On theta series attached to maximal lattices and their adjoints.

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Abstract. The space $M_k(N)^*$ spanned by theta series of adjoints of maximal even lattices of exact level N, determinant N^2 and dimension $2k$ has the Weierstrass property and hence allows us to define extremality for arbitrary squarefree level N. We find examples of such dual-extremal lattices. The space $M(N)^* = \oplus_k M_{2k}(N)^*$ is a free module of rank N over the ring of modular forms for the full modular group generated by homogeneous elements of weight ≤ 10.

Keywords: theta series, modular forms, Weierstrass property, free module, dual-extremal lattices.
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1 Introduction

This paper studies maximal even lattices from the geometric, arithmetic and analytic point of view. It is interesting to find even lattices L such that the dual lattice $L^\#$ has the highest possible minimum. The most promising candidates for L are clearly the maximal even lattices.

The maximal even lattices L of level N are characterized by the arithmetic property that the discriminant group $L^\#/L$ is an anisotropic quadratic abelian group of exponent N. If $m := \dim(L) = 2k$ is even, then this property can be translated into transformation rules for the theta series of L under the Atkin-Lehner involutions for all prime divisors of N (Theorem 3.1). If $\det(L) = N^2$ then the theta series of the adjoint lattice $\sqrt{N}L^\#$ lies in the space $M_k(N)^*$ introduced in [1]. This space has the Weierstrass property as defined in Definition 5.1 and hence allows to define extremality. The even lattice L is called dual-extremal if the theta series $\theta(\sqrt{N}L^\#)$ of the adjoint lattice is the extremal modular form in $M_k(N)^*$. The dual-extremal lattices of level N are the maximal even lattices of level N for which the minimum $\min_Q$ of the quadratic form on the adjoint lattice is $\geq \dim(M_k(N)^*)$. Remark 6.5 shows that in general this inequality may be strict. The dimension of $M_k(N)^*$ is calculated in [1]. It is interesting to note that for $k > 2$ the space $M_k(N)^*$ is spanned by theta series of adjoint lattices of even maximal lattices of level N, so this space is as small as it can be to obtain bounds on the minimum with the theory of modular forms.

The space $M(N)^* = \oplus M_{2k}(N)^*$ is a module over the ring of modular forms for the full modular group. Theorem 5.12 proves that $M(N)^*$ is a free $\mathbb{C}[E_4, E_6]$-module of rank N generated by homogeneous elements of weight ≤ 10.

Analogous results for maximal isotropic codes over fields $\mathbb{F}_q$, $q = 2, 3, 4$ are obtained in [19].

The last section of this paper lists some examples of dual-extremal lattices. The level 2 case is remarkable. Its connection to the notion of s-extremal (odd) unimodular lattices

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in [9] allows to prove that for a dual-extremal lattice L of level 2 and dimension $2k$ the minimum $\min_Q(\sqrt{2}L^{\#}) = \dim(M_k(2)^*)$. Also for $k \equiv 2 \pmod{12}$ the layers of $L^{\#}$ and of L all form spherical 5-designs (Proposition 6.1) and hence both lattices are strongly perfect (see [28]) and therefore local maxima of the sphere packing density function.

2 Preliminaries

2.1 Modular forms

For basic facts about modular forms we refer to [20]. We denote by $M_k(N)$ and $S_k(N)$ the spaces of modular forms and cusp forms of weight k for the congruence subgroup $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N}\}$. Throughout the paper, we assume N to be squarefree. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and any function f on the upper half plane $\mathbb{H}$ we define the slash operator $|_k$ by

$$(f\,|_k\,\gamma)(\tau) = det(\gamma)^{\frac{k}{2}}(c\tau+d)^{-k} f(\frac{a\tau+b}{c\tau+d}) \qquad (\tau \in \mathbb{H}).$$

For primes p we use the Hecke operators $T(p)$ (if $p \nmid N$), and $U(p)$ (for $p \mid N$) acting on $M_k(N)$ in the usual way. We also use the operator $V(p)$ defined by

$$f \longmapsto (f \mid V(p))(\tau) := f(p \cdot \tau).$$

Occasionally we need a variant $U(p)^0$ of the operator $U(p)$, defined for functions f on $\mathbb{H}$ periodic with respect to $p \cdot \mathbb{Z}$:

$$f(\tau) = \sum_n a_n e^{2\pi i \frac{n}{p}\tau} \longmapsto f \mid U^0(p)(\tau) = \sum_n a_{np} e^{2\pi i n\tau}.$$

Let p be a prime with $p \mid N$. We denote by ω_p^N any element of $SL(2,\mathbb{Z})$ satisfying

$$\omega_p^N \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod p$$

and

$$\omega_p^N \equiv I_2 \bmod \frac{N}{p}.$$

For such a matrix ω_p^N we put

$$W_p^N := \omega_p^N \cdot \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

and we recall that this defines an "Atkin-Lehner involution" on the space $M_k(N)$.

2.2 Lattices

We mainly consider even lattices L in some positive definite rational quadratic space (V, Q). Here L is called even, if $Q(L) \subset \mathbb{Z}$. Then L is automatically contained in its dual lattice $L^\# := \{x \in V \mid (x, \ell) \in \mathbb{Z} \text{ for all } \ell \in L\}$ where $(x, y) := Q(x + y) - Q(x) - Q(y)$ is the associated bilinear form. The minimal number $N \in \mathbb{N}$ such that the adjoint lattice $\sqrt{N}L^\# := (L^\#, NQ)$ is again even is called the level of L. We also define the minimum of the quadratic form

$$\min_Q(L) := \min\{Q(\ell) \mid 0 \neq \ell \in L\}.$$

Note that this is half of the usual minimum of the lattice.

For a quadratic space (V, Q) over $\mathbb{Q}$ we define the local Witt invariants $s_p(V)$ as in [26, p.80]. This normalization is very convenient for our purposes, in particular we will use the following lemma from [5].

Lemma 2.1. *Let L be an even lattice of level $N \cdot p$ with $p \nmid N$ in the quadratic space (V, Q) Then the following statements are equivalent*
i) $s_p(V) = 1$
ii) V carries (even) lattices of level N.
iii) If $L_p = L_p^{(0)} \perp L_p^{(1)}$ denotes the Jordan splitting of $L_p = L \otimes \mathbb{Z}_p$, then $L_p^{(1)}$ is an orthogonal sum of hyperbolic planes.

