# Hermitian modular forms congruent to 1 modulo $p$.

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**Abstract.** For any natural number $\ell$ and any prime $p \equiv 1 \pmod 4$ not dividing $\ell$ there is a Hermitian modular form of arbitrary genus $n$ over $L := \mathbb{Q}[\sqrt{-\ell}]$ that is congruent to 1 modulo $p$ which is a Hermitian theta series of an $O_L$-lattice of rank $p-1$ admitting a fixed point free automorphism of order $p$. It is shown that also for non-free lattices such theta series are modular forms.

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## 1. Introduction.

The purpose of the present note is to generalize the construction of Siegel modular forms that are congruent to 1 modulo a suitable prime $p$ given in [3] to the case of Hermitian modular forms over $L := \mathbb{Q}[\sqrt{-\ell}]$. For $\ell = 1$ and $\ell = 3$ this was done in [11], in fact we use the same strategy by constructing an even unimodular lattice $\Lambda$ as an ideal lattice in $K := L[\zeta_p]$ for any prime $p \equiv 1 \pmod 4$ not dividing $\ell$. The existence of $\Lambda$ essentially follows from class field theory and is predicted by [2, Théorème 2.3, Proposition 3.1 (1)] (see also [1, Corollary 2]). Since the ring of integers $O_L$ is in general not a principal ideal domain the lattice $\Lambda$ is not necessarily a free $O_L$-module. We are not aware of an explicit statement in the literature that the genus $n$ Hermitian theta series $\theta^{(n)}(\Lambda)$ of such a lattice $\Lambda$ is a modular form for the full modular group. Therefore the first section sketches a proof. In fact the proofs in the literature never seriously use the fact that the lattice is a free $O_L$-module. The next section applies the results of [2] and [1] to the special case of the field $K = \mathbb{Q}[\sqrt{-\ell}, \zeta_p]$ and proves the existence of a Hermitian $O_K$-lattice $\Lambda_h$ that is an even unimodular $\mathbb{Z}$-lattice (with respect to the trace of the Hermitian form). The invariance under $O_K$ yields both a Hermitian $O_L$-module structure on $\Lambda_h$ and an $O_L$-linear automorphism (the multiplication by the primitive $p$-th root of unity $\zeta_p \in O_K$) of order $p$ acting fixed point freely on $\Lambda_h \setminus \{0\}$. Therefore all but the first coefficient in $\theta^{(n)}(\Lambda_h)$ are multiples of $p$ yielding the desired Hermitian modular form.

## 2. Hermitian theta-series are Hermitian modular forms.

Let $\ell \in \mathbb{N}$ such that $-\ell$ is a fundamental discriminant (which means that either $\ell \equiv -1 \pmod 4$ is square-free or $\ell = 4m$, where $m \equiv 2$ or $1 \pmod 4$ is square-free). Let $L := \mathbb{Q}[\sqrt{-\ell}]$ be the imaginary quadratic number field of discriminant $-\ell$, with ring of integers $O_L$ and inverse different

$$O_L^* := \{a \in L \mid \mathrm{Tr}_{L/\mathbb{Q}}(aO_L) \subset \mathbb{Z}\} = \sqrt{-\ell}^{-1} O_L.$$

For $n \in \mathbb{N}$ let trace : $L^{n\times n} \to \mathbb{Q}$ denote the composition of the matrix trace with $\mathrm{Tr}_{L/\mathbb{Q}}$, the trace of $L$ over $\mathbb{Q}$. Let $J := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Then the **full modular group** $\Gamma_n := \{M \in SL_{2n}(O_L) \mid MJ\overline{M}^t = J\}$ is generated by

$$\Gamma_n = \langle \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}, J \mid B \in O_L^{n\times n} \text{ Hermitian } , U \in \mathrm{GL}_n(O_L)\rangle$$

(see [7], [5, Anhang V], [8]) for the proof that these matrices really generate) acts on the Hermitian half space

$$\mathcal{H}_n := \{Z \in \mathbb{C}^{n\times n} \mid 1/(2i)(Z - \overline{Z}^t) \text{ Hermitian positive definite } \}$$

by

$$Z \mapsto Z + B,\ Z \mapsto \overline{U}^t ZU,\ Z \mapsto -Z^{-1}$$

for the respective generators.

