Orthogonal Determinants of SL_3(q) and SU_3(q)

Linda Hoyer* and Gabriele Nebe†

Lehrstuhl für Algebra und Zahlentheorie, RWTH Aachen University, Germany

Abstract

We give a full list of the orthogonal determinants of the even degree indicator '+' ordinary irreducible characters of $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$.
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1 Introduction

Let G be a finite group and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation. We call ρ an orthogonal representation, if there is a symmetric, non-degenerate, $\rho(G)$-invariant bilinear form β on $\mathbb{C}^n$. It is well-known that an absolutely irreducible representation is orthogonal if and only if the associated character has Frobenius-Schur indicator '+', i.e. ρ is equivalent to a real representation. A character is called orthogonal if it is the character afforded by an orthogonal representation. The character χ is orthogonal, if and only if it is of the form

$$\chi = \sum_{i=1}^{r} a_i \chi_i^{(+)} + 2 \sum_{j=1}^{s} b_i \chi_j^{(-)} + \sum_{k=1}^{t} c_i (\chi_k^{(0)} + \overline{\chi_k}^{(0)}), \tag{1}$$

where $\chi_i^{(+)}$ (resp. $\chi_j^{(-)}$, resp. $\chi_k^{(0)}$) are irreducible characters of G with Frobenius-Schur indicator '+' (resp. '-', resp. '0'), and a_i, b_j, c_k are non-negative integers.

Let χ be an orthogonal character as in equation (1) and let $K = \mathbb{Q}(\chi)$ be the character field of χ. The main result of [8] shows that if the degree of all $\chi_i^{(+)}$ is even then there is a unique element $\det(\chi) := d \in K^\times/(K^\times)^2$, called the *orthogonal determinant* of χ, such that for all representations $\rho : G \to \mathrm{GL}_n(L)$ over a field

*linda.hoyer@rwth-aachen.de
†nebe@math.rwth-aachen.de

extension $L \supseteq K$ affording χ all non-degenerate, $\rho(G)$-invariant, symmetric bilinear forms β on L^n have the same determinant

$$\det(\beta) = d \cdot (L^\times)^2 \in L^\times/(L^\times)^2.$$

We call such a character *orthogonally stable*.

The orthogonal determinant of $2\chi_j^{(-)}$ is always 1 (see [13]) and for characters of the form $\chi_k^{(0)} + \overline{\chi_k}^{(0)}$ the orthogonal determinant can be obtained from the character values as given in Lemma 2.4 below. So it remains to deal with the indicator '+' characters in the sum (1). Put

$$\mathrm{Irr}^+(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi \text{ is an indicator '+' character of even degree}\}.$$

In a long term project the second author has developed theoretical and computational methods to calculate the orthogonal determinants of the small finite simple groups (see [2] for a survey). The goal of this paper is to determine the orthogonal determinants of the characters in $\mathrm{Irr}^+(G)$ for the two infinite series of finite groups of Lie type, $G = \mathrm{SL}_3(q)$ and $G = \mathrm{SU}_3(q)$, for all prime powers q. For $\mathrm{SL}_2(q)$ the orthogonal character table is already computed in [1]. An important subgroup to analyse the structure and the representations of a finite group G of Lie type is its standard Borel subgroup B. A character $\chi \in \mathrm{Irr}^+(G)$ is called *Borel stable*, if the restriction of χ to B is orthogonally stable. The structure of B as a semidirect product of a p-group and an abelian group allows us to determine the orthogonal determinants of all Borel stable characters of G (see Remark 4.2). For the groups $G = \mathrm{SL}_3(q)$ and $G = \mathrm{SU}_3(q)$ it turns out that those $\chi \in \mathrm{Irr}^+(G)$ that are not Borel stable occur nicely in a permutation character. For such characters Lemma 2.5 below can be used to determine their orthogonal determinants.

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2 Methods

This section collects some basic results about orthogonal determinants.

