Maximal finite subgroups and minimal classes

by Renaud COULANGEON and Gabriele NEBE

ABSTRACT. We apply Voronoi's algorithm to compute representatives of the conjugacy classes of maximal finite subgroups of the unit group of a maximal order in some simple $\mathbf{Q}$-algebra. This may be used to show in small cases that non-conjugate orders have non-isomorphic unit groups.
Keywords: unit groups of orders, Voronoi algorithm, minimal classes
MSC: primary: 20H05; secondary: 11F75, 20H10, 11H55

1. INTRODUCTION

Let A be some simple $\mathbf{Q}$-algebra and let Λ and Γ be two maximal orders in A. If A is not a division algebra, then the order Λ is generated by its unit group $\Lambda^\times$ as a $\mathbf{Z}$-lattice (see Lemma 2.1). So $\Lambda^\times$ and $\Gamma^\times$ are conjugate in $A^\times$ if and only if the two orders Λ and Γ are conjugate, which can be decided with the arithmetic theory of orders exposed in the next section. By the theorem of Skolem and Noether we hence have that the unit groups are *conjugate* if and only if Λ and Γ are isomorphic as orders over the center of A. The motivation of this paper is to develop tools for deciding whether the two unit groups are *isomorphic*, which is in general much more difficult than the conjugacy problem. In fact this innocent question was raised by Oliver Braun during his work on the paper [5] that grew out of his Bachelor thesis in Aachen supervised by the second author.

One invariant of the isomorphism class of $\Lambda^\times$ is the number of conjugacy classes of maximal finite subgroups. Our main result is that these maximal finite subgroups arise as automorphism groups of *well rounded minimal classes*, which will be defined in Section 5. The basic idea underlying this approach is already apparent in Ryškov's paper [16] on the computation of the finite subgroups of $\mathrm{GL}_n(\mathbf{Z})$. Nevertheless, whereas Ryškov's classify

all finite subgroups and then develop *ad hoc* arguments to determine the maximal ones, our method permits in principle to solve the problem directly. Precisely, a refinement of the classical Voronoi algorithm, involving Bergé-Martinet-Sigrist's equivariant version of Voronoi's theory [4], is applied to compute the cellular decomposition of a suitable retract of a cone of positive definite Hermitian forms, and therewith also the (finitely many) conjugacy classes of maximal finite subgroups of $\Lambda^\times$. As will be illustrated in Section 8, this turns out to be enough, in some cases, to distinguish between non-isomorphic unit groups. The argument can of course not be reversed: non-isomorphic unit groups might have the same conjugacy classes of maximal finite subgroups. Note also that, as in the classical case of $\mathrm{GL}_n(\mathbf{Z})$, the obtained cellular decomposition can be used to compute the integral homology of $\Lambda^\times$. The relevance of Voronoi theory in such homology computations was first highlighted in the works of Soulé [17, 18] and Ash [1, 2], and it has given rise since then to numerous developments (we refer the interested reader to P. Gunnels' appendix of [19] which provides an excellent survey on this topic, and to [13, 14] for recent related works on Bianchi groups).

The methods apply to arbitrary (semi)-simple $\mathbf{Q}$-algebras, though we are mainly interested in the case where A is a matrix ring over either a imaginary quadratic number field or a definite rational quaternion algebra. For these algebras we may ease these computations by adopting a projective notion of minimal vectors as exposed in Section 7.

2. Conjugacy classes of maximal orders

The theory in this section is well known and can be extracted from the two books [15] and [6]. However, we did not find a self-contained short exposition of the proof of Theorem 2.4, so we repeat the details here for the reader's convenience. Let A be a simple $\mathbf{Q}$-algebra. Then $A = M_n(K)$ for some rational division algebra K with center $Z(K)$. Let R be the maximal order in $Z(K)$ and choose some maximal R-order $\mathcal{O}$ in K. An $\mathcal{O}$-lattice L of rank n is a finitely generated $\mathcal{O}$-submodule of the right K-module $V := K^n$ that contains a K-basis. By Steinitz-theorem (see for instance [15, Theorem 4.13, Corollary 35.11]) there are right ideals $\mathfrak{c}_1, \dots \mathfrak{c}_n$ of $\mathcal{O}$ and a basis $(e_1, \dots, e_n)$ of V such that

$$L = e_1\mathfrak{c}_1 \oplus \dots \oplus e_n\mathfrak{c}_n.$$

The family $(\mathfrak{c}_i, e_i)_{1\le i\le n}$ is called a *pseudo-basis* of L. The *Steinitz-invariant* of L, denoted $\mathrm{St}(L)$, is the class

$$\mathrm{St}(L) := [\mathfrak{c}_1] + \cdots + [\mathfrak{c}_n]$$

in the group $\mathrm{Cl}(\mathcal{O})$ of stable isomorphism classes of right $\mathcal{O}$-ideals and does not depend of the choice of a pseudo-basis. By Eichler's theorem (see [15, Theorem (35.14)]) the reduced norm

$$\mathrm{nr} : \mathrm{Cl}(\mathcal{O}) \to \mathrm{Cl}_K(R)$$

induces a group isomorphism between $\mathrm{Cl}(\mathcal{O})$ and the ray class group $\mathrm{Cl}_K(R)$, the quotient of the ideal group of R modulo those principal ideals αR for which $\iota(\alpha) > 0$ for all real places ι of $Z(K)$ that ramify in K.

