

A GILBERT-VARSHAMOV TYPE BOUND FOR EUCLIDEAN PACKINGS.

GABRIELE NEBE AND CHAOPING XING

ABSTRACT. The present paper develops a method to obtain a Gilbert-Varshamov type bound for dense packings in the Euclidean spaces using suitable lattices. For the Leech lattice the obtained bounds are quite reasonable for large dimensions, better than the Minkowski-Hlawka bound but not as good as the lower bound given in [1]. In fact, no sphere packings have been described that beat our bounds.

1. INTRODUCTION

It is a classical problem to find dense sphere packings in Euclidean space, see for instance [2, 3, 4, 7] for an introduction to this topic.

For a real $b > 0$, let \mathcal{A}_b^m be the ball of radius b in \mathbb{R}^m defined by

$$\mathcal{A}_b^m := \{x := (x_1, \dots, x_m) \in \mathbb{R}^m : \|x\|^2 := \sum_{i=1}^m x_i^2 \leq b^2\}.$$

Then the volume of \mathcal{A}_b^m is equal to $V_m \cdot b^m$, where $V_m = \frac{\pi^{m/2}}{\Gamma(m/2+1)}$ denotes the volume of a unit ball in \mathbb{R}^m .

A packing in \mathbb{R}^m is a set \mathcal{P} of points in \mathbb{R}^m such that the Euclidean distance of \mathcal{P}

$$d(\mathcal{P}) := \inf_{\mathbf{u}, \mathbf{v} \in \mathcal{P}, \mathbf{u} \neq \mathbf{v}} d(\mathbf{u}, \mathbf{v})$$

is positive. It is clear that all balls with radius $d(\mathcal{P})/2$ and centers being points of \mathcal{P} are not overlapping. Let $\mathcal{U}(\mathcal{P})$ be the union of all such balls, i.e.,

$$\mathcal{U}(\mathcal{P}) = \{\mathbf{x} \in \mathbb{R}^m : \exists \mathbf{u} \in \mathcal{P} \text{ such that } d(\mathbf{x}, \mathbf{u}) \leq d(\mathcal{P})/2\}.$$

The density of \mathcal{P} is defined by

$$\Delta(\mathcal{P}) = \limsup_{b \rightarrow \infty} \frac{\text{vol}(\mathcal{U}(\mathcal{P}) \cap \mathcal{A}_b^m)}{\text{vol}(\mathcal{A}_b^m)},$$

where $\text{vol}(\mathcal{T})$ denotes the volume of a subset \mathcal{T} in \mathbb{R}^m . It is clear that $\Delta(\mathcal{P}) \leq 1$. We are interested in dense packings, i.e., we want to find a packing with density close to the quantity

$$\Delta_m := \limsup_{\mathcal{P}} \Delta(\mathcal{P}),$$

2000 AMS subject Classification: 11H31, 52C17, 11H71, 11H06.

where \mathcal{P} is extended over all packings in \mathbb{R}^m . Sometimes, it is more convenient to convert Δ_m into the center density which is defined by

$$\delta_m := \frac{\Delta_m}{V_m}.$$

The Gilbert-Varshamov (G-V, for short) bound is a benchmark for good codes in coding theory. The idea involved in the G-V bound can be employed for many other problems. In this paper, we use the idea to construct dense packings in \mathbb{R}^m by choosing a subset of certain lattice points. It turns out that the obtained G-V type bound is quite reasonable. For the orthogonal sum of copies of the Leech lattice, it is better than the Minkowski-Hlawka bound but not as good as the lower bound given in [1]. Up to now no sphere packings have been described that beat our bounds. A table for densities for large dimensions divisible by 720 is given in this paper.

2. GILBERT-VARSHAMOV TYPE BOUND

In this section we fix a lattice L of dimension m in Euclidean space \mathbb{R}^m such that all norms in L are integers, i.e. $\|\mathbf{c}\|^2 \in \mathbb{Z}$ for all $\mathbf{c} \in L$. For a point $\mathbf{c} \in L$, define the ball $\mathcal{B}_{L,k}(\mathbf{c})$ to be the set consisting of points in L with Euclidean distance from \mathbf{c} at most \sqrt{k} , i.e.,

$$\mathcal{B}_{L,k}(\mathbf{c}) := \{\mathbf{b} \in L : \|\mathbf{b} - \mathbf{c}\|^2 \leq k\}.$$

It is clear that $\mathcal{B}_{L,k}(\mathbf{c})$ is a finite set and its cardinality, denoted by $B_{L,k}$, is independent of its center \mathbf{c} . In fact, we have

$$(2.1) \quad B_{L,k} = 1 + \sum_{i=1}^k S_{L,i},$$

where $S_{L,i}$ is the number of lattice points of norm i in L .