Definition 2.1. *Let G be a finite group, $H \subseteq G$ a subgroup, and let χ be an orthogonal character of G. Then χ is called H-stable if the restriction $\mathrm{Res}_H^G(\chi)$ of χ to H is an orthogonally stable character of H.*

Lemma 2.2. *(see [9, Proposition 5.17 and Remark 5.21] for a more general statement of (ii))*

(i) If χ is H-stable, then χ is orthogonally stable and

$$\det(\chi) = \det(\mathrm{Res}_H^G(\chi)) \cdot (\mathbb{Q}(\chi)^\times)^2.$$

(ii) If $\chi = \sum_{i=1}^k \chi_i$ is the sum of orthogonally stable characters χ_i then χ is orthogonally stable. Moreover if $\mathbb{Q}(\chi_i) \subseteq \mathbb{Q}(\chi)$ for all i then

$$\det(\chi) = \prod_{i=1}^{k} \det(\chi_i) \cdot (\mathbb{Q}(\chi)^\times)^2.$$

The paper [7] gives an easy formula for the orthogonal determinant of orthogonally stable characters of p-groups. We only need the following special case:

Lemma 2.3. *(see [7, Corollary 4.4]) Let p be an odd prime and let χ be an orthogonally stable rational character of a finite p-group. Then $p-1$ divides $\chi(1)$ and $\det(\chi) = p^{\chi(1)/(p-1)} \cdot (\mathbb{Q}^\times)^2$.*

Lemma 2.4. *(see [9, Proposition 3.12]) Let $\psi = \chi + \overline{\chi}$ for some indicator '0' irreducible character χ. Let $K = \mathbb{Q}(\psi)$, $L = \mathbb{Q}(\chi)$, i.e. K is the maximal real subfield of the complex field L. Choose a totally positive $\delta \in K$ such that $L = K[\sqrt{-\delta}]$. Then*

$$\det(\psi) = \delta^{\chi(1)} \cdot (K^\times)^2.$$

We introduce the following notation: For $m \in \mathbb{N}$, $m > 2$ let $\mu_m := \exp\left(\frac{2\pi i}{m}\right) \in \mathbb{C}$ be the first primitive m-th root of unity. For arbitrary $j \in \mathbb{N}$, we put

$$\vartheta_m^{(j)} := \mu_m^j + \mu_m^{-j} \in \mathbb{R}.$$

For the special case that $L = \mathbb{Q}(\mu_m^j)$ in Lemma 2.4 we obtain $K = \mathbb{Q}(\vartheta_m^{(j)})$ and we can choose $\delta = 2 - \vartheta_m^{(2j)} = -(\mu_m^j - \mu_m^{-j})^2$.

Lemma 2.5. *Let G be a finite group acting on a finite set M and denote by V the associated rational permutation module. Define the G-invariant bilinear form $\beta : V \times V \to \mathbb{Q}$ by choosing M to be an orthonormal basis. Then $V_1 = \langle \sum_{m \in M} m \rangle$ and $V_1^\perp$ are G-invariant subspaces and*

$$\det(\beta_{|V_1^\perp}) = |M| \cdot (\mathbb{Q}^\times)^2.$$

Proof. It is clear that $\det(\beta) = \det(\beta_{|V_1}) \cdot \det(\beta_{|V_1^\perp}) = 1 \cdot (\mathbb{Q}^\times)^2$ and

$$\beta\left(\sum_{m \in M} m, \sum_{m \in M} m\right) = |M|,$$

from which the result follows. □

3 The orthogonal characters of $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$

Let p be a prime and let q be a power of p and put $\mathbb{F}_q^{n\times n}$ to denote the ring of $n\times n$ matrices over the finite field $\mathbb{F}_q$. The group $\mathrm{SL}_3(q)$ is

$$\mathrm{SL}_3(q) = \{A \in \mathbb{F}_q^{3\times 3} | \det(A) = 1\}.$$

The unitary group $\mathrm{SU}_3(q)$ is the stabiliser in $\mathrm{SL}_3(q^2)$ of a non-degenerate Hermitian form H on $\mathbb{F}_{q^2}^3$. Up to isometry there is a unique such form. We put

$$\Omega = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and denote by F the $\mathbb{F}_q$-linear map on $\mathbb{F}_{q^2}^{3\times 3}$ that raises each matrix entry to the q-th power. We choose H to be the Hermitian form associated to the matrix Ω. Then

$$\mathrm{SU}_3(q) := \{A \in \mathrm{SL}_3(q^2) | F(A)^{tr} \cdot \Omega \cdot A = \Omega\}.$$