If $n \ge 2$ (which we assume in the following) then, as a consequence of Corollary 35.13 of [15], two lattices $L_1, L_2 \le V$ are isomorphic as $\mathcal{O}$-modules, if and only if they have the same Steinitz-invariant. In particular, L is isomorphic to $L(\mathfrak{c})$ where

$$L(\mathfrak{c}) = e_1\mathcal{O} \oplus \ldots \oplus e_{n-1}\mathcal{O} \oplus e_n\mathfrak{c}$$

for any ideal $\mathfrak{c}$ with $[\mathfrak{c}] = \mathrm{St}(L)$. The endomorphism ring

$$\mathrm{End}_{\mathcal{O}}(L) = \{X \in M_n(K) \mid XL \subseteq L\}$$

is a maximal order in $\mathrm{End}_K(V) \cong A$. In fact any maximal order in A is obtained this way (see [15, Corollary 27.6]). If $[\mathfrak{c}] = \mathrm{St}(L)$ then $\mathrm{End}_{\mathcal{O}}(L)$ is conjugate in $\mathrm{GL}_n(K)$ to

$$\mathrm{End}_{\mathcal{O}}(L(\mathfrak{c})) = \Lambda(\mathfrak{c}) := \begin{pmatrix} \mathcal{O} & \ldots & \mathcal{O} & \mathfrak{c}^{-1} \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{O} & \ldots & \mathcal{O} & \mathfrak{c}^{-1} \\ \mathfrak{c} & \ldots & \mathfrak{c} & \mathcal{O}' \end{pmatrix}$$

where $\mathcal{O}' = O_l(\mathfrak{c}) = \{x \in K \mid x\mathfrak{c} \subseteq \mathfrak{c}\}$.

LEMMA 2.1. *For $n \ge 2$ any maximal order Λ in $A = M_n(K)$ is generated as a $\mathbf{Z}$-order by its unit group.*

Proof. Without loss of generality let $\Lambda = \Lambda(\mathfrak{c})$ and let

$$(x_1, \ldots, x_d),\ (y_1, \ldots, y_d),\ (z_1, \ldots, z_d)$$

be $\mathbf{Z}$-bases of $\mathcal{O}$, $\mathfrak{c}$, respectively $\mathfrak{c}^{-1}$. We denote by e_{ij} the matrix units in $M_n(K)$ having an entry 1 at i,j and 0 elsewhere, and $I_n = e_{11} + \ldots + e_{nn}$ the

unit matrix. Let X be the $\mathbf{Z}$-order spanned by $\Lambda(\mathfrak{c})^\times$. Since I_n and $I_n + x_k e_{ij}$ $\in \Lambda(\mathfrak{c})^\times$ we obtain that $x_k e_{ij} \in X$ for all $k = 1, \dots, d$, $1 \le i \ne j \le n-1$. Similarly $y_k e_{ni}$ and $z_j e_{in}$, as well as $y_k z_j e_{nn}$ and $z_j y_k e_{ii}$ are in X for all $i = 1, \dots n-1$, $k, j = 1, \dots, d$. As the $y_k z_j$ generate $\mathcal{O}'$ and the $z_j y_k$ generate $\mathcal{O}$ the order X contains $\Lambda(\mathfrak{c})$. □

COROLLARY 2.2. *Let Λ and Γ be two maximal orders in the simple algebra A and assume that A is not a division algebra. Then $\Lambda^\times$ and $\Gamma^\times$ are conjugate in $A^\times$ if and only if Λ and Γ are conjugate.*

A separating invariant of the conjugacy classes of maximal orders in A can be constructed in a suitable class group of the center of A.

DEFINITION 2.3. Let $\mathrm{Cl}_K(n) := \mathrm{Cl}_K(R)/\langle \mathrm{nr}(\mathfrak{a})^n \mid \mathfrak{a} \trianglelefteq \mathcal{O} \rangle$ denote the quotient of the ray class group $\mathrm{Cl}_K(R)$ defined above modulo the n-th powers of the reduced norms of the two-sided $\mathcal{O}$-ideals.

Note that the subgroup $\langle \mathrm{nr}(\mathfrak{a})^n \mid \mathfrak{a} \trianglelefteq \mathcal{O} \rangle$ can be obtained from the discriminant of K. In particular it does not depend on the choice of the maximal order $\mathcal{O}$. Also if K is commutative then $\mathrm{Cl}_K(n) = \mathrm{Cl}(K)/\mathrm{Cl}(K)^n$ is just the class group of K modulo the n-th powers.

THEOREM 2.4. *Let $A = M_n(K)$ be a simple $\mathbf{Q}$- algebra and $\mathcal{O}$ a maximal order in K. For any two right $\mathcal{O}$-ideals $\mathfrak{c}$ and $\mathfrak{c}'$, the corresponding maximal orders $\Lambda(\mathfrak{c})$ and $\Lambda(\mathfrak{c}')$ are conjugate in $A^\times = \mathrm{GL}_n(K)$ if and only if $\mathrm{nr}([\mathfrak{c}]) = \mathrm{nr}([\mathfrak{c}'])$ in $\mathrm{Cl}_K(n)$.*

Proof. We use the approach in [6, Section VI.8]. Let $\Gamma := M_n(\mathcal{O}) = \Lambda(\mathcal{O})$. Then any other maximal order in A arises as the left order of some Γ-right ideal, in particular

$$\Lambda(\mathfrak{c}) = O_l(I(\mathfrak{c})) = \{a \in A \mid aI(\mathfrak{c}) \subseteq I(\mathfrak{c})\} \text{ where } I(\mathfrak{c}) = \begin{pmatrix} \mathcal{O} & \dots & \mathcal{O} \\ \vdots & \dots & \vdots \\ \mathcal{O} & \dots & \mathcal{O} \\ \mathfrak{c} & \dots & \mathfrak{c} \end{pmatrix}.$$

Two left orders $O_l(I)$ and $O_l(I')$ are conjugate, if and only if $I' = aIJ$ for some $a \in A^\times$ and some two-sided fractional Γ-ideal J. By Morita theory any two-sided Γ-ideal J is of the form $J = M_n(\mathfrak{a})$ for some two-sided $\mathcal{O}$-ideal

$\mathfrak{a}$ in K. By [15, Lemma (35.8)], the reduced norm of $J = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^n, \mathfrak{a}^n)$ equals $\mathrm{nr}(\mathfrak{a})^n \in \mathrm{Cl}_K(R)$ and the reduced norm of $I(\mathfrak{c}) = \mathrm{Hom}_{\mathcal{O}}(L(\mathcal{O}), L(\mathfrak{c}))$ is $\mathrm{nr}(\mathfrak{c})$. By [15, Theorem 35.14] the reduced norm is injective, so

$$I(\mathfrak{c}) = aI(\mathfrak{c}')M_n(\mathfrak{a}) \text{ for some } a \in A^\times \quad \text{if and only if} \quad \mathrm{nr}(\mathfrak{c}) = \mathrm{nr}(\mathfrak{c}')\,\mathrm{nr}(\mathfrak{a})^n.$$