Let $(V, h)$ be a finite dimensional positive definite Hermitian vector space over $L$. Recall that an $O_L$-lattice $\Lambda \subset V$ is a finitely generated $O_L$-submodule of $V$ that spans $V$ as a vector space. The $O_L$-dual lattice

$$\Lambda^* := \{v \in V \mid h(v, \Lambda) \subset O_L\}$$

is again an $O_L$-lattice in $V$. The Hermitian theta series of the lattice $\Lambda$ is

$$\theta^{(n)}(\Lambda)(Z) := \sum_{(x_1,\ldots,x_n)\in\Lambda^n} \exp(2\pi i \,\mathrm{trace}(h(x_i, x_j)Z)).$$

This section extends the results in [4] to not necessarily free Hermitian $O_L$-lattices in $(V, h)$. Note that we use a different scaling for the Hermitian form resulting in the additional factor of 2 in the definition of the Hermitian Siegel theta series. It is already stated in [4] that the authors restrict to free lattices "for convenience" and that the same results hold in the more general context.

**Theorem 2.1.** *Let $(V, h)$ be a positive definite Hermitian space of dimension $d$ over $L$. Let $\Lambda_h \subset V$ be an $O_L$-lattice such that*

$$\Lambda_h^* = \sqrt{-\ell}\Lambda_h = (O_L^*)^{-1}\Lambda_h.$$

*Then its Hermitian theta series $\theta^{(n)}(\Lambda_h)$ is a Hermitian modular form for the full modular group $\Gamma_n$.*

Proof. For $x := (x_1, \ldots, x_n) \in \Lambda_h^n$ the Hermitian matrix $H := H_x := (h(x_i, x_j)) \in (O_L^*)^{n\times n}$, so for any Hermitian matrix $B \in O_L^{n\times n}$ the trace trace$(HB)$ is in $\mathbb{Z}$. This shows the invariance of $\theta^{(n)}(\Lambda_h)$ under $Z \mapsto Z + B$. Similarly

$$\mathrm{trace}(H_x\overline{U}^t ZU) = \mathrm{trace}(UH_x\overline{U^t}Z) = \mathrm{trace}(H_{xU}Z)$$

so the transformation $Z \mapsto \overline{U}^t ZU$ for $U \in \mathrm{GL}_n(O_L)$ just changes the order of summation in $\theta^{(n)}(\Lambda_h)$. It remains to prove the theta-transformation formula

$$(\star) \quad \theta^{(n)}(\Lambda_h)(-Z^{-1}) = \det(Z/i)^d \theta^{(n)}(\Lambda_h)(Z)$$

also for non-free $O_L$-lattices $\Lambda_h$ of dimension $d$ that satisfy $\Lambda_h^* = (O_L^*)^{-1}\Lambda_h$. But Poisson summation only depends on the abelian group structure, not on the underlying module, so the proof from [9, p. 110-112] can be adopted to the situation here (for details we refer to [6]). Indeed, as in the usual proof, by the Identity Theorem it suffices to prove $(\star)$ for $Z = iY$, $Y$ Hermitian positive definite. Let $\varphi : \mathbb{R}^{2dn} \to \mathbb{C}^{d\times n}$ be the obvious isomorphism and consider $\Lambda_h^n$ as a lattice $\tilde{\Lambda}$ in $\mathbb{C}^{d\times n}$ choosing coordinates with respect to an orthonormal basis of $(\mathbb{C}^d, h)$. Then there is some $F \in \mathbb{R}^{2dn\times 2dn}$ such that $\tilde{\Lambda} = \varphi(F\mathbb{Z}^{2dn})$ and $H_{\varphi(x)} = \overline{\varphi(Fx)}^{tr}\varphi(Fx)$. Then