The letter G will always denote one of $\mathrm{SL}_3(q)$ or $\mathrm{SU}_3(q)$. The full character table of G was first calculated in [11] in 1973 by Simpson and Frame. By 'Ennola duality' (see [4] for the statement and [6] for a proof),

$$"\mathrm{SU}_3(q) = \mathrm{SL}_3(-q)",$$

the irreducible characters of $\mathrm{SU}_3(q)$ can be obtained from the ones of $\mathrm{SL}_3(q)$ by formally replacing every instance of q by $-q$, so that there is a single generic character table giving the character table for both groups introducing an additional parameter $\varepsilon = +1$ for $G = \mathrm{SL}_3(q)$ and $\varepsilon = -1$ for $G = \mathrm{SU}_3(q)$.

In this notation the center of G is the group of scalar matrices in G and hence of order $\gcd(q-\varepsilon, 3)$. In particular the set $\mathrm{Irr}^+(G)$ is the set of irreducible orthogonal characters of even degree of the group

$$\mathrm{PSL}_3(q) = \mathrm{SL}_3(q)/Z(\mathrm{SL}_3(q)) \text{ and } \mathrm{PSU}_3(q) = \mathrm{SU}_3(q)/Z(\mathrm{SU}_3(q)).$$

It is well-known that the groups $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q)$ are simple groups for all prime powers q, apart from $q = 2$, where $\mathrm{PSU}_3(2)$ is solvable. The irreducible characters of $\mathrm{PSU}_3(q)$ and $\mathrm{PSL}_3(q)$ are the irreducible characters of G that are constant on the center.

Gow [5] showed that almost all characters of $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q)$ have Schur index 1; the exception is the character of degree $q^2 - q$ of $\mathrm{PSU}_3(q)$, which has Schur index 2 and Frobenius-Schur indicator '-'. Additionally, the results in [10] allow us to obtain the character fields from some combinatorial description. For cyclotomic numbers we use the notation from Lemma 2.4 and for the naming convention of the irreducible characters we follow [11, Table 2]. Then the set $\mathrm{Irr}^+(G)$ is given as follows:

Theorem 3.1. *The following table includes all $\chi \in \mathrm{Irr}^+(G)$, their degrees $\chi(1)$ and character fields $\mathbb{Q}(\chi)$:*

χ	u	$\chi(1)$	$\mathbb{Q}(\chi)$
χ_{qs}	—	$q(q+\varepsilon)$	$\mathbb{Q}$
χ_{q^3}	—	q^3	$\mathbb{Q}$
$\chi_{st'}^{(u)}$	$0 \le u \le 2$	$1/3(q+\varepsilon)(q^2+\varepsilon q+1)$	$\mathbb{Q}$
$\chi_{st}^{(u,-u,0)}$	$1 \le u < q-\varepsilon$, $u \notin \{(q-\varepsilon)/3, (q-\varepsilon)/2, 2(q-\varepsilon)/3\}$	$(q+\varepsilon)(q^2+\varepsilon q+1)$	$\mathbb{Q}(\vartheta_{q-\varepsilon}^{(u)})$
$\chi_{rt}^{((q-\varepsilon)u)}$	$1 \le u < q+\varepsilon$	$(q-\varepsilon)(q^2+\varepsilon q+1)$	$\mathbb{Q}(\vartheta_{q+\varepsilon}^{(u)})$