3 Lattices maximal at p and their theta series

We assume that L is an even lattice in a positive definite quadratic space (V, Q) of dimension $m = 2k$. We denote by N the (exact) level of L. We put $D = det(L)$; then $(-1)^k D$ is a discriminant (i.e. it is congruent 1 or 0 mod 4) and we denote by $(-1)^k d$ the corresponding fundamental discriminant (= a discriminant of a quadratic number field or equal to 1). Note that d is odd because N is squarefree.

We consider the theta series

$$\theta(L)(\tau) := \sum_{x \in L} e^{2\pi i Q(x) \cdot \tau} = \sum_{x \in L} q^{Q(x)}$$

for $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$. Let p be a prime with $p \mid N$.

We recall the transformation properties of $\theta(L)$ under ω_p^N:

$$\theta(L) \mid_k \omega_p^N = \gamma_p(d_p) s_p(V) D_p^{-\frac{1}{2}} \theta(L^{\#,p})$$

Here $L^{\#,p} = L^\# \cap \mathbb{Z}[\frac{1}{p}] \cdot L$ is the lattice dualized only at p, $s_p(V)$ is the Witt invariant and γ_p depends only on $d_p \cdot (\mathbb{Q}_p^\times)^2$, more precisely, $\gamma_p(1) = 1$ and for odd primes p, $\delta \in \mathbb{Z}_p^\times$

$$\gamma_p(\delta) = 1, \qquad \gamma_p(\delta \cdot p) = (\delta_p, p)_p \cdot (-i)^{\frac{p(p-1)}{2}}$$

For details see [6, Lemma 8.2], [5], or in more classical language, [12], for the explicit determination of γ_p see [8]. We do not need the more complicated γ_2 here.

Theorem 3.1. *Let p be a prime divisor of N with $p \mid\mid N$.*

$$\begin{array}{ccc} L_p & \textit{is maximal} & \Longleftrightarrow \\ \theta(L) \mid_k \omega_p^N \mid U^o(p) & = & -\gamma_p(d)p^{-1}d_p^{\frac{1}{2}}\,\theta(L) \end{array}$$

We remark here that the statement of the theorem is local; actually the assumption that N is squarefree is not necessary here.
Proof. "$\Longleftarrow$": The transformation properties of theta series imply

$$\theta(L) \mid_k \omega_p^N = \gamma_p(d_p)s_p(V)D_p^{-\frac{1}{2}}\theta(L^{\#,p})$$

Comparing constant terms on the right sides implies

$$s_p(V) = -1, \qquad D_p = p^2 \cdot d_p^{-1}.$$

In any case, (V, Q) does not carry a p-unimodular lattice and $D_p = p^2$ or $D_p = p$.
"$\Longrightarrow$": Suppose that L_p is maximal, in particular, V_p does not carry a lattice, which is unimodular (at p), hence $s_p(V) = -1$. The local lattice L_p has a decomposition

$$L_p = L_p^{(0)} \perp L_p^{(1)}$$

such that $L_p^{(0)}$ is unimodular and the lattice $\sqrt{p}^{-1}L_p^{(1)}$ is anisotropic mod p and of rank 1 or 2. This implies that any vector in $L_p^{\#}$ with length in $\mathbb{Z}_p$, is already in the sublattice L_p, which implies the global statement

$$\theta(L^{\#,p}) \mid U^0(p) = \theta(L).$$

Taking into account that $s_p(V_p) = -1$ and using the transformation formula from above, we therefore obtain

$$\theta(L) \mid_k \omega_p^N \mid U^0(p) = -\gamma_p(d)D_p^{-\frac{1}{2}}\theta(L)$$

Moreover, D_p is either p or p^2, i.e. $D_p = p^2 \cdot d_p^{-1}$. The assertion follows. □

Remark 3.2. *We can more generally consider theta series with harmonic polynomials of degree ν,*

$$\theta_P(L) := \sum_{x \in L} P(x)e^{2\pi i Q(x)\cdot\tau}.$$

Then we obtain again

$$\theta_P(L) \mid_{k+\nu} \omega_p^N \mid U^o(p) = -\gamma_p(d)p^{-1}d_p^{\frac{1}{2}}\,\theta_P(L)$$

provided that L_p is maximal and $p \mid\mid N$.

Remark 3.3. *Theorem 3.1 covers all maximal lattices except those where the fundamental discriminant d is divisible by 2 (where the level N is divisible by 4 and 8 respectively).*

We will mainly consider lattices which are maximal at all primes p. Concerning the existence we state

Proposition 3.4. *Suppose that N is squarefree; then there is an even maximal lattice of even rank $m = 2k$ with $\det(L) = N^2$ if and only if $m \equiv 4 \pmod 8$ and the number of prime divisors of N is odd or $8 \mid m$ and the number of prime divisors of N is even.*

Proof. Let (V, Q) be a quadratic space over $\mathbb{Q}$ possibly carrying such a lattice. Then we have for finite primes

$$s_p(V) = -1 \iff p \mid N$$

and

$$s_\infty(V) = \begin{cases} -1 & \text{if} \quad m \equiv 4 \pmod 8 \\ 1 & \text{if} \quad 8 \mid m \end{cases}$$

By the product formula for the Witt invariant, the number of prime divisors has to be odd ($m \equiv_4 \pmod 8$) or even (if $8 \mid m$). In the other direction we prefer to give an explicit construction: For N squarefree with an odd number of prime divisors, we choose a maximal order $\mathcal{O}(N)$ in the quaternion algebra over $\mathbb{Q}$ ramified exactly in the primes dividing N. We view it as usual as quadratic space (with the norm form). If $m \equiv 4 \pmod 8$ we may then take $\mathcal{O}(N) \oplus M$ as an example and for $8 \mid m$ we take $\mathcal{O}(N_1) \oplus \mathcal{O}(N_2) \oplus M$. Here M is an appropriate even unimodular lattice and $N = N_1 \cdot N_2$ is a decomposition of N into factors with an odd number of prime factors. The maximality of these lattices is then easily checked locally. □

4 The space $M_k(N)^*$

4.1 Definition and basic properties

The space of interest for us is (for any squarefree $N > 1$ and even weight k)

$$M_k(N)^* = \{f \in M_k(N) \mid \forall p \mid N : f \mid W_p^N + p^{1-\frac{k}{2}} f \mid U(p) = 0\}.$$

The subspace $S_k(N)^*$ of cuspforms in $M_k(N)^*$ was investigated in [1]. We recall some properties from there:

1) The definition may be rephrased in terms of the "trace"-operator (familiar from the theory of newforms [17]):

$$\forall p \mid N : \quad \text{trace}^N_{\frac{N}{p}}(f \mid W_p^N) = 0.$$

We recall that $\text{trace}^N_{\frac{N}{p}} : M_k(N) \longrightarrow M_k(\frac{N}{p})$ is defined by $f \longmapsto \sum_\gamma f \mid_k \gamma$, where γ runs over $\Gamma_0(N)\backslash\Gamma_0(\frac{N}{p})$; using explicit representatives for the γ we obtain the expression $\text{trace}^N_{\frac{N}{p}}(f) = f + p^{1-\frac{k}{2}} f \mid W_p^N \mid U(p)$.

