a module over the ring $\mathrm{Mat}_m(R)$ of $m \times m$-matrices over R. This gives rise to a notion of a matrix ring of a form ring (see [9, Section 1.10]) such that $R^m \otimes \rho$ is a representation of this matrix ring. The genus-m Clifford-Weil group is

$$\mathcal{C}_m(\rho) := \mathcal{C}(R^m \otimes \rho) \text{ (see [9, Definition 5.3.4]).}$$

Denote by $K[\mathbf{x}^{(\mathbf{m})}]$ the symmetric algebra of V^m. Theorem 6.3 has the following immediate consequence.

Corollary 6.5. *Assume that ρ satisfies the Weight Enumerator Conjecture* 5.1 *and that* $\mathbf{a}$ *satisfies the sign condition. Then the space of invariants of* $\mathcal{C}_m(\rho)$ *in* $K[\mathbf{x}^{(\mathbf{m})} \circ \Gamma]_{\mathbf{d}}$ *is spanned by the* Γ*-conjugate complete weight enumerators of genus* m *of codes of Type* $\rho^{\mathbf{a}}$*.*

Remark 6.6. The map Φ_m mapping $x_{(v_1,\ldots,v_m)^{tr}}$ to $x_{(v_1,\ldots,v_{m-1})^{tr}}$ if $v_m = 0$ and to 0 if $v_m \neq 0$ defines a ring epimorphism

$$\Phi_m : K[\mathbf{x}^{(\mathbf{m})} \circ \Gamma] \to K[\mathbf{x}^{(\mathbf{m-1})} \circ \Gamma]$$

that preserves the grading from Section 2 and maps $\mathrm{ccwe}_m(C)$ to $\mathrm{ccwe}_{m-1}(C)$ for any code C in $\rho^{\mathbf{a}}$. In particular it induces an epimorphism from the space of invariants of $\mathcal{C}_m(\rho)$ in $K[\mathbf{x}^{(\mathbf{m})} \circ \Gamma]_{\mathbf{d}}$ to the space of invariants of $\mathcal{C}_{m-1}(\rho)$ in $K[\mathbf{x}^{(\mathbf{m-1})} \circ \Gamma]_{\mathbf{d}}$.
If all self-dual isotropic codes in $\rho^{\mathbf{a}}$ are generated by $m-1$ elements then this epimorphism is in fact an isomorphism. In this case two codes are equivalent (see Definition 6.2) if and only if they have the same genus-m weight enumerator and these weight enumerators of the equivalence classes of codes of Type $\rho^{\mathbf{a}}$ form a basis of the space of invariants of degree $\mathbf{d}$.

7. Complex conjugate invariants

In this section we give the most important example where all a_i from Remark 4.5 are ± 1. As γ_{-1} is the complex conjugation $^{-}$ and the field F_1 from Definition 5.2 is totally real, we always have $\epsilon(\iota, -1) = 1$ for all symmetric idempotents $\iota \in R$ and hence the sign condition is fulfilled.

In particular the main result of [1] is an immediate consequence of Theorem 6.3. Whereas [1] only considers doubly even, self-dual binary codes Theorem 6.3 allows to consider complex conjugate invariants for more general Clifford-Weil groups.

In view of possible applications to complex projective designs we are mainly interested in codes of Type $\rho^{(1^N,(-1)^N)}$ which we call Type (N, N) for short. As we want to have as few invariants as possible we consider those representations ρ where the quadratic group $\rho_\Phi(\Phi)$ is as large as possible: For self-dual codes over fields of odd characteristic we focus on self-dual codes that contain the all-ones vector $\mathbf{1} = (1, \ldots, 1)$ and in even characteristic on the generalized doubly even codes.

7.1. Codes of Type (N, N)

7.1.1. The trivial codes. The standard norm $z_m := \|\ \|^2$ on $\mathbb{C}^{v^m}$ is an invariant for the full unitary group $\mathrm{U}_{v^m}(\mathbb{C})$. In fact z_m^N is the genus-m complex conjugate weight enumerator of the code $T_{N,N}$ with generator matrix $(I_N|I_N)$, the trivial code of Type (N, N).

7.1.2. Decomposable codes. For codes C_i of Type (N_i, N_i), $i \in \{1, 2\}$, the orthogonal sum $C_1 \oplus C_2$ is a code of Type (N_1+N_2, N_1+N_2). If $(A_i|B_i)$ is a generator matrix for C_i then $C_1 \oplus C_2$ has generator matrix $\begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \end{pmatrix}$.

Clearly the genus-m weight enumerator of $C_1 \oplus C_2$ is just the product of the genus-m weight enumerators of C_1 and C_2.

In the following we only list representatives of the equivalence classes of indecomposable codes, i.e., codes that cannot be written as a non-trivial orthogonal sum.

7.1.3. The doubling construction. There is another general construction of codes in $\rho^{(N,N)}$ which we call the doubling construction:
Let C be a self-orthogonal isotropic code C in ρ^N and put $D := C^\perp \supseteq C$. Then

$$\mathrm{Double}(C) := \{(d, d+c) \mid d \in D, c \in C\}$$

is a code of Type $\rho^{(N,N)}$. If C is a self-dual isotropic code in ρ^N, i.e. $C = D$, then $\mathrm{Double}(C) = \{(c_1, c_2) \mid c_1, c_2 \in C\}$.