One important invariant of L is its discriminant D , defined as the determinant of a Gram matrix of L with equals the square of the volume of a fundamental domain of L in \mathbb{R}^m (see [3]). It measures the number of lattice points contained in a unit ball.

Lemma 2.1. *Let L be a lattice in \mathbb{R}^m with discriminant D . Then*

$$\lim_{b \rightarrow \infty} \frac{B_{L,b}}{V_m b^{m/2}} = \frac{1}{\sqrt{D}},$$

where V_m is the volume of the unit ball in \mathbb{R}^m .

Proof. Clearly \sqrt{D} is the volume of the Dirichlet domain

$$\mathcal{D}_L = \{x \in \mathbb{R}^m \mid \|x\| < \|x - \ell\| \forall 0 \neq \ell \in L\}$$

around 0. For a fixed b , the $B_{L,b}$ translates $\ell + \mathcal{D}_L$ with $\ell \in \mathcal{B}_{L,b}(0)$ tile a subset of volume $B_{L,b}\sqrt{D}$ of \mathbb{R}^m which tends to be the ball around 0 with radius \sqrt{b} when b tends to infinity. \square

Theorem 2.2. *Let $L \leq \mathbb{R}^m$ be a lattice of discriminant D , such that all norms in L are integral. Then for any $r \geq 1$, one has*

$$(2.2) \quad \Delta_m \geq \frac{1}{\sqrt{D}} \times \frac{1}{B_{L,r}} \times \left(\frac{\sqrt{r+1}}{2}\right)^m \times V_m,$$

hence

$$(2.3) \quad \delta_m \geq \frac{1}{2^m \sqrt{D}} \max_{r \geq 1} \left(\frac{(r+1)^{m/2}}{B_{L,r}} \right).$$

Proof. For a sufficiently large integer $b > 0$ choose an arbitrary point \mathbf{c}_1 from $\mathcal{B}_{L,b}(\mathbf{0})$. Pick up a point \mathbf{c}_2 from $\mathcal{B}_{L,b}(\mathbf{0}) \setminus \mathcal{B}_{L,r}(\mathbf{c}_1)$. Then pick up a point \mathbf{c}_3 from $\mathcal{B}_{L,b}(\mathbf{0}) \setminus \mathcal{B}_{L,r}(\mathbf{c}_1) \cup \mathcal{B}_{L,r}(\mathbf{c}_2)$.

This procedure constructs a subset $C := \{\mathbf{c}_1, \dots, \mathbf{c}_{M+1}\}$ of $\mathcal{B}_{L,b}(\mathbf{0})$ with

$$M := \left\lfloor \frac{B_{L,b} - 1}{B_{L,r}} \right\rfloor$$

such that

$$\mathbf{c}_i \in \mathcal{B}_{L,b}(\mathbf{0}) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_{L,r}(\mathbf{c}_j)$$

for any $1 \leq i \leq M+1$. This is because the set $\mathcal{B}_{L,b}(\mathbf{0}) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_{L,r}(\mathbf{c}_j)$ is not empty due to the fact that

$$\left| \mathcal{B}_{L,b}(\mathbf{0}) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_{L,r}(\mathbf{c}_j) \right| \geq |\mathcal{B}_{L,b}(\mathbf{0})| - \left| \bigcup_{j=1}^M \mathcal{B}_{L,r}(\mathbf{c}_j) \right| \geq B_{L,b} - M \cdot B_{L,r} > 0.$$

Since norms in L are integral and $\|\mathbf{c}_i - \mathbf{c}_j\|^2 > r$ for $i \neq j$ by construction, any two distinct points in the set C have Euclidean distance at least $\sqrt{r+1}$. Hence the balls around distinct \mathbf{c}_j with radius $\sqrt{r+1}/2$ are disjoint. When b tends to ∞ this yields

$$\begin{aligned} \Delta_m &\geq \lim_{b \rightarrow \infty} \frac{M}{V_m b^{m/2}} \times \left(\frac{\sqrt{r+1}}{2} \right)^m \times V_m \\ &= \lim_{b \rightarrow \infty} \frac{B_{L,b}}{V_m b^{m/2}} \times \frac{1}{B_{L,r}} \times \left(\frac{\sqrt{r+1}}{2} \right)^m \times V_m \\ &= \frac{1}{\sqrt{D}} \times \frac{1}{B_{L,r}} \times \left(\frac{\sqrt{r+1}}{2} \right)^m \times V_m. \end{aligned}$$

The desired result follows. \square

Remark 2.3. As r tends to infinity, the right hand side of the inequality (2.2) tends to $\frac{1}{2^m}$ yielding asymptotically the same bound as the Minkowski-Hlawka bound, $\frac{1}{m} \log_2(\Delta_m) \geq -1$. However, for smaller values of r the right hand side may be bigger, which yields a slight improvement of this bound.