2) When we compare the definition of $S_k(N)^*$ with the characterization of newforms in terms of traces, we see that $S_k(N)^*$ satisfies half of the conditions describing newforms, see [17] for details. In particular, the space of newforms of level N is contained in $S_k(N)^*$ and in fact it is easy to see from the theory of newforms that each eigenvalue system for the

collection $\{T(p) \in End(S_k(N)) \mid p$ coprime to $N\}$ occurs with multiplicity one in $S_k(N)^*$. More precisely, $S_k(N)^*$ can be built out of the spaces of newforms of level $M \mid N$ as follows: For a normalized Hecke eigenform $f = \sum_n a_f(n)q^n$ in $S_k(M)^{new}$ we put

$$f^{(N)}(\tau) := \sum_{d \mid \frac{N}{M}} \mu(d) \frac{d a_f(d)}{\sigma_1(d)} f(d \cdot \tau)$$

By the same reasoning as in [1], section 2.1, remark 2, we see that this defines an element of $S_k(N)^*$. We put

$$S_k(M)^{new,N} := \mathbb{C}\{f_i^{(N)}\},$$

where f_i runs over the normalized Hecke eigenforms in $S_k(M)^{new}$. Then

$$S_k(N)^* = \oplus_{M|N} S_k(M)^{new,N}.$$

3) We computed the dimension of this space for even $k \geq 2$

$$\dim S_k(N)^* = \frac{(k-1)N}{12} - \frac{1}{2} - \frac{1}{4}\left(\frac{-1}{(k-1)N}\right) - \frac{1}{3}\left(\frac{-3}{(k-1)N}\right).$$

4) It is easy to see that $S_k(N)^*$ has codimension one in $M_k(N)^*$, so there is only one Eisenstein series in this space. Actually, we can (at least for $k \geq 4$) compute the Eisenstein series in $M_k(N)^*$ explicitly from the level one Eisenstein series E_k by the same reasoning as above:

$$E_k^{(N)} := \sum_{d|N} \mu(d) \frac{d\sigma_{k-1}(d)}{\sigma_1(d)} E_k(d \cdot \tau).$$

4.2 The basis problem for $M_k(N)^*$

We want to span this space $M_k(N)^*$ by appropriate theta series. In [3] we already proved that $S_k(N)^{new}$ is always generated by linear combinations of theta series of quadratic forms from any fixed genus of quadratic forms with (exact) level N and determinant D such that $p^2 \mid D$ and $p^m \nmid D$. The machinery developed in [3], section 8 can also be applied to oldforms in $M_k(N)$.

Theorem 4.1. *Suppose that the data $m = 2k > 4, N$ admit the existence of a genus $\mathfrak{G}$ of maximal lattices of determinant N^2 and rank m. Then*

$$M_k(N)^* = \Theta(\mathfrak{G}^*),$$

where $\mathfrak{G}^*$ is the genus adjoint to $\mathfrak{G}$ and $\Theta(\mathfrak{G}^*)$ denotes the $\mathbb{C}$-vector space generated by the theta series $\theta(L)$, $L \in \mathfrak{G}^*$.

The statement above is false for $m = 4$ unless $S_k(N)^* = S_k(N)^{new}$, as follows from the work of Eichler [7] and Hijikata-Saito [10] on the basis problem. Anyway, our proof would not work here (because of convergence reasons and because here (and only here) the genus of maximal lattices is equal to its adjoint genus).

Before we sketch the proof of this theorem, we recall from Theorem 3.1 that the inclusion

$$\Theta(\mathfrak{S}^*) \subseteq M_k(N)^*$$

holds. To simplify the exposition, we only consider the case $N = p$. We have to study the map

$$\Lambda : \begin{cases} S_k(p) & \longrightarrow & \Theta(\mathfrak{S}^*) \\ g & \longmapsto & \sum_i \frac{1}{m(L_i)} < g, \theta(L_i) > \theta(L_i) \end{cases}$$

Here $m(L)$ is the number of automorphisms of the lattice L and the L_i run over representatives of the classes in the genus $(\mathfrak{S}^*)$; the bracket $<,>$ denotes the Petersson product for modular forms. It is a general fact ("pullback formulas" for Eisenstein series) that this map can also be described completely in terms of Hecke operators, the explicit form of the contribution of the bad place p depends on the genus at hand, see [3].

The case of newforms of level p was discussed in [3].

We just have to add for a Hecke eigenform f of level one an explicit description of the map Λ for the two-dimensional space

$$M(f) := \mathbb{C}\{f, f \mid V(p)\}.$$

Indeed, it is of the form

$$\begin{pmatrix} \Lambda(f) \\ \Lambda(f \mid V(p)) \end{pmatrix} = c \cdot L_2(f, 2k-2) \cdot \mathcal{A}_p \cdot \begin{pmatrix} f \\ f \mid V(p) \end{pmatrix}.$$

Here c is an unimportant constant, $L_2(f,s)$ denotes the symmetric square L-function attached to f and $\mathcal{A}_p$ is a certain 2×2-matrix (involving the "Satake parameters" α_p and β_p of f) which can be computed from [3]. The inclusion $\Theta(\mathfrak{S}^*) \subseteq M_k(p)^*$ already implies that the image of $M(f)$ under Λ is at most one-dimensional. An inspection of $\mathcal{A}_p$ shows that it is always different from the zero matrix (i.e. of rank one), in other words, $M(f)$ will always be mapped onto the one-dimensional space $\mathbb{C} \cdot f^{(p)} \subseteq M_k(p)^*$.