- *For q odd and $G = \mathrm{SL}_3(q)$, $\mathrm{Irr}^+(G) = \{\chi_{qs}, \chi_{st'}^{(u)}, \chi_{st}^{(u,-u,0)}, \chi_{rt}^{((q-\varepsilon)u)}\}$.*
- *For q odd and $G = \mathrm{SU}_3(q)$, $\mathrm{Irr}^+(G) = \{\chi_{st'}^{(u)}, \chi_{st}^{(u,-u,0)}, \chi_{rt}^{((q-\varepsilon)u)}\}$.*
- *For q even and $G = \mathrm{SL}_3(q)$, $\mathrm{Irr}^+(G) = \{\chi_{qs}, \chi_{q^3}\}$.*
- *For q even and $G = \mathrm{SU}_3(q)$, $\mathrm{Irr}^+(G) = \{\chi_{q^3}\}$.*

Note that the characters $\chi_{st'}^{(u)}$ only exist for $3|q-\varepsilon$.

4 Results

Let $G = \mathrm{SL}_3(q)$ or $G = \mathrm{SU}_3(q)$. Let

$$B := \left\{ \begin{pmatrix} d & a & b \\ 0 & e & c \\ 0 & 0 & f \end{pmatrix} \in G \right\} \text{ and } U := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in G \right\}.$$

Then U is the unipotent radical of B and a Sylow p-subgroup of G, and $B = N_G(U) = U \rtimes T$ is a (standard) Borel subgroup, where $T := \{\mathrm{diag}(d,e,f) \in G\}$ is a maximal torus. Denote by $W = N_G(T)/T$ the Weyl group of G.

We need an explicit notation for $\mathrm{Irr}(T)$:
For $G = \mathrm{SL}_3(q)$ we fix a generator t of $\mathbb{F}_q^\times$. Then the torus

$$T = \{t_{a,b} := \mathrm{diag}(t^a, t^{-a-b}, t^b) \mid a, b \in \{0, \dots, q-2\}\}$$

is isomorphic to $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ and

$$\mathrm{Irr}(T) = \{\alpha_1^{u_1}\alpha_2^{u_2} : t_{a,b} \mapsto \mu_{q-1}^{au_1+bu_2} \mid u_1, u_2 \in \{0, \dots, q-2\}\}. \tag{2}$$

For $G = \mathrm{SU}_3(q)$ the torus is isomorphic to $\mathbb{F}_{q^2}^\times =: \langle\tau\rangle$. So

$$T = \{\tau_a := \mathrm{diag}(\tau^a, \tau^{(q-1)a}, \tau^{-qa}) \mid a \in \{0, \dots, q^2 - 2\}\}$$

and

$$\mathrm{Irr}(T) = \{\alpha^u : \tau_a \mapsto \mu_{q^2-1}^{au} \mid u \in \{0, \dots, q^2 - 2\}\}. \tag{3}$$

To unify notation we put

$$\theta^{(u)} := \begin{cases} \alpha_1^u \alpha_2^{-u} & G = \mathrm{SL}_3(q) \\ \alpha^u & G = \mathrm{SU}_3(q). \end{cases} \tag{4}$$

Recall that a B-stable character as in Definition 2.1 is also called a *Borel stable* character of G.

Remark 4.1. *By Lemma 2.2 (i) the orthogonal determinant of a Borel stable character χ of G is determined by the restriction of χ to B. Decompose this restriction as*

$$\mathrm{Res}_B^G(\chi) = \chi_T + \chi_U \tag{5}$$

where χ_T is the character of T on the U-fixed space, also known as the Harish-Chandra restriction of χ. In particular its degree is $\chi_T(1) = \langle \mathrm{Res}_U^G(\chi), \mathbf{1}_U\rangle$.