3. NUMERICAL RESULTS.

In applications one usually chooses orthogonally decomposable lattices

$$L = L_1^{n_1} \perp L_2^{n_2} \perp \dots \perp L_s^{n_s} \leq \mathbb{R}^m$$

of dimension $m = \sum_{i=1}^s n_i \dim(L_i)$ and discriminant $D = \prod_{i=1}^s D_i^{n_i}$ where D_i denotes the discriminant of L_i . We obtained the good results by choosing

$$L = \frac{1}{\sqrt{2}} \Lambda_{24}^{n_1} \perp \frac{1}{\sqrt{2}} \mathbf{A}_2^{n_2} \perp \mathbb{Z}^{n_3}$$

with $m = 24n_1 + 2n_2 + n_3$, $0 \leq n_2 \leq 11$, $0 \leq n_3 \leq 1$ where $\Lambda_{24} \leq \mathbb{R}^{24}$ denotes the Leech lattice and \mathbf{A}_2 the 2-dimensional hexagonal lattice. Then $D_1 = 2^{-24}$,

$D_2 = \frac{3}{4}$ and $D_3 = 1$ and so $\log_2(\frac{1}{\sqrt{D}}) = 12n_1 + n_2\alpha$ with $\alpha = (1 - \log_2(\sqrt{3}))$ and hence $\log_2(\delta_m) \geq b_0(m)$ where

$$(3.1) \quad b_0(24n_1 + 2n_2 + n_3) = 12n_1 + \alpha n_2 + \max_{r \geq 1} \left((12n_1 + n_2 + \frac{n_3}{2}) \log_2(\frac{r+1}{4}) - \log_2(B_{L,r}) \right).$$

The following table lists a few examples (truncated to the second decimal place).