Remark 4.2. *The case of an arbitrary squarefree number N goes along the same line (Kronecker products of such 2×2-matrices have then to be considered). A more detailed analysis of these matrices $\mathcal{A}_p$ for arbitrary genera $\mathfrak{S}$ will be given elsewhere [4].*

By the same reasoning (or by applying the Fricke involution $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ to both sides of the theorem) we obtain

Corollary 4.3. *Under the same assumptions as in the theorem we have*

$$M_k(N)_* = \Theta(\mathfrak{S}),$$

where

$$\begin{aligned} M_k(N)_* &:= M_k(N)^* \mid_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \\ &= \{f \in M_k(N) \mid \forall p \mid N : \mathit{trace}^N_{\frac{N}{p}}(f) = 0\}. \end{aligned}$$

Remark: Both the theorem and the corollary are remarkable because they describe precisely the "old" part of $\Theta(\mathfrak{S}^*)$ and $\Theta(\mathfrak{S})$. From the point of view of [5] it may be of interest to study the trace of such an oldform: We consider the simplest case, $N = p$ and $f \in S_k(1)$ is a normalized Hecke eigenform. Then

$$\mathrm{trace}_1^p(f^{(p)}) = \mathrm{trace}_1^p(f - \frac{p}{p+1} a_f(p) f \mid V(p)) = \lambda \cdot f$$

with $\lambda = p+1-\frac{p}{p+1}a_f(p)^2 p^{-k+1}$. By an elementary estimate (see e.g. [16]) $|a_f(p)| < p^{\frac{k}{2}}(1+\frac{1}{p})$ and therefore λ cannot be zero. On the other hand, $f^{(p)}$ is a linear combination of the $\theta(L)$ with $L \in \mathfrak{S}^*$. The trace of such theta series is not understood at all, see [5]. The situation is completely different for $f^{(p)} \mid W_p^p \in S_k(p)_*$: this function is in $\Theta(\mathfrak{S})$ and the traces of the theta series are all zero. This fits well with the fact that $tr_p^p(f^{(p)} \mid W_p^p) = 0$.

5 Extremality

5.1 Generalities on analytic extremality

Definition 5.1. *A subspace $\mathcal{M} \subseteq M_k(N)$ has the* Weierstrass property $(\mathcal{W})$ *if the projection $\mathcal{M} \longrightarrow \mathbb{C}^r$ to the first $r = dim\mathcal{M}$ coefficients of the Fourier expansion*

$$f = \sum_{n \geq 0} a_n q^n \longmapsto (a_0, a_1, \ldots, a_{r-1})$$

is injective. If this holds, the unique element

$$F = F_{\mathcal{M}} \in \mathcal{M}$$

with Fourier expansion

$$F = 1 + \sum_{n \geq r} a_n q^n$$

is called the extremal modular form *in $\mathcal{M}$.*

If $\mathcal{M}$ contains (say, by definition) only modular forms with vanishing Fourier coefficient a_0, the definition of "Weierstrass property" has to be modified in the obvious way. Note that $(\mathcal{W})$ holds for $\mathcal{M}$ iff $(\mathcal{W})$ holds for the cuspidal subspace of $\mathcal{M}$, provided that the codimension of the cuspidal part in $\mathcal{M}$ is one.
The notion "Weierstrass property" is motivated by the connection of this property with ∞ being a Weierstrass points on the modular curve $X_0(N)$ if $\mathcal{M} = S_2(N)$, see e.g. [25].
Suppose now that we have a lattice L such that $\theta(L) \in \mathcal{M}$ for a space $\mathcal{M}$ with property $(\mathcal{W})$. Then we may call the lattice L analytically extremal with respect to $\mathcal{M}$ if

$$\theta(L) = F_{\mathcal{M}}.$$

In particular, such an analytically extremal lattice satisfies

$$\min_Q(L) \geq \dim(\mathcal{M}).$$

In this generality this definition was introduced in [27].

Of course these notions only make sense, if we know interesting classes of such distinguished subspaces $\mathcal{M}$.

Example 5.2. *(1) Clearly, for any lattice L, the one-dimensional space $\mathcal{M} := \mathbb{C} \cdot \theta(L)$ has the property $(\mathcal{W})$ and then L is extremal with respect to this space.*
(2) The full space $M_k(1)$ of modular forms of level 1 has the Weierstrass property and the well-known Leech lattice is then an $\mathcal{M} = M_{12}(1)$- extremal lattice.
(3) The spaces of modular forms for the Fricke groups considered by Quebbemann [23, 24] in his work on modular lattices all have the Weierstrass property.

5.2 Analytic extremality with respect to $M_k(N)^*$

In general, neither the spaces $S_k(N)$ nor $S_k(N)^{new}$ (or versions of it appropriately enlarged by some Eisenstein series) have the Weierstrass property. In the case of squarefree level N we showed in [1] that the intermediate space $S_k(N)^*$ (and therefore also $M_k(N)^*$) has the property $(\mathcal{W})$, therefore there is an extremal modular form

$$F_{N,k} := F_{M_k(N)^*}$$

in this case.

Definition 5.3. *A maximal lattice L of level N and determinant N^2 in dimension $m = 2k$ is called* dual-extremal, *if $\theta(\sqrt{N}L^\#) = F_{N,k}$.*

Remark 5.4. *(1) Our definition allows to define analytic extremality for all squarefree levels. This is in contrast to the situation studied by Quebbemann [23, 24]; we also note that (again in contrast to [23, 24]) we cannot give an explicit construction for the extremal modular form.*
(2) The additional information $\Theta(\mathfrak{S}^*) = M_k(N)^*$ is not necessary for the definition of dual-extremal lattices; it shows however that the space $\mathcal{M} = M_k(N)^*$ is the appropriate (smallest) space with Weierstrass property containing all the theta series for lattices adjoint to maximal ones.
(3) The notion of (analytic) extremality has a geometric counterpart. Whereas extremal lattices are those for which the theta series has certain analytic extremality properties, extreme lattices are those that realise a local maximum of the sphere packing density function on the space of all lattices in n-dimensional euclidean space. The dual-extreme *lattices are the local maxima of the* Bergé-Martinet*-function*

$$\gamma'(L) = \sqrt{\gamma(L)\gamma(L^\#)} = 2(min_Q(L)min_Q(L^\#))^{1/2}$$

simultaneously considering both lattices, L and its dual. So one could expect that the notion of dual-extremality describes the analytic counterpart of this definition, which is in general not the case, since only the theta series of the dual lattice of a maximal lattice satisfies an analytic extremality condition. However, the dual-extremal lattices of level 2 and dimension

$24\ell + 4$ have the property that both lattices, L and its dual, realise a local maximum of the sphere-packing density and hence those dual-extremal lattices are also dual-extreme (cf. Proposition 6.1).
(4) Using an R-basis of $\bigoplus M_k(N)_*$ for $N = 2$ and $N = 3$ as given in Section 5.4 we checked with MAGMA [18] that the first 200 coefficients of the extremal modular form $f = 1 + aq^{\dim(M_k(N)^*)} + \ldots \in M_k(N)^*$ and of $f(-\frac{1}{Nz}) \in M_k(N)_*$ are even non-negative integers and that $a > 0$ for $k \leq 242$, $k \equiv 2 \pmod 4$ and $N = 2, 3$.

