On the minimum of an Hermitian tensor product.

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ABSTRACT. Using the Hermitian tensor product description of the extremal even unimodular lattice of dimension 72 described in [6] we show its extremality with the methods described in [2].
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MSC: primary: 11H06, secondary: 11H31, 11H50, 11H55, 11H56, 11H71

1 An Hermitian tensor product construction of Γ.

Throughout the paper let α be a generator of the ring of integers in the imaginary quadratic number field $\mathbb{Q}[\sqrt{-7}]$ with $\alpha^2 - \alpha + 2 = 0$ and $\beta := \overline{\alpha} = 1 - \alpha$ its complex conjugate. Then $\mathbb{Z}[\alpha]$ is an Euclidean domain with Euclidean minimum $\frac{4}{7}$.

Let (P, h) be an Hermitian $\mathbb{Z}[\alpha]$-lattice, so P is a free $\mathbb{Z}[\alpha]$-module and $h : P \times P \to \mathbb{Q}[\alpha]$ a positive definite Hermitian form. One example of such a lattice is the Barnes lattice P_b with Gram matrix

$$\begin{pmatrix} 2 & \alpha & -1 \\ \beta & 2 & \alpha \\ -1 & \beta & 2 \end{pmatrix}$$

Then P_b is Hermitian unimodular, $P_b = P_b^* := \{v \in \mathbb{Q}P_b \mid h(v, \ell) \in \mathbb{Z}[\alpha] \text{ for all } \ell \in P_b\}$ and has Hermitian minimum $\min(P_b) := \min\{h(v, v) \mid 0 \neq v \in P_b\} = 2$. By [5] the lattice P_b is the unique densest 3-dimensional Hermitian $\mathbb{Z}[\alpha]$-lattice.

Michael Hentschel [3] classified all Hermitian $\mathbb{Z}[\alpha]$-structures on the even unimodular $\mathbb{Z}$-lattices of dimension 24 using the Kneser neighbouring method [4] to generate the lattices and checking completeness with the mass formula. In particular there are exactly nine such $\mathbb{Z}[\alpha]$ structures (P_i, h) $(1 \leq i \leq 9)$ such that $(P_i, \text{trace}_{\mathbb{Z}[\alpha]/\mathbb{Z}} \circ h) \cong \Lambda$ is the Leech lattice. The paper [6] investigates the nine 36-dimensional Hermitian $\mathbb{Z}[\alpha]$-lattice R_i defined by $(R_i, h) := P_b \otimes_{\mathbb{Z}[\alpha]} P_i$ and shows that exactly one of them has minimum 4 and hence gives rise to an extremal even unimodular $\mathbb{Z}$-lattice in dimension 72. The proof uses computer calculations within the set of minimal vectors of the Leech lattice. The purpose of the present note is to give a new computational proof of the extremality of this lattice using its structure as a Hermitian tensor product.

2 Bounds for the minimum of the Hermitian tensor products.

To derive lower bounds for the minimum of the Hermitian lattices $R_i := P_i \otimes_{\mathbb{Z}[\alpha]} P_b$ we use [2, Proposition 3.2]. Any vector in $z \in R_i$ is a sum of tensors of the form $v \otimes w$ with $v \in P_i$ and $w \in P_b$. The minimal number of summands in such an expression is called the rank of z. Clearly the rank of any vector is less than the mimimum of the dimension of the two tensor factors.

Proposition 2.1. *([2, Proposition 3.2]) Let L and M be Hermitian lattices and denote by $d_r(L)$ the minimal determinant of a rank r sublattice of L. Then for any vector $z \in L \otimes_{\mathbb{Z}[\alpha]} M$ of rank r one has*

$$h(z,z) \geq r d_r(L)^{1/r} d_r(M)^{1/r}.$$

Remark 2.2. *(a) $d_1(P_b) = 2$.*
(b) $d_2(P_b) = 2$.
(c) $d_3(P_b) = \det(P_b) = 1$.

Proposition 2.3. *Let (P,h) be a Hermitian $\mathbb{Z}[\alpha]$ lattice with $\min(P,h) = 2$. Then*
(a) $d_1(P) = \min(P) = 2$.
(b) $d_2(P) \geq \frac{12}{7}$.
(c) $d_3(P) \geq 1$ and $d_3(P) = 1$ if and only if P contains a sublattice isometric to P_b.

Proof. (b) In the proof of [2, Lemma 4.2.2] it is shown that the determinant $\det(M)$ of a $\mathbb{Z}[\alpha]$-lattice M of rank 2 satisfies

$$\det(M) \geq \frac{3}{7} \min(M)^2.$$

If M is a sublattice of P, then $\min(M) \geq 2$ and hence $\det(M) \geq \frac{12}{7}$.
(c) By the thesis of Bertrand Meyer [5], there are 2 perfect Hermitian forms in dimension 3 over $\mathbb{Z}[\alpha]$. Both forms are eutactic and hence extreme. In particular P_b is the globally densest 3-dimensional Hermitian $\mathbb{Z}[\alpha]$-lattice and the Hermitian Hermite constant of $\mathbb{Z}[\alpha]$ is therefore $\gamma_3(\mathbb{Z}[\alpha]) = 2$. Now let M be a sublattice of rank 3 of P. Then $\min(M) \geq 2$ and hence $\det(M) \geq 1$ and $\det(M) = 1$ if and only if $M \cong P_b$. □

Theorem 2.4. *The minimum of the Hermitian lattices R_i is either 3 or 4. The number of vectors of norm 3 in R_i is equal to the representation number of P_i for the sublattice P_b. In particular $\min(R_i) = 4$ if and only if the Hermitian Leech lattice P_i does not contain a sublattice isomorphic to P_b.*

Proof. The proof follows from [2, Proposition 3.2] (see above). Let $z \in P_i \otimes_{\mathbb{Z}[\alpha]} P_b$ be a non-zero vector of rank $r = 1, 2, 3$.
If $r = 1$, then $z = v \otimes w$ and $h(z,z) \geq \min(P_i)\min(P_b) = 4$.
If $r = 2$ then $h(z,z) \geq 2\sqrt{2}\sqrt{\frac{12}{7}} > 3$, so $h(z,z) \geq 4$.
If $r = 3$, then $h(z,z) \geq 3d^{1/3}$ where $d = d_3(P_i)$. Since $h(z,z) \in \mathbb{Z}$ this implies that $h(z,z) \geq 3$ and $h(z,z) \geq 4$ if $d_3(P_i) > 1$. □

Remark 2.5. *With MAGMA ([1]) we computed the number of sublattices isomorphic to P_b in the lattices P_i. Only one of them, P_1, does not contain such a sublattice, so $d_3(P_1) > 1$ and hence $\min(P_1 \otimes_{\mathbb{Z}[\alpha]} P_b) \geq 4$.*

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