| m | $b_0(m)$ | r_0 | m | $b_0(m)$ | r_0 | m | $b_0(m)$ | r_0 |
|------|----------|-------|------|----------|-------|------|----------|-------|
| 1440 | 3176.35 | 161 | 1441 | 3178.99 | 162 | 1442 | 3181.81 | 163 |
| 1443 | 3184.46 | 165 | 1444 | 3187.29 | 167 | 1445 | 3189.95 | 169 |
| 1446 | 3192.78 | 171 | 1447 | 3195.46 | 174 | 1448 | 3198.31 | 178 |
| 1449 | 3201.01 | 184 | 1450 | 3203.88 | 192 | 1451 | 3206.63 | 209 |
| 1452 | 3209.54 | 218 | 1453 | 3212.34 | 235 | 1454 | 3215.27 | 240 |
| 1455 | 3218.10 | 253 | 1456 | 3221.03 | 257 | 1457 | 3223.88 | 268 |
| 1458 | 3226.82 | 270 | 1459 | 3229.68 | 281 | 1460 | 3232.62 | 282 |
| 1461 | 3235.50 | 291 | 1462 | 3238.44 | 292 | 1463 | 3241.32 | 301 |
| 1464 | 3246.59 | 164 | 1465 | 3249.24 | 165 | 1466 | 3252.07 | 166 |
| 1467 | 3254.73 | 168 | 1468 | 3257.57 | 170 | 1469 | 3260.24 | 172 |
| 1470 | 3263.09 | 174 | 1471 | 3265.78 | 177 | 1472 | 3268.64 | 181 |
| 1473 | 3271.36 | 188 | 1474 | 3274.24 | 195 | 1475 | 3277.00 | 212 |
| 1476 | 3279.92 | 221 | 1477 | 3282.74 | 238 | 1478 | 3285.67 | 244 |
| 1479 | 3288.52 | 258 | 1480 | 3291.46 | 261 | 1481 | 3294.32 | 273 |
| 1482 | 3297.27 | 275 | 1483 | 3300.15 | 285 | 1484 | 3303.10 | 287 |
| 1485 | 3305.99 | 296 | 1486 | 3308.94 | 297 | 1487 | 3311.83 | 306 |
| 1488 | 3317.11 | 167 | 1489 | 3319.77 | 168 | 1490 | 3322.61 | 169 |
| 1491 | 3325.29 | 171 | 1492 | 3328.14 | 173 | 1493 | 3330.82 | 175 |
| 1494 | 3333.68 | 177 | 1495 | 3336.38 | 180 | 1496 | 3339.26 | 184 |
| 1497 | 3341.98 | 191 | 1498 | 3344.88 | 198 | 1499 | 3347.66 | 216 |
| 1500 | 3350.59 | 225 | 1501 | 3353.41 | 242 | 1502 | 3356.36 | 248 |
| 1503 | 3359.21 | 262 | 1504 | 3362.17 | 265 | 1505 | 3365.05 | 277 |
| 1506 | 3368.01 | 279 | 1507 | 3370.89 | 290 | 1508 | 3373.86 | 291 |
| 1509 | 3376.75 | 301 | 1510 | 3379.72 | 302 | 1511 | 3382.62 | 311 |
| 1512 | 3387.90 | 170 | 1513 | 3390.58 | 171 | 1514 | 3393.43 | 172 |
| 1515 | 3396.12 | 174 | 1516 | 3398.98 | 176 | 1517 | 3401.68 | 178 |
| 1518 | 3404.55 | 180 | 1519 | 3407.27 | 184 | 1520 | 3410.15 | 187 |
| 1521 | 3412.89 | 194 | 1522 | 3415.80 | 201 | 1523 | 3418.58 | 219 |
| 1524 | 3421.53 | 229 | 1525 | 3424.36 | 246 | 1526 | 3427.32 | 252 |
| 1527 | 3430.19 | 266 | 1528 | 3433.16 | 269 | 1529 | 3436.04 | 282 |
| 1530 | 3439.02 | 284 | 1531 | 3441.91 | 295 | 1532 | 3444.89 | 296 |
| 1533 | 3447.80 | 306 | 1534 | 3450.77 | 307 | 1535 | 3453.69 | 316 |
| 1536 | 3458.97 | 173 | 1537 | 3461.66 | 174 | 1538 | 3464.53 | 175 |
| 1539 | 3467.23 | 177 | 1540 | 3470.10 | 178 | 1541 | 3472.81 | 181 |
| 1542 | 3475.70 | 183 | 1543 | 3478.42 | 187 | 1544 | 3481.32 | 190 |
| 1545 | 3484.07 | 197 | 1546 | 3486.99 | 205 | 1547 | 3489.79 | 223 |
| 1548 | 3492.74 | 232 | 1549 | 3495.59 | 250 | 1550 | 3498.56 | 256 |
| 1551 | 3501.44 | 270 | 1552 | 3504.42 | 274 | 1553 | 3507.31 | 286 |
| 1554 | 3510.30 | 288 | 1555 | 3513.20 | 299 | 1556 | 3516.19 | 301 |

| m | $b_0(m)$ | r_0 | | m | $b_0(m)$ | r_0 | | m | $b_0(m)$ | r_0 |
|------|----------|-------|--|------|----------|-------|--|------|----------|-------|
| 1557 | 3519.11 | 311 | | 1558 | 3522.10 | 311 | | 1559 | 3525.02 | 321 |
| 1560 | 3530.32 | 175 | | 1561 | 3533.02 | 177 | | 1562 | 3535.90 | 178 |
| 1563 | 3538.60 | 180 | | 1564 | 3541.49 | 181 | | 1565 | 3544.21 | 184 |
| 1566 | 3547.11 | 186 | | 1567 | 3549.84 | 190 | | 1568 | 3552.76 | 193 |
| 1569 | 3555.52 | 201 | | 1570 | 3558.45 | 208 | | 1571 | 3561.26 | 226 |
| 1572 | 3564.22 | 236 | | 1573 | 3567.08 | 254 | | 1574 | 3570.06 | 260 |
| 1575 | 3572.95 | 274 | | 1576 | 3575.94 | 278 | | 1577 | 3578.85 | 290 |
| 1578 | 3581.84 | 293 | | 1579 | 3584.76 | 304 | | 1580 | 3587.76 | 305 |

The Minkowski-Hlawka bound (see for instance [3, page 14]) yields that $\Delta_m \geq \frac{\zeta(m)}{2^{m-1}}$. The paper [1] improves this bound and shows that

$$\Delta_m \geq (m - 1) \frac{\zeta(m)}{2^{m-1}}$$

and hence

$$\begin{aligned} \log_2(\delta_m) &\geq \text{ball}(m) = \text{mh}(m) + \log_2(m - 1) \\ &\text{with } \text{mh}(m) = \log_2(\zeta(m)) + \log_2((m/2)!) + 1 - m + (m/2) \log_2(\pi) \end{aligned}$$