□

3. Positive cones

Let K be some rational division algebra and $A = M_n(K)$. Then $A_{\mathbf{R}} := A \otimes_{\mathbf{Q}} \mathbf{R}$ is a semi-simple real algebra, hence a direct sum of matrix rings over one of $\mathbf{H}$, $\mathbf{R}$ or $\mathbf{C}$. It carries a canonical involution that we use to define symmetric elements. Let d denote the degree of K, so $d^2 = \dim_{Z(K)}(K)$, and let

$\iota_1, \ldots, \iota_s$	be the real places of $Z(K)$ that ramify in K,
$\sigma_1, \ldots, \sigma_r$	the real places of $Z(K)$ that do not ramify in K
$\tau_1, \ldots, \tau_t$	the complex places of $Z(K)$.

Then

$$K_{\mathbf{R}} := K \otimes_{\mathbf{Q}} \mathbf{R} \cong \bigoplus_{i=1}^{s} M_{d/2}(\mathbf{H}) \oplus \bigoplus_{i=1}^{r} M_d(\mathbf{R}) \oplus \bigoplus_{i=1}^{t} M_d(\mathbf{C}).$$

The "canonical" involution * (depending on the choice of this isomorphism) is defined on any simple summand of $K_{\mathbf{R}}$ to be transposition for $M_d(\mathbf{R})$, transposition and complex (respectively quaternionic) conjugation for $M_d(\mathbf{C})$ and $M_{d/2}(\mathbf{H})$. The resulting involution on $K_{\mathbf{R}}$ is again denoted by *. As usual it defines a mapping $^\dagger : M_{m,n}(K_{\mathbf{R}}) \to M_{n,m}(K_{\mathbf{R}})$ by applying * to the entries and then transposing the $m \times n$-matrices. In particular this defines an involution † on $A_{\mathbf{R}} = M_n(K_{\mathbf{R}})$. In general this involution will not fix the set A.

Definition 3.1. $\Sigma := \mathrm{Sym}(A_{\mathbf{R}}) := \left\{F \in A_{\mathbf{R}} \mid F^\dagger = F\right\}$ is the $\mathbf{R}$-linear subspace of symmetric elements of $A_{\mathbf{R}}$. It supports the positive definite inner product

$$\langle F_1, F_2 \rangle := \mathrm{trace}(F_1 F_2)$$

where trace is the reduced trace of the semi simple $\mathbf{R}$-algebra $A_{\mathbf{R}}$. The real vector space Σ contains the open real cone of positive elements

$$\mathcal{P} := \{(q_1, \ldots, q_s, f_1, \ldots, f_r, h_1, \ldots, h_t) \in \Sigma \mid q_i, f_j, h_k \text{ pos. def.}\}.$$

Let V be the simple left A-module K^n. Then $V_{\mathbf{R}} := V \otimes_{\mathbf{Q}} \mathbf{R} = K_{\mathbf{R}}^n$ and for any $x \in V_{\mathbf{R}}$ the matrix $xx^\dagger$ lies in Σ. The following lemma is easily checked:

LEMMA 3.2. *Any $F \in \Sigma$ defines a quadratic form on $V_{\mathbf{R}}$ by:*

$$F[x] := \langle F, xx^\dagger \rangle \in \mathbf{R} \text{ for all } x \in V_{\mathbf{R}}.$$

This quadratic form is positive definite if and only if $F \in \mathcal{P}$.

As a consequence, with a slight abuse of language, we will sometimes refer to elements of Σ as *forms*.

4. MINIMAL VECTORS

Let $A = M_n(K)$ for some division algebra K. As before we fix some maximal order $\mathcal{O}$ in K and choose some right $\mathcal{O}$-lattice L in the simple left A-module $V = K^n$. Then $\Lambda := \mathrm{End}_{\mathcal{O}}(L)$ is a maximal order in A with unit group $\Lambda^\times := \mathrm{GL}(L) = \{a \in A \mid aL = L\}$.

Following [2], we will define the L-minimum of a form $F \in \mathcal{P}$ with respect to a *weight*.

DEFINITION 4.1. A weight φ on L is a $\mathrm{GL}(L)$-invariant map from the projective space $\mathbf{P}(K^n)$ to the positive reals, such that $\max_{x \in \mathbf{P}(K^n)} \varphi(x) = 1$.

A natural choice for the weight is $\varphi_0(x) = 1$ for all $x \in K^n - \{0\}$. However, another rather standard choice for φ is possible, which allows for definitions having a natural geometric interpretation and somehow simplify the computations, at least in the case of imaginary quadratic fields or definite quaternion algebras (see Section 7). Roughly speaking, this alternative weight is given by the inverse of the gcd of the coefficients of a vector in K^n with respect to a given pseudo-basis of the lattice L. To be more precise, we need the following definition

DEFINITION 4.2. Let $L = e_1\mathfrak{c}_1 \oplus \ldots \oplus e_n\mathfrak{c}_n$. To any $\ell = \sum_{i=1}^n e_i\ell_i \in L - \{0\}$ we associate the integral left $\mathcal{O}$-ideal

$$\mathfrak{a}_\ell := \sum_{i=1}^n \mathfrak{c}_i^{-1}\ell_i$$

as well as its norm

$$\mathrm{N}(\mathfrak{a}_\ell) := |\mathcal{O}/\mathfrak{a}_\ell| = \mathrm{N}_{Z(K)/\mathbf{Q}}\left(\mathrm{nr}\,(\mathfrak{a}_\ell)^d\right).$$

LEMMA 4.3.

(a) $\mathrm{N}(\mathfrak{a}_\ell) \geq 1$ *for all* $\ell \in L - \{0\}$.

(b) For any $\lambda \in K^\times$ *and* $\ell = \sum_{i=1}^n e_i\ell_i \in L - \{0\}$, *one has* $\mathfrak{a}_{\ell\lambda} = \mathfrak{a}_\ell\lambda$.