Note that for odd primes p the character χ_U of B is U-stable. As p does not divide the discriminant of $\mathbb{Q}(\chi)$ for all $\chi \in \mathrm{Irr}^+(G)$ we have $\mathbb{Q}(\chi_U) = \mathbb{Q}$ so Lemma 2.3 gives the determinant of χ_U:

Remark 4.2. *Let q be odd. If χ_T is orthogonally stable then*

$$\det(\chi) = \det(\chi_T) p^{\chi_U(1)/(p-1)}$$

Note that T is abelian and so χ_T is a sum of linear characters. If these characters are complex (i.e. of indicator '0') then χ_T is orthogonally stable and its determinant can be computed from Lemma 2.4. In fact the irreducible constituents of χ_T can be obtained from the action of the Weyl group W on $\mathrm{Irr}(T)$. It is well-known that

$$W \cong \begin{cases} S_3 & \text{for } G = \mathrm{SL}_3(q) \\ C_2 & \text{for } G = \mathrm{SU}_3(q) \end{cases}$$

Let $\theta \in \mathrm{Irr}(T)$. Then θ can also be considered as a character of B. A character $\chi \in \mathrm{Irr}(G)$ is said to be in the principal series if χ appears in $\mathrm{Ind}_B^G(\theta)$ for some $\theta \in \mathrm{Irr}(T)$.

We will need a special case of the well-known Mackey formula for Harish-Chandra induction and restriction:

Lemma 4.3. *(see [3, Theorem 5.2.1])*

$$(\mathrm{Ind}_B^G(\theta))_T = \sum_{w \in W} w \cdot \theta.$$

Corollary 4.4. *Let $\chi \in \mathrm{Irr}(G)$. Then $0 \leq \chi_T(1) \leq |W|$, with $\chi_T(1) = 0$ if and only if χ is not in the principal series, and $\chi_T(1) = |W|$ if and only if $\chi = \mathrm{Ind}_B^G(\theta)$ for some $\theta \in \mathrm{Irr}(T)$.*

Explicit calculations with the character table [11, Table 2] now give rise to the following propositions:

Proposition 4.5. *Let $G = \mathrm{SL}_3(q)$. The only characters in $\mathrm{Irr}^+(G)$ which are not in the principal series are $\chi_{rt}^{((q-1)u)}$ for q odd.*

(a) $\theta^{(0)} = \mathbf{1}_T$ is the trivial character and

$$\mathrm{Ind}_B^G(\theta^{(0)}) = \mathbf{1}_G + 2\chi_{qs} + \chi_{q^3}.$$

(b) For $1 \leq u < q-1$, $u \notin \{(q-1)/3, (q-1)/2, 2(q-1)/3\}$, we have that

$$\mathrm{Ind}_B^G(\theta^{(u)}) = \chi_{st}^{(u,-u,0)}$$

and $(\chi_{st}^{(u,-u,0)})_T(1) = 6$, i.e. the U-fixed space in $\chi_{st}^{(u,-u,0)}$ has dimension 6.

(c) For $j \in \{(q-1)/3, 2(q-1)/3\}$, we have that

$$\mathrm{Ind}_B^G(\theta^{(j)}) = \sum_{u=0}^{2} \chi_{st'}^{(u)}$$

where $(\chi_{st'}^{(u)})_T(1) = 2$ for $u = 0, 1, 2$.

Proposition 4.6. *Let $G = \mathrm{SU}_3(q)$. The only characters in $\mathrm{Irr}^+(G)$ which are not in the principal series are $\chi_{st}^{(u,-u,0)}$ and $\chi_{st'}^{(u)}$ for q odd.*

(a) $\theta^{(0)} = \mathbf{1}_T$ is the trivial character and

$$\mathrm{Ind}_B^G(\theta^{(0)}) = \mathbf{1}_G + \chi_{q^3}.$$

(b) For $1 \leq u < q+1$, we have that

$$\mathrm{Ind}_B^G(\theta^{((q+1)u)}) = \chi_{rt}^{((q+1)u)}$$

and $(\chi_{rt}^{((q+1)u)})_T(1) = 2$.