$$\theta^{(n)}(\Lambda_h)(iY) = \sum_{g\in\mathbb{Z}^{2dn}} \psi(g)$$

where

$$\psi : \mathbb{R}^{2dn} \to \mathbb{C}, x \mapsto \exp(-2\pi\,\mathrm{trace}(\overline{\varphi(Fx)}^{tr}\varphi(Fx)Y)).$$

The condition $\Lambda_h^* = (O_L^*)^{-1}\Lambda_h$ implies that $|\det(F)| = 1$ and we can apply the usual Poisson summation to get the result as in [9, pp. 110-112]. □

## 3. Congruences of Hermitian theta-series.

Let $p$ be a prime $p \equiv 1 \pmod 4$ such that $\ell$ is not a multiple of $p$. This section constructs a Hermitian $O_L$-lattice $(\Lambda, h)$ of rank $p-1$ admitting an automorphism of order $p$ such that the $\mathbb{Z}$-lattice $(\Lambda, \mathrm{Tr}_{L/\mathbb{Q}}(h))$ is a positive definite even unimodular lattice. The existence of such a lattice follows from the much more general result [2, Théorème 2.3] together with [2, Proposition 3.1] which are based on Artin's reciprocity law in global class field theory (see [10, Theorem (V.3.5)]). For our special case it is however more convenient to use [1, Corollary 2], which is essentially a consequence of [2, Théorème 2.3].

To this aim we consider the number field $K = \mathbb{Q}[\sqrt{-\ell}][\zeta_p] = LM$ with $M = \mathbb{Q}[\zeta_p]$, where $\zeta_p = \exp(\frac{2\pi i}{p})$ is a primitive $p$-th root of unity. Then $K$ is an abelian number field of degree $2(p-1)$ over $\mathbb{Q}$ which is a multiple of 8. The field $K$ is totally complex and admits an involution $\overline{\phantom{x}}$, the complex conjugation, with fixed field $F$ the totally real subfield of $K$.

The following lemma is well known.

**Lemma 3.1.** *$K/F$ is unramified at all finite primes.*

<u>Proof.</u> The discriminant $d_{K/F}$ of $K/F$ divides the discriminant of any $F$-basis of $K$ that consists of integral elements. For $B_1 = (1, \sqrt{-\ell})$ one finds $d_{B_1} = \det(\mathrm{Tr}_{K/F}(b_i b_j)) = -4\ell$ and for $B_2 = (1, \zeta_p)$ one get $d_{B_2} = \zeta_p^{-2}(\zeta_p^2 - 1)^2$ which generates an ideal of norm $p^2$ in $F$. Since $p$ is an odd prime not dividing $\ell$, the gcd of these two discriminants is 1 and hence $d_{K/F} = 1$ which implies the lemma.
□

Since all real embeddings of $F$ extend to complex embeddings of $K$ and $[K : \mathbb{Q}] = 2(p-1) \equiv 0 \pmod 8$ [1, Corollary 2] yields the existence of a fractional $O_K$-ideal $\mathcal{A}$ in $K$ and a totally positive element $a \in F$ such that the $O_K$-module $\mathcal{A}$ together with the symmetric integral bilinear form

$$b_a : \mathcal{A} \times \mathcal{A} \to \mathbb{Z},\ (x, y) \mapsto \mathrm{trace}_{K/\mathbb{Q}}(ax\overline{y})$$

is an even unimodular $\mathbb{Z}$-lattice $\Lambda := (\mathcal{A}, b_a)$. This means that $b_a(x, x) \in 2\mathbb{Z}$ for all $x \in \mathcal{A}$ and

$$\Lambda^{\#} := \{x \in K \mid b_a(x, y) \in \mathbb{Z} \text{ for all } y \in \mathcal{A}\} = \Lambda.$$