(c) If $g \in \mathrm{GL}(L)$ *and* $\ell = \sum_{i=1}^n e_i\ell_i \in L - \{0\}$, *then* $\mathfrak{a}_{g\ell} = \mathfrak{a}_\ell$.

Proof. (a) is clear, because all $\mathfrak{c}_i^{-1}\ell_i$ are integral left $\mathcal{O}$-ideals, and (b) is straightforward.
To see (c) write $ge_i = \sum_{j=1}^n e_j g_{ji}$. Since $gL \subseteq L$ we get $g_{ji} \in \mathfrak{c}_j\mathfrak{c}_i^{-1}$. Then $g\ell = \sum_{j=1}^n e_j(\sum_{i=1}^n g_{ji}\ell_i)$ and

$$\mathfrak{a}_{g\ell} = \sum_{j=1}^n \mathfrak{c}_j^{-1}\sum_{i=1}^n g_{ij}\ell_i \subseteq \sum_{j,i}\mathfrak{c}_j^{-1}\mathfrak{c}_j\mathfrak{c}_i^{-1}\ell_i \subseteq \mathfrak{a}_\ell.$$

One obtains equality by applying $g^{-1} \in \mathrm{GL}(L)$. □

Now for any $x \in K^n$, we can find $\lambda \in K - \{0\}$ such that $x\lambda \in L$. It follows from the previous lemma that the class of $\mathrm{nr}(\mathfrak{a}_{x\lambda})$ in $\mathrm{Cl}_K(R)$ does not depend on the choice of an element λ with this property. Consequently, if we define the norm of a class in $\mathrm{Cl}_K(R)$ as the smallest possible norm of an integral ideal in that class, we can associate to x a well-defined quantity N_x by the formula

$$N_x = \mathrm{N}([\mathrm{nr}\,(\mathfrak{a}_{x\lambda})]) = \min_{\substack{I\subset\mathcal{O}\\ [\mathrm{nr}(I)]=[\mathrm{nr}(\mathfrak{a}_{x\lambda})]}} \mathrm{N}_{Z(K)/\mathbf{Q}}\left(\mathrm{nr}\,(I)^d\right),$$

where as before λ is any element in $K - \{0\}$ such that $x\lambda \in L$. This in turn can be used to define a weight φ_1 on K^n setting

$$\varphi_1(x) = N_x^{-2/[K:\mathbf{Q}]} \tag{4.1}$$

(that this is indeed a weight follows immediately from Lemma 4.3).

REMARK 4.4. As explained in [2], the space of weights is isomorphic to $\mathbf{R}^{h_K-1}$, where h_K stands for the class number of K. In particular, the trivial weight φ_0 is the only possible choice if $h_K = 1$ (and $\varphi_1 = \varphi_0$ in that case).

Having fixed a weight φ on L, we can define the minimum of a form an its set of minimal vectors as follows:

DEFINITION 4.5. The L-minimum of $F \in \mathcal{P}$ with respect to the weight φ is

$$\min_L(F) := \min_{\ell \in L - \{0\}} \varphi(\ell) F[\ell].$$

The set of minimal vectors of F in L is defined as

$$S_L(F) := \{\ell \in L - \{0\} \mid \varphi(\ell) F[\ell] = \min_L(F)\}.$$

REMARK 4.6. The set $S_L(F)$ is finite. Indeed, let $m := \min\{\varphi(\ell) \mid \ell \in L \setminus \{0\}\}$. Then $m > 0$ as φ takes only finitely many positive real values, so $S_L(F) \subset \left\{\ell \in L \mid F[\ell] \leq m^{-1} \min_L(F)\right\}$ which is a finite set and can be computed as the set of vectors of small length in a $\mathbf{Z}$-lattice.

5. MINIMAL CLASSES

We keep the general assumptions of the previous section: K is a division algebra, $\mathcal{O}$ a maximal order in K and L a right $\mathcal{O}$-lattice in K^n, on which a weight φ is fixed.

DEFINITION 5.1. Two elements F_1 and $F_2 \in \mathcal{P}$ are called *minimally equivalent with respect to L and φ*, if $S_L(F_1) = S_L(F_2)$. We denote by

$$\mathrm{Cl}_L(F_1) := \{F \in \mathcal{P} \mid S_L(F) = S_L(F_1)\}$$

the *minimal class* of F_1. If $C = \mathrm{Cl}_L(F_1)$ is a minimal class then we define $S_L(C) = S_L(F_1)$ the associated set of minimal vectors. A minimal class C is called *well rounded*, if $S_L(C)$ contains a K-basis of V. The form $F \in \mathcal{P}$ is called *perfect with respect to L*, if $\mathrm{Cl}_L(F) = \{aF \mid a \in \mathbf{R}, a > 0\}$.

REMARK 5.2. Note that minimal classes and all subsequent definitions in this section actually *depend on* the weight φ, although we do not indicate it systematically in our notations. No inconstancy can arise from this, since we work with fixed weight φ (and fixed lattice L).

The group $\mathrm{GL}_n(K)$, and hence also its subgroup $\mathrm{GL}(L)$, acts on Σ by $(F, g) \mapsto g^\dagger F g$ (where we embed A into $A_{\mathbf{R}}$ to define the multiplication). Two forms in Σ are called *L-isometric*, if they are in the same $\mathrm{GL}(L)$-orbit. For $F \in \mathcal{P}$ we denote by

$$\mathrm{Aut}_L(F) := \left\{g \in \mathrm{GL}(L) \mid g^\dagger F g = F\right\}$$

the *automorphism group* of F. Then $\mathrm{Aut}_L(F)$ is always a finite subgroup of $\mathrm{GL}(L)$. The group $\mathrm{GL}(L)$ acts on the set of minimal classes. Two minimal classes are called *equivalent*, if they are in the same orbit under this action. The stabiliser of a minimal class is called the *automorphism group* of the class,

$$\mathrm{Aut}_L(C) = \{g \in \mathrm{GL}(L) \mid gS_L(C) = S_L(C)\}.$$

LEMMA 5.3. *(see [3, Proposition 2.9]) Let C be a well rounded minimal class. Then the* canonical form $T_C := \sum_{x \in S_L(C)} xx^\dagger \in \mathcal{P}$ *is positive definite and* $\mathrm{Aut}_L(C) = \mathrm{Aut}_L(T_C^{-1})$. *Two well rounded minimal classes C and C' are equivalent, if and only if T_C^{-1} and $T_{C'}^{-1}$ are L-isometric.*