Example 5.5. *Let D be a rational definite quaternion algebra ramified exactly at the prime p. Then any lattice L of level p in the quadratic space (D, n), where n is the norm form, is a maximal even lattice. These lattices L are fractional left-ideals for some maximal order in D. The non principal L satisfy $\min_n(L) \geq 2$. If the class number (the number of isomorphism classes of left-ideals for a fixed maximal order in D) is two, then $\dim M_2(p)^* = 2$ since $S_2(p)^* = S_2(p)^{new}$ and any non-principal L is dual-extremal. Note that the definite quaternion algebras over $\mathbb{Q}$ with class number two are classified by the work of Kirschmer and Voight [13] confirming the work by Pizer [22]: $N = p \in \{11, 17, 19\}$ and $N = 2{\cdot}3{\cdot}5, N = 2 \cdot 3 \cdot 7, N = 2 \cdot 3 \cdot 13, N = 2 \cdot 5 \cdot 7$. The condition $S_2(N)^* = S_2(N)^{new}$, which is quite special for the case $m = 4$, is automatically satisfied if $N = p$, but never in the other cases of class number two as can be seen by evaluating the dimension formula for $S_2(N)^*$.*

5.3 A remark about extremal modular forms of level p and weight divisible by $p - 1$

Proposition 5.6. *Let p be a prime. Assume that the weight k is divisible by $p - 1$. Then any modular form $f \in M_k(1)$ with Fourier expansion*

$$f \equiv 1 + \sum_{n \geq d} a_n q^n \bmod p \qquad (d = \dim M_k(1))$$

satisfies

$$f \equiv 1 \bmod p$$

Corollary 5.7. *Let $p \geq 5$ be a prime and let k be a multiple of $p - 1$. Then the extremal modular form $g \in M_k(p)^*$ satisfies*

$$g \equiv 1 \bmod p$$

Proof. (of Proposition 5.6) There exists a modular form $\mathcal{E}$ of weight k with $\mathcal{E} \equiv 1 \bmod p$. For $p \geq 5$ we may take an appropriate power of the Eisenstein series E_{p-1} of weight $p - 1$. For $p = 2$ or $p = 3$ we can take a suitable monomial $E_4^\alpha \cdot E_6^\beta$. Therefore we can write f as

$$f = \mathcal{E} + F$$

with

$$F = \sum_{n \geq 1} b_n q^n$$

such that the first $d-1$ coefficients b_i are congruent zero mod p. For $1 \le i \le d-1$ we choose $f_j \in S_k(1)$ with integral Fourier coefficients $c_{i,n}$ such that for $1 \le i, j \le d-1$

$$c_{i,j} = \delta_{i,j}$$

Such cusp forms always exist, see e.g. [15, Theorem 4.4].

Then

$$f = \mathcal{E} + \sum a_i f_i + H$$

such that the first d Fourier coefficients of H are zero, hence H is identically zero. The assertion follows. □

To prove the corollary we note that (by [1]) g is equivalent mod p to a modular form $G \in M_{k+(p-1)(k-1)}(1)$ provided that $p \ge 5$. We apply Proposition 5.6 to this G.

Remark: Using a suitable interpretation of the congruence of modular forms, it is not necessary in the statements above to assume that the Fourier coefficients of the modular forms are rational.

Remark: It would be desirable to include the cases $p = 2$ and $p = 3$ in the corollary.

5.4 $M(N)_*$ as a module over the ring of modular forms of level one.

The orthogonal sum of a maximal lattice with an even unimodular lattice is again a maximal lattice. This elementary observation corresponds to fact that $M(N)_* = \bigoplus_k M_k(N)_*$ is a module over the ring of modular forms of level one. The corresponding module structure for $M(N)^*$ is defined by multiplying $f \in M_k(N)^*$ with $g \mid_\ell \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in \mathbb{C}[[q^N]]$ for $g \in M_\ell(1)$.

From the dimension formula we obtain

$$\dim M_k(N)^* =: d_k(N) = \frac{(k-1)N}{12} + \frac{1}{2} - \frac{1}{4}\left(\frac{-1}{(k-1)N}\right) - \frac{1}{3}\left(\frac{-3}{(k-1)N}\right)$$

for even $k \ge 2$ and $\dim M_0(N)^* = 1$. The next remark follows from the periodicity of the Jacobi symbol.

Remark 5.8. *$d_{k+12}(N) = d_k(N) + N$ for all even $k \ge 2$.*

In connection with theta series of lattices of prime level N (or more general when N is a product of an odd number of primes, see Proposition 3.4) the structure of the direct sum $M'(N)^* := \oplus_{k \equiv 2 \pmod 4} M_k(N)^*$ as a module over the graded ring $R := \oplus_{4|k} M_k(1) = \mathbb{C}[\theta(\mathbb{E}_8), \Delta]$ is of interest.