**Corollary 3.2.** *The $O_L$-lattice $\Lambda_h := (\mathcal{A}, h(x, y) := \mathrm{Tr}_{K/L}(ax\overline{y}))$ is a Hermitian $O_L$-lattice with automorphism $x \mapsto \zeta_p x$ of order $p$ such that $\Lambda_h^* = (O_L^*)^{-1}\Lambda_h$.*

<u>Proof.</u> Since $\mathcal{A}$ is an ideal of $K$, the multiplication by $\zeta_p \in O_K$ preserves the lattice $\mathcal{A}$. It also respects the Hermitian form $h$, because

$$h(\zeta_p x, \zeta_p y) = \mathrm{Tr}_{K/L}(a\zeta_p x\overline{\zeta_p y})) = \mathrm{Tr}_{K/L}(a\zeta_p\zeta_p^{-1}x\overline{y})) = h(x, y).$$

The fact that $\Lambda_h^* = (O_L^*)^{-1}\Lambda_h$ follows from the unimodularity of the integral lattice $\Lambda$: For $y \in K$ we obtain

$$b_a(x, y) = \mathrm{trace}_{L/\mathbb{Q}}(h(x, y)) \in \mathbb{Z} \text{ for all } x \in \mathcal{A} \Leftrightarrow h(x, y) \in O_L^* \text{ for all } x \in \mathcal{A}$$

using the fact that $\mathcal{A}$ is an $O_L$-module and $h$ is Hermitian over $O_L$. Hence $\Lambda_h^* = (O_L^*)^{-1}\Lambda^{\#} = (O_L^*)^{-1}\Lambda_h$. □

Together this implies the existence of a Hermitian modular form of weight $p-1$ that is congruent to 1 modulo p for more general imaginary quadratic number fields than those treated in [11]:

**Theorem 3.3.** *Let $L = \mathbb{Q}[\sqrt{-\ell}]$ be an imaginary quadratic number field ($-\ell$ a fundamental discriminant) and let $p$ be a prime $p \equiv 1 \pmod 4$ not dividing $\ell$. Then for arbitrary genus $n \geq 1$ there is a Hermitian modular form*

$$F_{p-1}^{(n)} \in M_{p-1}(\Gamma_n)$$

*of weight $p-1$ for the full modular group $\Gamma_n$ such that*

$$F_{p-1}^{(n)} \equiv 1 \pmod p.$$

Proof. Corollary 3.2 constructs a Hermitian $O_L$-lattice $\Lambda_h$ of rank $p-1$ admitting an automorphism of order $p$ (which necessarily acts fixed point freely) such that $\Lambda_h^* = (O_L^*)^{-1}\Lambda_h$. By Theorem 2.1 its Siegel theta series is a Hermitian modular form for the full modular group. Since $\Lambda_h$ admits a fixed point free automorphism of order $p$, all the representation numbers

$$R_A := |\{(x_1,\ldots,x_n) \in \Lambda^n \mid (h(x_i,x_j)) = A\}|$$

for any non-zero Hermitian matrix $A \in L^{n\times n}$ are multiples of $p$ and hence

$$F_{p-1}^{(n)} := \theta_{\Lambda_h}^{(n)} \equiv 1 \pmod p$$

provides the desired Hermitian modular form. □

Since the root lattice $E_8$ is the unique even unimodular lattice of dimension 8, we obtain the following corollary.

**Corollary 3.4.** *Let $\ell \in \mathbb{N}$ be not a multiple of 5. Then the root lattice $E_8$ has a Hermitian structure as a lattice $\Lambda_h$ over the ring of integers of $\mathbb{Q}[\sqrt{-\ell}]$ such that* $\mathrm{Aut}(\Lambda_h)$ *contains an element of order* 5.

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