Proof. The proof is similar to the one in [3]. The well roundedness of C implies that the rank of T_C is maximal: The mapping $(,) : V \times V \to K_{\mathbf{R}}, (x,y) := x^\dagger y$ is Hermitian and non-degenerate. Let $\{x_1, \ldots, x_n\} \subset S_L(C)$ be a K-basis of V, then for any $v \in V$

$$\sum_{i=1}^{n} x_i x_i^\dagger v = \sum_{i=1}^{n} x_i (x_i, v) = 0 \text{ if and only if } v \in V^\perp = \{0\}$$

so the kernel of the positive semidefinite matrix $\sum_{i=1}^n x_i x_i^\dagger$ is $\{0\}$, therefore T_C is invertible and hence in $\mathcal{P}$. Clearly $\mathrm{Aut}_L(C) \subseteq \mathrm{Aut}_L(T_C^{-1})$. To see the converse put $s := |S_L(C)|$ and let $S \in M_{n,s}(K)$ be a matrix whose columns are the elements of $S_L(C)$, in particular $T_C = SS^\dagger$. Take some $g \in \mathrm{Aut}(T_C^{-1}) = \{g \in \mathrm{GL}(L) \mid gT_C g^\dagger = T_C\}$ and put $S' := gS$. Then $S'(S')^\dagger = T_C = SS^\dagger$ and for any $F \in \mathcal{P}$

$$(\star) \qquad \sum_{y \in \mathrm{cols}(S')} F[y] = \mathrm{trace}((S')^\dagger F S') = \langle S'(S')^\dagger, F\rangle = \langle SS^\dagger, F\rangle = \sum_{x \in S_L(C)} F[x].$$

If x is some column of S and $y := gx$, then $\varphi(y) = \varphi(x)$, because of the $\mathrm{GL}(L)$-invariance of φ. Moreover $\varphi(y)F[y] \geq \varphi(x)F[x] = \min_{\ell \in L - \{0\}} \varphi(\ell)F[\ell]$, whence $F[y] \geq F[x]$, with equality if and only if $y \in S_L(C)$. So we can only have equality in $(\star)$ if $S_L(C) = \{\mathrm{cols}(S')\}$ and hence $g \in \mathrm{Aut}_L(C)$. □

6. MAXIMAL FINITE SUBGROUPS OF $\mathrm{GL}(L)$

In this section we will use variants of the Voronoi algorithm to compute a set of representatives of the conjugacy classes of maximal finite subgroups of

$\mathrm{GL}(L)$. The known methods (see e.g. [12]) start with the list of all conjugacy classes of finite subgroups of $\mathrm{GL}_n(K)$. For each group G they compute the invariant lattices to find the $\mathrm{GL}(L)$-conjugacy classes of subgroups in the class of G. In particular for reducible groups G this set of invariant lattices is infinite and one needs to use the action of $N_{\mathrm{GL}_n(K)}(G)$. Also it seems to be difficult to restrict to one isomorphism class of $\mathcal{O}$-lattices L.

Here we will start with some lattice L and use the tessellation of the cone of positive definite hermitian forms into L-minimal classes to obtain a list of subgroups of $\mathrm{GL}(L)$ that contains representatives of all conjugacy classes of maximal finite subgroups of $\mathrm{GL}(L)$. To check maximal finiteness and also conjugacy of the groups in the list, we use a relative version of Voronoi's theory.

DEFINITION 6.1. Let $G \leq \mathrm{GL}(L)$ be some finite subgroup. Let $\mathcal{F}(G) := \left\{F \in \Sigma \mid g^\dagger F g = F \text{ for all } g \in G\right\}$ denote the space of G-invariant Hermitian forms. It contains the cone $\mathcal{F}_{>0}(G) := \mathcal{F}(G) \cap \mathcal{P}$ of positive definite G-invariant forms. For $F \in \mathcal{F}_{>0}(G)$ the *G-minimal class* of F is $\mathrm{Cl}_L(F) \cap \mathcal{F}(G)$. A form $F \in \mathcal{F}_{>0}(G)$ is called *G-perfect with respect to L*, if $\mathrm{Cl}_L(F) \cap \mathcal{F}(G) = \{aF \mid a \in \mathbf{R}_{>0}\}$.

LEMMA 6.2. *Let*

$$\pi_G : \Sigma \to \mathcal{F}(G), F \mapsto \frac{1}{|G|} \sum_{g \in G} g^\dagger F g$$

be the usual averaging operator and C be a G-invariant minimal class. Then

$$C \cap \mathcal{F}(G) = \pi_G(C).$$

Proof. Since $\pi_G(F) = F$ for all G-invariant forms, it is clear that $C \cap \mathcal{F}(G) \subseteq \pi_G(C)$. So let $F \in C$. Then $S_L(F) = S_L(C)$. Since $S_L(C)$ is G-invariant, $S_L(C) = S_L(g^\dagger F g)$ for any $g \in G$. As $\pi_G(F)$ is a sum of positive definite forms, also $S_L(\pi_G(F)) = S_L(C)$ and so $\pi_G(F) \in C$. □

As in the classical case, Voronoi's algorithm, as described *e.g.*in [11] can be adapted to the case of G-invariant forms to compute the G-perfect forms and the cellular decomposition of $\mathcal{F}_{>0}(G)$ into G-minimal classes up to the action of the normaliser (see for instance [3, Theorem 2.4] for details on this procedure in the classical case).