Theorem 5.9. *$M'(N)^*$ is a free R-module of rank N generated by homogeneous elements of weight ≤ 14.*

Proof. We first note that Remark 5.8 implies that $(\sum_{k \equiv 2 \pmod 4} d_k(N) x^k)(1-x^4)(1-x^{12}) =$

$$d_2(N)x^2 + (d_6(N) - d_2(N))x^6 + (d_{10}(N) - d_6(N))x^{10} + (d_{14}(N) - d_{10}(N) - d_2(N))x^{14}.$$

so we obtain in total at least $d_{14}(N) - d_2(N) = N$ generators of the R-module $M'(N)^*$.
On the other hand, by the Weierstrass property of $M_k(N)^*$, there are $d_k(N)$ elements in $M_k(N)^*$ whose Fourier development starts with $1, q, q^2, \ldots, q^{d_k(N)-1}$. If $d_k(N) \leq N$ then these are clearly linearly independent over $\mathbb{C}((q^N))$ and hence also over the quotient field of R.
Taking the series $p_i = q^i + \ldots \in M^*_{k_i}(N)$ of minimal weight $k_i \equiv 2 \pmod 4$ for $i = 0, \ldots, N-1$ we hence obtain a sequence $(p_0, \ldots, p_{N-1})$ which is R-linear independent and generates an R-submodule $\langle p_0, \ldots, p_{N-1}\rangle_R$ of $M'(N)$ whose Hilbert series is at least as big as the one of $M'(N)$. Therefore

$$M'(N) = Rp_0 \oplus Rp_1 \oplus \ldots \oplus Rp_{N-1}.$$

□

An analogous proof shows

Theorem 5.10. *$\bigoplus M_{4k}(N)^*$ is a free R-module of rank N generated by homogeneous elements of weight ≤ 16.*

From the point of view of modular forms the full space

$$M(N)^* := \bigoplus M_{2k}(N)^*$$

deserves attention as well as a module over the full graded ring $M(1) = \mathbb{C}[E_4, E_6]$ of modular forms of level one.

Lemma 5.11. *Fix N and put $d_k := d_k(N) = \dim(M_k(N)^*)$. Then $d_0 = 1$ and for all even $k \geq 2$*

$$d_{10+k} - d_{6+k} - d_{4+k} + d_k = 0.$$

Moreover $d_{10} + d_8 - d_4 - d_2 = N$.

Proof. The first statement follows from the multiplicativity of the Jacobi symbol and the fact that

$$\left(\frac{-1}{(k-1)}\right) + \left(\frac{-1}{(k+9)}\right) - \left(\frac{-1}{(k+3)}\right) - \left(\frac{-1}{(k+5)}\right) = 0 \text{ for all even } k \geq 2$$

because this Jacobi symbol has period 4 and

$$\left(\frac{-3}{(k-1)}\right) + \left(\frac{-3}{(k+9)}\right) - \left(\frac{-3}{(k+3)}\right) - \left(\frac{-3}{(k+5)}\right) = 0 \text{ for all even } k \geq 2$$

because this symbol has period 6.
Similarly we obtain $d_{10} + d_8 - d_4 - d_2 = N$. □

In the same spirit as Theorem 5.9 we now can prove

Theorem 5.12. *$M(N)^*$ is a free $\mathbb{C}[E_4, E_6]$-module of rank N generated by homogeneous elements of weight ≤ 10.*

Proof. Put $d_k := d_k(N)$. From Lemma 5.11 $(\sum d_{2k}x^{2k})(1-x^4)(1-x^6) =$

$$d_0 + d_2x^2 + (d_4 - d_0)x^4 + (d_6 - d_2 - d_0)x^6 + (d_8 - d_4 - d_2)x^8 + (d_{10} - d_6 - d_4 + d_0)x^{10}$$

so we obtain in total at least $d_{10} + d_8 - d_4 - d_2 = N$ generators of the $\mathbb{C}[E_4, E_6]$-module $M(N)^*$ of weight ≤ 10. We now proceed as in the proof of Theorem 5.9. □

6 Examples for dual-extremal maximal lattices.

This section lists some examples of dual-extremal maximal lattices of small level N and small dimension m. For $N = 2$ and $N = 3$, one may deduce the classification of all dual-extremal lattices from suitable known classifications of unimodular lattices. For the higher levels $N \geq 5$ we use Kneser's neighboring method [14] to list the whole genus of maximal lattices together with the mass formula to double check the completeness of the result. The computer calculations were performed with MAGMA [18]. Gram matrices for the new lattices are available in [21].

6.1 $N = 2$.

Let L be a maximal 2-elementary lattice of exact level 2 and even dimension $m := \dim(L) = 2k \equiv 4 \pmod 8$. Then L is the even sublattice of an odd unimodular lattice M and $L^{\#} = M \cup v + M$ where $2v \in M$ is a characteristic vector of M, i.e. $(2v, x) \equiv (x, x) \pmod 2$ for all $x \in M$. If $\mu = \min(M) = 2\min_Q(M)$ and 4σ is the minimal norm of a characteristic vector in M, then $4\sigma \equiv m \pmod 8$ and $\min_Q(\sqrt{2}L^{\#}) = \min(\mu, \sigma)$. Philippe Gaborit proved in [9] that for $m \neq 23$

$$\mu + \frac{\sigma}{2} \leq 1 + \frac{m}{8} \quad (\star).$$

Lattices achieving this bound are called s-extremal. We use $(\star)$ to show that dual-extremal lattices L satisfy $\min_Q(L^{\#}) = \frac{1}{2}\lfloor \frac{k+4}{6} \rfloor$.

Proposition 6.1. *Let L be a dual-extremal maximal lattice of level 2 and dimension $m = 24\ell + 4$. Then $L^{\#}$ has minimum $1/2 + \ell$ and all layers of L and of $L^{\#}$ form spherical 5-designs. In particular L and $L^{\#}$ are strongly perfect. If M is one of the three odd unimodular lattices with even sublattice L, then M is s-extremal of minimum $1/2 + \ell$.*

Proof. Let $\mu := 2\min_Q(M)$ and $\sigma := 2\min_Q(L^{\#} - M)$. Since L is dual-extremal μ and σ are both $\geq 1 + 2\ell$. By the bound in [9] we obtain $\mu + \frac{\sigma}{2} \leq \frac{3}{2} + 3\ell$ hence $\mu = \sigma = 1+2\ell$. The design property follows from the fact that $\dim(M_{12\ell+2}(2)^*) = \dim(M_{12\ell+4}(2)^*) = \dim(M_{12\ell+6}(2)^*) = 2\ell + 1$. □

Similarly we obtain

Proposition 6.2. *Let L be a dual-extremal maximal lattice of level 2 and dimension $m = 24\ell - 4$ and let M be one of the three odd unimodular lattices with even sublattice L. Then M is s-extremal of minimum ℓ. The minimum of Q on $L^{\#} - M$ is $\ell + 1/2$ and the minimal vectors of $L^{\#}$ (which are also those of L and those of M) form a spherical 3-design, which means that $L^{\#}$, L and M are all strongly eutactic. The lattice M is s-extremal.*