The trivial code $T_{N,N}$ is the double of the zero code, $T_{N,N} = \mathrm{Double}(\langle 0^N \rangle)$.

7.2. Enumeration of equivalence classes of codes

In this section we sketch some methods to enumerate all self-dual isotropic codes in V^N.

7.2.1. The mass formula. In many situations the unitary group (in the examples below the orthogonal group of the quadratic space $\mathbf{1}^\perp/\langle \mathbf{1} \rangle$) acts transitively on the set of codes of Type (N,N); their number t_N is the number of isotropic subspaces in a certain finite geometry. If $C_1, \ldots, C_{h_N}$ represent the $\mathrm{Sym}_N \times \mathrm{Sym}_N$-orbits of codes of Type (N,N) then we obtain the following mass formula:

$$\frac{t_N}{N! \cdot N!} = \sum_{i=1}^{h_N} \frac{1}{|\,\mathrm{Aut}(C_i)|}$$

where $\mathrm{Aut}(C_i)$ is the stabiliser in $\mathrm{Sym}_N \times \mathrm{Sym}_N$ of the code $C_i \leq V^{N+N}$. This formula can be used to check the completeness of a list of pairwise inequivalent codes but also yields a method to enumerate these codes by partitioning the orbit of the unitary group into $\mathrm{Sym}_N \times \mathrm{Sym}_N$-orbits.

7.2.2. Kneser neighbors. A more efficient method to enumerate self-dual codes is the Kneser neighbor method [5] (see also [7] for an application to codes and [10] for a survey).

Starting from one self-dual code C (e.g. $C = T_{N,N}$) one enumerates all $\mathrm{Aut}(C)$-orbits of neighbors, i.e. those equivalence classes of codes D of Type ρ that intersect C in a maximal subcode. Continuing with the neighbors one successively enumerates all codes up to equivalence.

7.2.3. Using projections. Denote by $\pi_1 : V^{N+N} \to V^N$ the projection onto the first N components and by π_2 the projection onto the last N coordinates. For a self-dual isotropic code C of Type (N,N) the kernel of π_2 is the dual of the image of π_1.

There is a shortcut to classify the codes $C \leq \mathbb{F}_q^{N+N}$ of Type (N,N) for which $\pi_1(C) = \mathbb{F}_q^N$. If the involution is trivial then each such code has a generator matrix $(I_N|A)$ with $AA^{tr} = I_N$. Equivalence of codes translates into the action of $\mathrm{Sym}_N \times \mathrm{Sym}_N$ on $\mathrm{O}_N(\mathbb{F}_q) := \{A \in \mathbb{F}_q^{N\times N} \mid AA^{tr} = I_N\}$ by $A \cdot (g,h) := g^{-1}Ah$. Representatives of the Sym_N-double cosets in $\mathrm{O}_N(\mathbb{F}_q)$ hence yield generator matrices for representatives of the equivalence classes of codes. If one only wants to classify codes that contain the all-ones vector of length $2N$, one needs to additionally assume that sum of the rows of A is the all-ones vector $\mathbf{1}$ of length N, i.e. $\mathbf{1}A = \mathbf{1}$. As $\mathbf{1}g = \mathbf{1}$ for all permutation matrices $g \in \mathrm{Sym}_N$ this property is invariant on the Sym_N-double cosets.

7.3. Doubly even binary self-dual codes

This is the most important example and has been considered in [1]. The Type of these codes is denoted by 2_{II} in [9, Section 2.3.2], has conductor 8, and the associated Clifford-Weil group is the complex Clifford group. The indecomposable codes of length (N,N), i.e. of Type $2_{\mathrm{II}}^{(N,N)}$ for $N = 1,2,3,4,5$ are $T_{1,1}$ and $\mathrm{Double}(\langle(1,1,1,1)\rangle)$ which is called $g_{4,4}$ in [1].

7.4. Doubly even Euclidean self-dual codes over $\mathbb{F}_4$

Doubly-even euclidean self-dual codes over fields of characteristic 2 have been studied in [8]. Their Type q_{II}^E is discussed in [9, Theorem 2.3.2]. For $q = 4$ the conductor is 4. Enumerating the equivalence classes of codes in $(4_{\mathrm{II}}^E)^{N,N}$ for $N = 1,2,3,4$ we obtain the following indecomposable ones:

Length	number	indecomposables
(1,1)	1	$T_{1,1}$
(2,2)	1	
(3,3)	2	$\mathrm{Double}(\langle(1,\omega,\omega^2)\rangle)$
(4,4)	5	$\mathrm{Double}(\langle(1,1,1,1)\rangle)$, $\mathrm{Double}(Q_4)$, $C_{4,4}(\mathbb{F}_4)$