PROPOSITION 6.3. *Let $G \leq \mathrm{GL}(L)$ be finite. Then there exists at least one G-perfect form with respect to L.*

Proof. We will show that $L-\{0\}$ is discrete and admissible in the sense of [11, Definition 1.4]. Then by [11, Proposition 1.8] there exists a G-perfect form. Moreover, [11, Theorem 1.9] tells us that the Voronoi domains of the G-perfect forms form an exact tessellation of $\mathcal{F}(G^\dagger)_{>0}$.
Clearly $L-\{0\}$ is discrete in $V_{\mathbf{R}} := V \otimes_{\mathbf{Q}} \mathbf{R}$. For the admissibility we need to show that for any $F \in \partial\mathcal{P}$, the boundary of $\mathcal{P}$, and any $\epsilon > 0$ there is some $\ell \in L-\{0\}$, such that $\varphi(\ell)F[\ell] < \epsilon$. If $F \in \partial\mathcal{P}$, it is positive semidefinite, so

$$\{x \in V_{\mathbf{R}} \mid F[x] = 0\} = \{x \in V_{\mathbf{R}} \mid Fx = 0\} \leq V_{\mathbf{R}}$$

is a subspace. In particular the volume of the convex set

$$\mathcal{K}_\epsilon := \{x \in V_{\mathbf{R}} \mid F[x] < \epsilon\} = -\mathcal{K}_\epsilon$$

is infinite and by Minkowski's convex body theorem $\mathcal{K}_\epsilon$ contains some $0 \neq \ell \in L$. Then $F[\ell] < \epsilon$ and hence also $\varphi(\ell)F[\ell] < \epsilon$ since $\varphi(\ell) \leq 1$.
□

LEMMA 6.4. *Let $G \leq \mathrm{GL}(L)$ be finite. Then any G-perfect form F with respect to L is well rounded.*

Proof. The proof is similar to the classical case. Assume that $\langle S_L(F)\rangle_K \neq V$. Then there is some linear form $H \in V^* = K^n$ so that $Hx = 0$ for all $x \in S_L(F)$. Let

$$F_0 := \frac{1}{|G|}\sum_{g\in G} g^\dagger H^\dagger H g.$$

Since $S_L(F)$ is G-invariant, $x^\dagger F_0 x = 0$ for all $x \in S_L(F)$, so $S_L(F+\epsilon F_0) \supseteq S_L(F)$ for all $\epsilon > 0$ with equality, if ϵ is small enough. So $F + \epsilon F_0 \in \mathrm{Cl}_L(F) \cap \mathcal{F}_{>0}(G)$ contradicting the assumption that F is G-perfect with respect to L. □

THEOREM 6.5. *Let $G \leq \mathrm{GL}(L)$ be some maximal finite subgroup of $\mathrm{GL}(L)$. Then $G = \mathrm{Aut}_L(C)$ for some well rounded minimal class C with respect to L, such that $C \cap \mathcal{F}(G)$ spans a subspace of $\mathcal{F}(G)$ of dimension 1.*

Proof. The group G always fixes some G-perfect form F with respect to L. Let $C := \mathrm{Cl}_L(F)$. Then $S_L(C) = S_L(F)$ is G-invariant, so $G \leq \mathrm{Aut}_L(C)$. By Lemma 6.4 C is well rounded, so $\mathrm{Aut}_L(C)$ is finite and the maximality of G implies that $G = \mathrm{Aut}_L(C)$. □

With Theorem 6.5 we obtain a finite list of finite subgroups of $\mathrm{GL}(L)$ that contains a system of representatives of conjugacy classes of maximal finite subgroups. We need to be able to test maximal finiteness and conjugacy in $\mathrm{GL}(L)$ of such groups $\mathrm{Aut}_L(C)$.

PROPOSITION 6.6. *Let $G \leq \mathrm{GL}(L)$ be some finite subgroup. Then the maximal finite subgroups H of $\mathrm{GL}(L)$ that contain G are of the form $H = \mathrm{Aut}_L(C_G)$ for some G-minimal class C_G.*

Proof. Let H be some maximal finite subgroup of $\mathrm{GL}(L)$ that contains G. By Theorem 6.5 there is some G-invariant L-minimal class C such that $H = \mathrm{Aut}_L(C)$. By Lemma 6.2 $S_L(C) = S_L(C_G)$ for the G-minimal class $C_G = \pi_G(C)$ and $H = \mathrm{Aut}_L(C_G)$. □

REMARK 6.7. To test whether two maximal finite subgroups G_1, G_2 of $\mathrm{GL}(L)$ are conjugate, one computes a system of representatives R_i of the $N_{\mathrm{GL}(L)}(G_i)$-orbits of G_i-perfect forms and then checks whether a given form in R_1 is L-isometric to some form in R_2. Since $G_i = \mathrm{Aut}_L(F_i)$ for all $F_i \in R_i$, any isometry yields a conjugating element.

7. IMAGINARY QUADRATIC FIELDS AND DEFINITE QUATERNION ALGEBRAS

In this section we will assume that K is either the field of rational numbers, an imaginary quadratic number field or a definite quaternion algebra over the rationals. These are exactly the cases, where $K_{\mathbf{R}}$ is a division algebra and $\mathrm{Sym}(K_{\mathbf{R}}) = \mathbf{R}$. We thus have in those cases (and in those cases only) the noteworthy property that

$$\forall \lambda \in K_{\mathbf{R}}, \forall x \in V_{\mathbf{R}} \quad F[x\lambda] = \lambda\lambda^{\dagger} F[x]. \tag{7.1}$$

As a consequence, it is more natural and more efficient to compute minima with respect to the weight φ_1 defined in the previous subsection, because of the following lemma

LEMMA 7.1. *For any $F \in \mathcal{P}$ one has*

$$\min_L(F) := \min_{\ell \in L - \{0\}} \frac{F[\ell]}{\mathrm{N}(\mathfrak{a}_\ell)^{2/[K:\mathbf{Q}]}}$$

where the minimum on the left hand side is computed with respect with the weight $\varphi_1(\ell) = N_\ell^{-2/[K:\mathbf{Q}]}$.