Proof. Let $\mu := 2\min_Q(M)$ and $\sigma := 2\min_Q(L^{\#} - M)$. Since L is dual-extremal μ and σ are both $\geq 2\ell$. Since $\sigma \equiv \frac{m}{4} \pmod 2$ it is odd $\sigma \geq 2\ell + 1$. By the bound $(\star)$ above we obtain $\mu + \frac{\sigma}{2} \leq \frac{1}{2} + 3\ell$ hence $\mu = 2\ell$, $\sigma = 1 + 2\ell$ and M is s-extremal. □

Proposition 6.3. *Let L be a dual-extremal maximal lattice of level 2 and dimension $m = 24\ell + 12$. Then $\min_Q(L^{\#}) = \ell + 1/2$.*

Proof. Let M be one of the three odd unimodular lattices with even sublattice L. Let $\mu := 2\min_Q(M)$ and $\sigma := 2\min_Q(L^\# - M)$. Since L is dual-extremal, $\min(\mu,\sigma) \geq 2\ell+1$. By Gaborit's bound $\mu + \frac{\sigma}{2} \leq 3\ell + 2 + \frac{1}{2}$. If $\min(\mu,\sigma) \geq 2\ell + 2$, then $\mu + \frac{\sigma}{2} \geq 3\ell + 3$ contradicting the bound above. □

Corollary 6.4. *A dual-extremal lattice L of level 2 and dimension $2k \equiv 4 \pmod 8$ satisfies* $\min_Q(\sqrt{2}L^\#) = \lfloor \frac{k+4}{6} \rfloor$.

6.1.1 $m = 4$

Here the root lattice $\mathbb{D}_4$ is the unique maximal 2-elementary lattice and dual-extremal.

6.1.2 $m = 12$

The two root lattices $\mathbb{D}_4 \perp \mathbb{E}_8$ and $\mathbb{D}_{12}$ are all maximal 2-elementary lattices and both are dual-extremal.

6.1.3 $m = 20$

Let L be a maximal 2-elementary lattice of dimension 20. Then $L \perp \mathbb{D}_4$ is contained in some even unimodular lattice U of dimension 24. Since L is maximal it is the orthogonal complement $\mathrm{Comp}(\mathbb{D}_4)$ of $\mathbb{D}_4$ in U and $L^\#$ is the projection of U to $\mathbb{D}_4^\perp$. Since $\min_Q(\sqrt{2}L^\#) \geq 2$, all roots $u \in U$ with $Q(u) = 1$ are either in $\mathbb{D}_4$ or perpendicular to this sublattice. Hence $\mathbb{D}_4$ is an orthogonal summand of the root system of U, which is therefore either $\mathbb{D}_4^6$ or $\mathbb{D}_4 \perp \mathbb{A}_5^4$. Both lattices U contain a unique $\mathrm{Aut}(U)$-orbit of such sublattices $\mathbb{D}_4$ yielding the two dual-extremal 2-elementary lattices of dimension 20.

6.1.4 $m = 28$

Let L be a maximal 2-elementary lattice of dimension 28 and M be an odd unimodular lattice containing L. If L is dual-extremal, then $\min_Q(L^\#) \geq 3/2$ and hence M has Q-minimum $3/2$. The 28-dimensional unimodular lattices of Q-minimum $3/2$ are all classified in [2]. There are 38 isometry classes of such lattices, two of which have a characteristic vector of norm 4. The other 36 lattices give rise to 31 even sublattices L which are all dual-extremal. By Proposition 6.1 the 6720 minimal vectors of $L^\#$ as well as all layers of L and $L^\#$ form spherical 5-designs and hence $L^\#$ is a strongly perfect lattice (see [28]). The next dimension where such a phenomenon occurs is $m = 52$, where $\min_Q(L^\#) = 5/2$. Then any unimodular sublattice M (with even sublattice L) is an s-extremal lattice of Q-minimum $5/2$ in the sense of [9]. Up to now, no such lattice is known.

6.2 $N = 3$.

A dual-extremal lattice L of dimension $m = 2k \equiv 4 \pmod 8$ satisfies $\min_Q(\sqrt{3}L^\#) \geq \frac{k+2}{4}$

6.2.1 $m = 4$.

Here $\mathbb{A}_2 \perp \mathbb{A}_2$ is the unique maximal 3-elementary lattice and this is dual-extremal.

6.2.2 $m = 12$.

The 3-elementary maximal lattices are $\mathbb{A}_2 \perp \mathbb{A}_2 \perp \mathbb{E}_8$ and $\mathbb{E}_6 \perp \mathbb{E}_6$, the latter is dual-extremal.

6.2.3 $m = 20$.

Let L be a dual-extremal 3-elementary lattice of dimension 20. Then $L \perp \mathbb{A}_2 \perp \mathbb{A}_2$ is contained in an even unimodular lattice of dimension 24. As for $N = 2$ the dual-extremality of L implies that the root system of U is $\mathbb{A}_2^{12}$ and there is a unique such lattice L.

6.2.4 $m = 28$.

Let L be a dual-extremal 3-elementary lattice of dimension 28 and let U be an even unimodular lattice of dimension 32 containing $L \perp \mathbb{A}_2 \perp \mathbb{A}_2$. Then $2\min_Q(L^{\#}) \geq 8/3 > 2$ implies that L has no roots and that the root system of U is $\mathbb{A}_2 \perp \mathbb{A}_2$. By [11] the mass of such lattices U is > 41610 so there are more than $72 \cdot 41610$ such lattices. Every lattice $L \perp \mathbb{A}_2 \perp \mathbb{A}_2$ is contained in 8 unimodular lattices, so it follows from the discussion below that there are at least $9 \cdot 41610$ dual-extremal lattices. The lattice $L^{\#}$ is the projection of U to $(\mathbb{A}_2 \perp \mathbb{A}_2)^{\perp}$, so

$$L^{\#} = \{x \in (\mathbb{A}_2 \perp \mathbb{A}_2)^{\perp} \mid \text{ there is some } z \in (\mathbb{A}_2 \perp \mathbb{A}_2)^{\#} \text{ such that } y := x + z \in U\}.$$

Here we may assume that z is minimal in its class modulo $\mathbb{A}_2 \perp \mathbb{A}_2$. Then $(z, z) \in \{0, \frac{2}{3}, \frac{4}{3}\}$. If $x \neq 0$ then $(y, y) \geq 4$ and $(x, x) = (y, y) - (z, z) \geq 4 - \frac{4}{3} = \frac{8}{3}$. This shows that for all these lattices U the orthogonal L of the root sublattice of U is dual-extremal.