For dimensions $m = 720n$ we compare our bounds $b_1(m)$ and $b_2(m)$ obtained by taking $L = \frac{1}{\sqrt{2}}\Lambda_{24}^{30n}$ respectively $L = \frac{1}{\sqrt{2}}\Lambda_{72}^{10n}$ (for some putative extremal even unimodular lattice Λ_{72}) to these two bounds. This shows that our lower bounds are better than the Minkowski Hlawka bound, but not as good as the ones obtained by Keith Ball. The results (rounded to 10 decimal places) were calculated with MAGMA [5] using the modular forms package.

| m | $b_1(m)$ | r_1 | $b_2(m)$ | r_2 | $\text{mh}(m)$ | $\text{ball}(m)$ |
|-------|-------------|-------|-------------|-------|----------------|------------------|
| 1440 | 3176.358513 | 161 | 3177.780626 | 128 | 3173.388428 | 3183.879279 |
| 2160 | 5391.441687 | 249 | 5392.867890 | 196 | 5388.598648 | 5399.674795 |
| 2880 | 7782.965065 | 339 | 7784.398557 | 265 | 7780.204931 | 7791.696283 |
| 3600 | 10305.69411 | 430 | 10307.13442 | 334 | 10302.99474 | 10314.80812 |
| 4320 | 12932.98468 | 522 | 12934.43130 | 404 | 12930.33297 | 12942.40945 |
| 5040 | 15647.23054 | 615 | 15648.68263 | 474 | 15644.61786 | 15656.91678 |
| 5760 | 18435.91780 | 708 | 18437.37474 | 545 | 18433.33806 | 18445.82966 |
| 6480 | 21289.68966 | 803 | 21291.15088 | 615 | 21287.13832 | 21299.79988 |
| 7200 | 24201.28315 | 898 | 24202.74824 | 686 | 24198.75675 | 24211.57033 |
| 7920 | 27164.89608 | 993 | 27166.36461 | 757 | 27162.39186 | 27175.34296 |
| 8640 | 30175.78597 | 1089 | 30177.25765 | 829 | 30173.30171 | 30186.37836 |
| 9360 | 33230.00377 | 1185 | 33231.47830 | 900 | 33227.53761 | 33240.72975 |
| 10080 | 36324.20987 | 1282 | 36325.68702 | 972 | 36321.76029 | 36335.05935 |
| 10800 | 39455.54308 | 1379 | 39457.02261 | 1044 | 39453.10876 | 39466.50737 |
| 11520 | 42621.52459 | 1477 | 42623.00632 | 1115 | 42619.10439 | 42632.59612 |
| 12240 | 45819.98605 | 1574 | 45821.46984 | 1188 | 45817.57900 | 45831.15820 |
| 12960 | 49049.01471 | 1672 | 49050.50042 | 1260 | 49046.61995 | 49060.28161 |
| 13680 | 52306.91065 | 1771 | 52308.39816 | 1332 | 52304.52742 | 52318.26710 |
| 14400 | 55592.15318 | 1869 | 55593.64235 | 1404 | 55589.78081 | 55603.59449 |
| 15120 | 58903.37385 | 1968 | 58904.86459 | 1477 | 58901.01173 | 58914.89580 |

A MAGMA program producing this table is available from [6].

REFERENCES

- [1] Keith Ball, *A lower bound for the optimal density of lattice packings*. Internat. Math. Res. Notices 1992, no. 10, 217–221
- [2] J. W. S. Cassels, “An Introduction to the Geometry of Numbers,” Springer-Verlag, New York, 1971.
- [3] J. H. Conway and N. J. A. Sloane, “Sphere Packings, Lattices and Groups,” Second Edition, Springer-Verlag, New York, 1992.
- [4] P. M. Gruber, C. G. Lekkerkerker, “Geometry of numbers.” Second edition. North-Holland Publishing Co., Amsterdam, 1987.
- [5] J. Cannon et al., *The Magma Computational Algebra System for Algebra, Number Theory and Geometry*, published electronically at <http://magma.maths.usyd.edu.au/magma/>.
- [6] Gabriele Nebe’s homepage, <http://www.math.rwth-aachen.de/homes/Gabriele.Nebe/>
- [7] C. A. Rogers, “Packing and Covering,” Cambridge Univ. Press, 1964.
- [8] N. J. A. Sloane, *Table of Densest Packings Presently Known*, [Online] <http://www.research.att.com/~njas/lattices/density.html>.

LEHRSTUHL D FÜR MATHEMATIK, RWTH AACHEN, GERMANY
E-mail address: `nebe@math.rwth-aachen.de`

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE 117543,
REPUBLIC OF SINGAPORE.

E-mail address: `matxcp@nus.edu.sg`