Proof. The inequality $\min_L(F) \geq \min_{\ell \in L-\{0\}} \frac{F[\ell]}{\mathrm{N}(\mathfrak{a}_\ell)^{2/[K:\mathbf{Q}]}}$ is clear, since $N_\ell \leq \mathrm{N}(\mathfrak{a}_\ell)$ for every $\ell \in L-\{0\}$. In the opposite direction, every $\ell \in L-\{0\}$, there exists $\lambda \in K-\{0\}$ such that $\mathfrak{a}_{\ell\lambda} = \mathfrak{a}_\ell \lambda \subset \mathcal{O}$ and $\mathrm{N}(\mathfrak{a}_\ell\lambda) = \mathrm{N}([\mathfrak{a}_\ell]) = N_\ell$ (in particular, $\ell\lambda \in L$) . Using (7.1), we see that $\frac{F[\ell]}{\mathrm{N}(\mathfrak{a}_\ell)^{2/[K:\mathbf{Q}]}} = \frac{F[\ell\lambda]}{\mathrm{N}(\mathfrak{a}_{\ell\lambda})^{2/[K:\mathbf{Q}]}} = \varphi_1(\ell\lambda)F[\ell\lambda] \geq \min_L(F)$, whence the conclusion taking the minimum of the left-hand side over $L-\{0\}$. □

REMARK 7.2. The reformulation given in Lemma 7.1 of the minimum of a form with respect to φ_1 has two noteworthy applications

1. It can be interpreted in terms of *minimal distance to cusps* as explained in [10] (see also [8, chapter 7]).
2. One can easily deduce from this that the Voronoi complex will depend only on the Steinitz class of L modulo nth powers (see [5, Theorem 3.8]).

8. EXAMPLES

We will use the method from the previous section to compute the conjugacy classes of maximal finite subgroups of $\mathrm{GL}(L)$ for imaginary quadratic number fields K. This is an invariant of the isomorphism class of $\mathrm{GL}(L)$ and will show that for small examples these groups are not isomorphic.

Example 1

Let $K := \mathbf{Q}[\sqrt{-15}]$, $\mathcal{O} = O_K = \mathbf{Z}[\frac{1+\sqrt{-15}}{2}]$, $n = 2$. Then $\mathrm{Cl}(K) = \{[O_K], [\wp_2]\}$ where $\wp_2$ is some prime ideal dividing 2, so there are two isomorphism classes of O_K-lattices in K^2 : one corresponding to the lattice L_0 with Steinitz-invariant $[O_k]$ and the other one to the lattice L_1 with Steinitz-invariant $[\wp_2]$. For both lattices the perfect forms are given in [5].

For both lattices L, Table 1 lists the $\mathrm{GL}(L)$-orbits of well rounded minimal classes C according to their perfection corank together with their stabilizers $G = \mathrm{Aut}_L(C)$. The two classes of perfection corank 0 contain the perfect forms. The third column gives the dimension of $\pi_G(C)$. If this dimension is one, then $\pi_G(C) \subset \langle F \rangle$ for some G-perfect form F, the next column gives the automorphism group $\mathrm{Aut}_L(F)$ and the last one indicates whether G is maximal finite.

$\underline{L = L_0}$:

The two groups $G = D_8$ and $G = D_{12}$ are absolutely irreducible maximal finite subgroups of $\mathrm{GL}_2(K)$. Since $\dim(\mathcal{F}(G)) = 1$ for both groups and C_i and D_i are inequivalent $(i = 1, 2)$ one gets 2 conjugacy classes of

TABLE 1
Well rounded minimal classes for $K = \mathbf{Q}[\sqrt{-15}]$

$L = L_0$				
C	$G = \mathrm{Aut}_L(C)$	$\dim(\pi_G(C))$	$\mathrm{Aut}_L(F)$	maximal
perf. corank = 0				
P_1	C_6	1	C_6	no
P_2	C_4	1	C_4	no
perf. corank = 1				
C_1	D_{12}	1	D_{12}	yes
C_2	D_{12}	1	D_{12}	yes
C_3	C_2	2		no
C_4	C_2	2		no
perf. corank = 2				
D_1	D_8	1	D_8	yes
D_2	D_8	1	D_8	yes
D_3	V_4	1	V_4	yes
D_4	V_4	1	V_4	yes
$L = L_1$				
C	$G = \mathrm{Aut}_L(C)$	$\dim(\pi_G(C))$	$\mathrm{Aut}_L(F)$	maximal
perf. corank = 0				
P	$C_3 : C_4$	1	$C_3 : C_4$	yes
perf. corank = 1				
C_1	D_8	1	D_8	yes
C_2	D_8	1	D_8	yes
C_3	D_{12}	1	D_{12}	yes
perf. corank = 2				
D	V_4	1	V_4	yes

maximal finite subgroups of both isomorphism types D_8 and D_{12}. To prove that $G := \mathrm{Aut}_L(D_3)$ is maximal finite, we compute the well rounded G-minimal classes, using Voronoi's algorithm and starting with the G-perfect form $F \in \pi_G(D_3)$. $S_L(F) = \{\pm v_1, \pm v_2\}$ with $\mathfrak{a}_{v_1} = O_K$, $\mathfrak{a}_{v_2} = \wp_2$. Therefore both minimal vectors are G-eigenvectors and the G-Voronoi domain has 2 faces, both of which are dead ends (see [9, Definition 13.1.7]). So F is the unique G-perfect form and there are no other well rounded G-minimal classes. The situation is the same for $\mathrm{Aut}_L(D_4)$. The two G-perfect forms in D_3 and D_4 (rescaled to have minimum 1) are Galois conjugate but not L-isometric, with shows that $\mathrm{Aut}_L(D_3)$ and $\mathrm{Aut}_L(D_4)$ are not conjugate in $\mathrm{GL}(L)$.

The proof that $G := \mathrm{Aut}_L(P_i)$ is not maximal finite is similar for both cases $i = 1, 2$. The space of invariant forms has dimension 2, there are two G-orbits on $S_L(P_i)$, so the G-Voronoi domain of P_i has two faces, corresponding to 1-dimensional G-minimal classes with automorphism group D_{12} (for P_1) resp. D_8 (for P_2). One checks for $i = 1, 2$ that $\mathrm{Aut}_L(P_i)$ is properly contained in these groups.