We list these results and the ones found for level $N = 5, 7, 11$ resp. $N = 6, 10$ in the following tables, with lines labeled by the level N and columns labeled by the dimension m. Each entry is the triple (h, h_{ext}, min) giving the class number h of the genus of maximal lattices, the number h_{ext} of isometry classes of dual-extremal maximal lattices as well as the minimum $\min_Q(\sqrt{N}L^{\#})$. A "$\cdot$" instead of h indicates that we did not compute the full genus. Note that for dimension $m = 4$, the classification follows from Example 5.5.

m	4	12	20	28
$N = 2$	$(1, 1, 1)$	$(2, 2, 1)$	$(18, 2, 2)$	$(\cdot, 31, 3)$
$N = 3$	$(1, 1, 1)$	$(2, 1, 2)$	$(\cdot, 1, 3)$	$(\cdot, \geq 9 \cdot 41610, 4)$
$N = 5$	$(1, 1, 1)$	$(5, 2, 2)$	$(329, 2, 4)$	
$N = 7$	$(1, 1, 1)$	$(12, 0, 4)$		
$N = 11$	$(3, 1, 2)$	$(36, 2, 5)$	$(\cdot, \geq 1, 10)$	

Remark 6.5. *It is interesting to note that for level $N = 11$ and dimension $m = 20$, the extremal theta series is $1+132q^{10}+660q^{12}+1320q^{13}+2640q^{14}+\ldots$ so any dual-extremal lattice L satisfies $\min_Q(\sqrt{11}L^{\#}) = 10 > \dim(M_{10}(11)^*) = 9$. So Corollary 6.4 does not hold in general for arbitrary levels. Note that here the 132 minimal vectors of L form a spherical 2-design. We constructed such a lattice L as the orthogonal complement $L = \mathrm{Comp}(D) \leq \Lambda_{24}$ in the Leech lattice, where D is the dual-extremal lattice of level 11 and dimension 4.*

m	8	16
$N=6$	$(3,1,2)$	$(45,2,4)$
$N=10$	$(6,1,3)$	$(228,7,6)$

References

[1] T. Arakawa, S. Böcherer, Vanishing of certain spaces of elliptic modular forms and some applications.J.reine angew.Math.559, 25-51(2003)

[2] R. Bacher, B. Venkov, Réseaux entiers unimodulaires sans racines en dimensions 27 et 28. in Réseaux euclidiens, designs sphériques et formes modulaires, 212–267, Monogr. Enseign. Math., 37, Enseignement Math., Geneva, 2001.

[3] S. Böcherer, The genus version of the basis problem I. In: Automorphic Forms and Zeta Functions. (Editors: S.Böcherer, T.Ibukiyma, M.Kaneko, F.Sato). World Scientific 2006

[4] S. Böcherer, The genus version of the basis problem II: The case of oldforms. In preparation.

[5] S. Böcherer, J. Funke, R. Schulze-Pillot, Trace operators and theta series. J. Number Theory 78, 119-139 (1999)

[6] S. Böcherer, R. Schulze-Pillot, Siegel modular forms and theta series attached to quaternion algebras. Nagoya Math. J. 121, 35-96 (1991)

[7] M. Eichler, The basis problem for modular forms and the traces of the Hecke operators. In: Modular Functions of one variable I. Lecture Notes in Math. 320. Springer 1973

[8] J. Funke, Spuroperator und Thetareihen quadratischer Formen. Diplomarbeit Universität Köln, 1994

[9] P. Gaborit, A bound for certain s-extremal lattices and codes. Archiv der Mathematik 89 (2007) 143-151.

[10] H. Hijikata, H. Saito, On the representability of modular forms by theta series. In: Number Theory, Algebraic Geometry and Commutative Algebra (in honour of Y.Akizuki) Tokyo 1973

[11] O. King, A mass formula for unimodular lattices with no roots. Math. Comp. 72 (2003), 839-863.

[12] Y. Kitaoka, A remark on the transformation properties of theta functions associated to positive quadratic forms. J.Number Theory 12, 224-229(1980) Modular Forms. Springer 1989

[13] M. Kirschmer, J. Voight, Algorithmic enumeration of ideal classes for quaternion orders. SIAM Journal on Computing (to appear)

[14] M. Kneser, Klassenzahlen definiter quadratischer Formen, Archiv der Math. 8 (1957) 241-250.

[15] S.Lang, *Introduction to modular forms.* Grundlehren 222, Springer 1976

[16] W.Kohnen, A simple remark on eigenvalues of Hecke operators and Siegel modular forms. Abh.Math.Sem.Univ.Hamburg 57 (1987), 33-35

[17] W-C.W.Li, Newforms and functional equations. Math.Ann. 212, 285–315 (1975)

[18] *The Magma Computational Algebra System for Algebra, Number Theory and Geometry.* available via the magma home page http://www.maths.usyd.edu.au:8000/u/magma/

[19] Annika Meyer, On extremal maximal isotropic codes of Type I-IV. preprint, RWTH Aachen

[20] T. Miyake, *Modular Forms.* Springer 1989

[21] G. Nebe, N.J.A. Sloane, *A catalogue of lattices.* http://www.research.att.com/∼njas/lattices/

[22] A. K. Pizer, Type number of Eichler orders, J. reine u. angew. Mathematik 264, 76-102 (1973)

[23] H.-G. Quebbemann, Modular lattices in euclidean spaces. J.Number Theory 54, 190-202 (1995)

[24] H.-G. Quebbemann, Atkin-Lehner eigenforms and strongly modular lattices. Enseign. Math. 43, 55-65 (1997)

[25] D. Rohrlich, Weierstrass points and modular forms. Illinois J.Math. 29, 134-141 (1985)

[26] W. Scharlau, *Quadratic and Hermitian Forms.* Springer Grundlehren 270 (1985)

[27] R. Scharlau, R. Schulze-Pillot, Extremal Lattices. In: Algorithmic Algebra and Number Theory (Editors: B.H.Matzat, G.-M.-Greuel, G.Hiss). Springer 1998

[28] B. Venkov, Réseaux parfaits et designs sphériques. in Réseaux euclidiens, designs sphériques et formes modulaires, 10–86, Monogr. Enseign. Math., 37, Enseign. Math., Geneva, 2001.

[29] J.-L. Waldspurger, Engendrement par des séries de théta de certains espaces de formes modulaires. Inv. Math. 50, 135-168 (1979)