On the minimum of an Hermitian tensor product.

Renaud Coulangeon and Gabriele Nebe

ABSTRACT. Using the Hermitian tensor product description of the extremal even unimodular lattice of dimension 72 described in [6] we show its extremality with the methods described in [2]. Keywords: extremal even unimodular lattice, Hermitian tensor product.


1 An Hermitian tensor product construction of $\Gamma$.

Throughout the paper let $\alpha$ be a generator of the ring of integers in the imaginary quadratic number field $\mathbb{Q}[\sqrt{-7}]$ with $\alpha^2 - \alpha + 2 = 0$ and $\beta := \overline{\alpha} = 1 - \alpha$ its complex conjugate. Then $\mathbb{Z}[\alpha]$ is an Euclidean domain with Euclidean minimum $\frac{4}{7}$.

Let $(P, h)$ be an Hermitian $\mathbb{Z}[\alpha]$-lattice, so $P$ is a free $\mathbb{Z}[\alpha]$-module and $h : P \times P \rightarrow \mathbb{Q}[\alpha]$ a positive definite Hermitian form. One example of such a lattice is the Barnes lattice $P_b$ with Gram matrix

$$
\begin{pmatrix}
2 & \alpha & -1 \\
\beta & 2 & \alpha \\
-1 & \beta & 2
\end{pmatrix}
$$

Then $P_b$ is Hermitian unimodular, $P_b = P_b^* := \{ v \in \mathbb{Q}P_b \mid h(v, \ell) \in \mathbb{Z}[\alpha] \text{ for all } \ell \in P_b \}$ and has Hermitian minimum $\min(P_b) := \min\{ h(v, v) \mid 0 \neq v \in P_b \} = 2$. By [5] the lattice $P_b$ is the unique densest 3-dimensional Hermitian $\mathbb{Z}[\alpha]$-lattice.

Michael Hentschel [3] classified all Hermitian $\mathbb{Z}[\alpha]$-structures on the even unimodular $\mathbb{Z}$-lattices of dimension 24 using the Kneser neighbouring method [4] to generate the lattices and checking completeness with the mass formula. In particular there are exactly nine such $\mathbb{Z}[\alpha]$ structures $(P_i, h)$ (1 ≤ $i$ ≤ 9) such that $(P_i, \text{trace}_{\mathbb{Z}[\alpha]/\mathbb{Z}} \circ h) \cong \Lambda$ is the Leech lattice. The paper [6] investigates the nine 36-dimensional Hermitian $\mathbb{Z}[\alpha]$-lattice $R_i$ defined by $(R_i, h) := P_b \otimes_{\mathbb{Z}[\alpha]} P_i$ and shows that exactly one of them has minimum 4 and hence gives rise to an extremal even unimodular $\mathbb{Z}$-lattice in dimension 72. The proof uses computer calculations within the set of minimal vectors of the Leech lattice. The purpose of the present note is to give a new computational proof of the extremality of this lattice using its structure as a Hermitian tensor product.

2 Bounds for the minimum of the Hermitian tensor products.

To derive lower bounds for the minimum of the Hermitian lattices $R_i := P_i \otimes_{\mathbb{Z}[\alpha]} P_b$ we use [2, Proposition 3.2]. Any vector in $z \in R_i$ is a sum of tensors of the form $v \otimes w$ with $v \in P_i$ and $w \in P_b$. The minimal number of summands in such an expression is called the rank of $z$. Clearly the rank of any vector is less than the minimum of the dimension of the two tensor factors.
Proposition 2.1. ([2, Proposition 3.2]) Let $L$ and $M$ be Hermitian lattices and denote by $d_r(L)$ the minimal determinant of a rank $r$ sublattice of $L$. Then for any vector $z \in L \otimes \mathbb{Z}[\alpha] M$ of rank $r$ one has

$$h(z, z) \geq r d_r(L)^{1/r} d_r(M)^{1/r}.$$ 

Remark 2.2. (a) $d_1(P_b) = 2$.
(b) $d_2(P_b) = 2$.
(c) $d_3(P_b) = \det(P_b) = 1$.

Proposition 2.3. Let $(P, h)$ be a Hermitian $\mathbb{Z}[\alpha]$-lattice with $\min(P, h) = 2$. Then

(a) $d_1(P) = \min(P) = 2$.
(b) $d_2(P) \geq \frac{12}{7}$.
(c) $d_3(P) \geq 1$ and $d_3(P) = 1$ if and only if $P$ contains a sublattice isometric to $P_b$.

Proof. (b) In the proof of [2, Lemma 4.2.2] it is shown that the determinant $\det(M)$ of a $\mathbb{Z}[\alpha]$-lattice $M$ of rank 2 satisfies

$$\det(M) \geq \frac{3}{7} \min(M)^2.$$ 

If $M$ is a sublattice of $P$, then $\min(M) \geq 2$ and hence $\det(M) \geq \frac{12}{7}$.

(c) By the thesis of Bertrand Meyer [5], there are 2 perfect Hermitian forms in dimension 3 over $\mathbb{Z}[\alpha]$. Both forms are eutactic and hence extreme. In particular $P_b$ is the globally densest 3-dimensional Hermitian $\mathbb{Z}[\alpha]$-lattice and the Hermitian Hermite constant of $\mathbb{Z}[\alpha]$ is therefore $\gamma_3(\mathbb{Z}[\alpha]) = 2$. Now let $M$ be a sublattice of rank 3 of $P$. Then $\min(M) \geq 2$ and hence $\det(M) \geq 1$ and $\det(M) = 1$ if and only if $M \cong P_b$. □

Theorem 2.4. The minimum of the Hermitian lattices $R_i$ is either 3 or 4. The number of vectors of norm 3 in $R_i$ is equal to the representation number of $P_i$ for the sublattice $P_b$. In particular $\min(R_i) = 4$ if and only if the Hermitian Leech lattice $P_i$ does not contain a sublattice isomorphic to $P_b$.

Proof. The proof follows from [2, Proposition 3.2] (see above). Let $z \in P_i \otimes \mathbb{Z}[\alpha] P_b$ be a non-zero vector of rank $r = 1, 2, 3$.

If $r = 1$, then $z = v \otimes w$ and $h(z, z) \geq \min(P_i) \min(P_b) = 4$.

If $r = 2$ then $h(z, z) \geq 2 \sqrt{2} \sqrt{\frac{12}{7}} > 3$, so $h(z, z) \geq 4$.

If $r = 3$, then $h(z, z) \geq 3d^{1/3}$ where $d = d_3(P_i)$. Since $h(z, z) \in \mathbb{Z}$ this implies that $h(z, z) \geq 3$ and $h(z, z) \geq 4$ if $d_3(P_i) > 1$. □

Remark 2.5. With MAGMA ([1]) we computed the number of sublattices isomorphic to $P_b$ in the lattices $P_i$. Only one of them, $P_1$, does not contain such a sublattice, so $d_3(P_1) > 1$ and hence $\min(P_1 \otimes \mathbb{Z}[\alpha] P_b) \geq 4$.

References