$\underline{L = L_1:}$ As in the free case the uniform groups $\mathrm{Aut}_L(P)$ and $\mathrm{Aut}_L(C_i)$, $i = 1, 2, 3$ are maximal finite and represent distinct conjugacy classes. For the group $G = \mathrm{Aut}_L(D) \cong V_4$ we again have a unique G-perfect form F and the two L-minimal vectors of F are eigenvectors for G. So both faces of the G-Voronoi domain of F are dead ends and $G = \mathrm{Aut}_L(F)$ is maximal finite.

As $\mathrm{GL}_2(O_K)$ and $\mathrm{GL}(L_1)$ have different conjugacy classes of maximal finite subgroups one finds the following corollary.

COROLLARY 8.1. $\mathrm{GL}_2(O_K) = \mathrm{GL}(L_0)$ *and* $\mathrm{GL}(L_1)$ *are not isomorphic.*

Example 2

Table 2 lists the results of similar computations which we did for certain small imaginary quadratic fields. In particular we find

COROLLARY 8.2. *Let K be one of the six fields in Table 2. Then non-conjugate maximal orders in $M_2(K)$ have non-isomorphic unit groups.*

REFERENCES

[1] Avner Ash, *Cohomology of congruence subgroups* $\mathrm{SL}(n, \mathbf{Z})$, Math. Ann. **249** (1980), no. 1, 55–73.

[2] Avner Ash, *Small-dimensional classifying spaces for arithmetic subgroups of general linear groups*, Duke Math. J. **51** (1984), no. 2, 459–468.

TABLE 2
Number of conjugacy classes of maximal finite subgroups

	D_8	D_{12}	V_4	$\mathrm{SL}_2(3)$	Q_8	$C_3 : C_4$
$K = \mathbf{Q}[\sqrt{-15}]$						
$St(L) = [O_K]$	2	2	2	-	-	-
$St(L) = [\wp_2]$	2	1	1	-	-	1
$K = \mathbf{Q}[\sqrt{-5}]$						
$St(L) = [O_K]$	3	2	1	-	1	-
$St(L) = [\wp_2]$	1	2	1	1	-	-
$K = \mathbf{Q}[\sqrt{-6}]$						
$St(L) = [O_K]$	3	2	1	1	-	-
$St(L) = [\wp_2]$	1	1	2	-	1	1
$K = \mathbf{Q}[\sqrt{-10}]$						
$St(L) = [O_K]$	3	2	1	-	1	-
$St(L) = [\wp_2]$	1	-	3	1	-	2
$K = \mathbf{Q}[\sqrt{-21}]$						
$St(L) = [O_K]$	6	4	2	-	-	2
$St(L) = [\wp_2]$	2	-	6	-	-	-
$St(L) = [\wp_3]$	-	2	6	2	-	-
$St(L) = [\wp_5]$	-	-	8	-	2	-

[3] C. Batut, *Classification of quintic eutactic forms.* Math. Comp. 70 (2001) 395-417

[4] A.-M. Bergé, J. Martinet, and F. Sigrist, *Une généralisation de l'algorithme de Voronoĭ pour les formes quadratiques*, Astérisque (1992), no. 209, 12, 137–158, Journées Arithmétiques, 1991 (Geneva).

[5] O. Braun, R. Coulangeon, *Perfect Lattices for Imaginary Quadratic Number Fields.* `arXiv : 1304.0559 [math.NT]`

[6] M. Deuring, *Algebren.* Springer Berlin, Heidelberg, 1935

[7] M. Dutour Sikirić, G. Ellis, A. Schürmann, *On the integral homology of $PSL_4(\mathbf{Z})$ and other arithmetic groups.* J. Number Theory **131** (2011) 2368-2375.

[8] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998, Harmonic analysis and number theory.

[9] J. Martinet, *Perfect lattices in euclidean spaces.* Springer Grundlehren 327 (2003)

[10] Eduardo R. Mendoza, *Cohomology of* PGL_2 *over imaginary quadratic integers*, Bonner Mathematische Schriften 128, Universität Bonn Mathematisches Institut, Bonn, 1979.

[11] J. Opgenorth, *Dual cones and the Voronoi algorithm.* Exp. Math. **10** (2001) 599-608

[12] W. Plesken and M. Pohst, *On maximal finite irreducible subgroups of* $\mathrm{GL}(n, Z)$. *I. The five and seven dimensional case. II. The six dimensional case*, Bull. Amer. Math. Soc. **82** (1976), no. 5, 757–758.

[13] Alexander D. Rahm, *The homological torsion of* PSL_2 *of the imaginary quadratic integers*, Trans. Amer. Math. Soc. **365** (2013), no. 3, 1603–1635.

[14] Alexander D. Rahm and Mathias Fuchs, *The integral homology of* PSL_2 *of imaginary quadratic integers with nontrivial class group*, J. Pure Appl. Algebra **215** (2011), no. 6, 1443–1472.

[15] I. Reiner, *Maximal Orders.* Academic Press London (1975)

[16] S. S. Ryškov, *Maximal finite groups of $n \times n$ integral matrices and full integral automorphism groups of positive quadratic forms (Bravais types)*, Trudy Mat. Inst. Steklov. **128** (1972), 183–211, 261, Collection of articles dedicated to Academician Ivan Matveevič Vinogradov on his eightieth birthday, II.

[17] Christophe Soulé, *Cohomologie de* $SL_3(Z)$, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), no. 5, Ai, A251–A254.

[18] ———, *The cohomology of* $\mathrm{SL}_3(\mathbf{Z})$, Topology **17** (1978), no. 1, 1–22.

[19] William Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007, With an appendix by Paul E. Gunnells.

(Reçu le 9 avril 2013)

Renaud Coulangeon

Université Bordeaux 1
UFR de Mathématiques et Informatique
351, cours de la Libération
33405, Talence cedex
France
e-mail : Renaud.Coulangeon@math.u-bordeaux1.fr

Gabriele Nebe

Lehrstuhl D für Mathematik
RWTH Aachen university
Templergraben 64
52062 Aachen
Germany
e-mail : nebe@math.rwth-aachen.de