Here Q_4 has generator matrix $\begin{pmatrix} 1 & 0 & \omega & \omega^2 \\ 0 & 1 & \omega^2 & \omega \end{pmatrix}$ and the generator matrix of $C_{4,4}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & \omega & \omega^2 \\ 0 & 1 & 0 & 0 & 1 & 1 & \omega^2 & \omega \\ 0 & 0 & 1 & 0 & \omega^2 & \omega & 1 & 1 \\ 0 & 0 & 0 & 1 & \omega & \omega^2 & 1 & 1 \end{pmatrix}.$$

7.5. Ternary codes that contain the all-ones vector

The Type of self-dual ternary codes that contain the all-ones vector is 3_1^E, has conductor 3, and is described in [9, Section 7.4.1]. We obtain the following list of codes for small lengths:

Length	number	indecomposables
(1,1)	1	$T_{1,1}$
(2,2)	1	
(3,3)	2	$\mathrm{Double}(\langle(1,1,1)\rangle)$
(4,4)	3	$\langle I_4 \vert 2J_4 - I_4\rangle$

where the last code has generator matrix $(I_4 \vert 2J_4 - I_4)$ and J_4 is the all-ones matrix.

7.6. Quinary codes that contain the all-ones vector

We consider the analogous Type as in the previous section, where the underlying field is $\mathbb{F}_5$ and the conductor is 5. We obtain the following list for small lengths:

Length	number	indecomposables
(1,1)	1	$T_{1,1}$
(2,2)	1	
(3,3)	2	$\langle I_3 \vert 4J_3 - I_3\rangle$
(4,4)	5	$\langle I_4 \vert 3J_4 - I_4\rangle$, Double($\langle(1,2,3,4)\rangle$), $C_{4,4}(\mathbb{F}_5)$

where $C_{4,4}(\mathbb{F}_5)$ has generator matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 4 & 4 \\ 0 & 1 & 0 & 0 & 3 & 2 & 3 & 3 \\ 0 & 0 & 1 & 0 & 4 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 4 & 3 & 4 & 0 \end{pmatrix}.$$

8. A Schur-Weyl duality for Clifford-Weil groups

Let ρ be a finite representation of a form ring (R,Φ) with underlying left R-module V. Then the associated Clifford-Weil group $G := \mathcal{C}(\rho)$ acts on $\mathbb{C}V$ and by diagonal action on all tensor powers $W_N := \mathbb{C}V^N = \otimes^N \mathbb{C}V$. Let $W_N^* := \mathrm{Hom}(W_N,\mathbb{C})$ denote the dual space of W_N. Then W_N^* is also a $\mathbb{C}G$-module, where the action is given by

$$f \cdot g : w \mapsto f(wg^{-1}) \text{ for all } w \in W_N, g \in G, f \in W_N^*.$$

It is well known that the linear map

$$\varphi : W_N \otimes W_N^* \to \mathrm{End}_{\mathbb{C}}(W_N), \text{ defined by } (w,f) \mapsto (x \mapsto f(x)w)$$

for $w \in W_N$ and $f \in W_N^* = \mathrm{Hom}(W_N,\mathbb{C})$ is an isomorphism of vector spaces. Then φ is an isomorphism of $\mathbb{C}G$-modules where G acts on $\mathrm{End}_{\mathbb{C}}(W_N)$ by conjugation. The commuting algebra $\mathrm{End}_{\mathbb{C}G}(W_N)$ is the fixed space of the G-action on $\mathrm{End}_{\mathbb{C}}(W_N)$. If the Weight Enumerator Conjecture holds for ρ then by Theorem 6.3 this fixed space of $G = \mathcal{C}(\rho)$ is spanned by the images of the full weight enumerators of self-dual isotropic codes in $\rho^{(N,N)}$, so we have the following theorem.

Theorem 8.1. *Assume that the Weight Enumerator Conjecture holds for ρ. Let $C_1,\dots,C_{t_N}$ be a complete list of self-dual isotropic codes in $\rho^{(N,N)}$. Then*

$$(\varphi(\mathrm{fwe}^{(N,N)}(C_1)),\dots,\varphi(\mathrm{fwe}^{(N,N)}(C_{t_N})))$$

is a generating set of $\mathrm{End}_{\mathcal{C}(\rho)}(\mathbb{C}V^N)$

Let $g_N(\rho) \in \mathbb{N}$ be such that any code of Type (N,N) can be generated by $g_N(\rho)$ elements. Whenever $m \geq g_N(\rho)$ the genus-m full weight enumerators of codes of length (N,N) are linearly independent.

Corollary 8.2. *([4, Theorem 4.3]) It $m \geq g_N(\rho)$ then the set*

$$(\varphi(\mathrm{fwe}_m^{(N,N)}(C_1)), \ldots, \varphi(\mathrm{fwe}_m^{(N,N)}(C_{t_N})))$$

is a basis of $\mathrm{End}_{\mathcal{C}_m(\rho)}(\mathbb{C}(V^m)^N)$.

The number t_N is often well understood (see Section 7.2.1) and hence determines the dimension of the commuting algebra of the N-fold tensor representations of the genus-m Clifford-Weil groups. Note that these dimensions are independent of m for $m \geq g_N(\rho)$.

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