Proof. We will handle both $G = \mathrm{SL}_3(q)$ and $G = \mathrm{SU}_3(q)$ simultaneously. Let $d = \gcd(q-\varepsilon, 3)$. There are $2+d$ conjugacy classes of G which have a non-empty intersection with U: C_1, C_2 and $C_3^{(l)}$, $0 \le l \le d-1$, which can be characterised by $\mathrm{rank}(g_i - I_3) = i-1$ for $g_i \in C_i, 1 \le i \le 3$. We further calculate that $|C_1 \cap U| = 1$, and

$$|C_2 \cap U| = \begin{cases} 2q^2 - q - 1 & \text{for } G = \mathrm{SL}_3(q) \\ q-1 & \text{for } G = \mathrm{SU}_3(q) \end{cases}$$
$$|C_3^{(l)} \cap U| = \begin{cases} 1/d(q^3 - 2q^2 + q) & \text{for } G = \mathrm{SL}_3(q) \\ 1/d(q^3 - q) & \text{for } G = \mathrm{SU}_3(q). \end{cases}$$

We will, as an example, calculate $(\chi_{st}^{(u,-u,0)})_T(1) = \langle \mathrm{Res}_U^G(\chi_{st}^{(u,-u,0)}), \mathbf{1}_U \rangle$. For $\mathrm{SL}_3(q)$, we see that

$$(\chi_{st}^{(u,-u,0)})_T(1) = \frac{1}{q^3}\big((q+1)(q^2+q+1) + (2q^2-q-1)(2q+1) + (q^3-2q^2+q)\big) = 6,$$

whereas for $\mathrm{SU}_3(q)$, the calculation becomes

$$(\chi_{st}^{(u,-u,0)})_T(1) = \frac{1}{q^3}\big((q-1)(q^2-q+1) + (q-1)(2q-1) - (q^3-q)\big) = 0.$$

The rest of the propositions is handled analogously. $\square$

We are now ready to give the main result.

Theorem 4.7. *(i) Let q be odd.*

$\det(\chi)$ *for* $G = \mathrm{SL}_3(q)$	χ	$\det(\chi)$ *for* $G = \mathrm{SU}_3(q)$
$3q \cdot (\mathbb{Q}^\times)^2$	$\chi_{st'}^{(u)}$	$q \cdot (\mathbb{Q}^\times)^2$
$q(2-\vartheta_{q-1}^{(2u)}) \cdot (\mathbb{Q}(\vartheta_{q-1}^{(u)})^\times)^2$	$\chi_{st}^{(u,-u,0)}$	$q \cdot (\mathbb{Q}(\vartheta_{q+1}^{(u)})^\times)^2$
$q \cdot (\mathbb{Q}(\vartheta_{q+1}^{(u)})^\times)^2$	$\chi_{rt}^{((q-\varepsilon)u)}$	$q(2-\vartheta_{q-1}^{(2u)}) \cdot (\mathbb{Q}(\vartheta_{q-1}^{(u)})^\times)^2$

Additionally, for $G = \mathrm{SL}_3(q)$, $\det(\chi_{(qs)}) = (q^2+q+1) \cdot (\mathbb{Q}^\times)^2$.

(ii) Let q be even.

$\det(\chi)$ *for* $G = \mathrm{SL}_3(q)$	χ	$\det(\chi)$ *for* $G = \mathrm{SU}_3(q)$
$(q+1)(q^2+q+1) \cdot (\mathbb{Q}^\times)^2$	χ_{q^3}	$(q^3+1) \cdot (\mathbb{Q}^\times)^2$

Additionally, for $G = \mathrm{SL}_3(q)$, $\det(\chi_{(qs)}) = (q^2+q+1) \cdot (\mathbb{Q}^\times)^2$.

Proof. We first deal with the case that q is odd: Then, for all $\chi \in \mathrm{Irr}^+(G)\backslash\{\chi_{qs}\}$, the value $\chi_U(1)/(p-1)$ is even if and only if q is an even power of p. So Lemma 2.3 gives us that $\det(\chi_U) = q \cdot (\mathbb{Q}^\times)^2$.

For the characters $\chi_{st}^{(u,-u,0)}$ of $\mathrm{SL}_3(q)$ Lemma 4.3 and Proposition 4.5 give that

$$(\chi_{st}^{(u,-u,0)})_T = \theta_1 + \overline{\theta}_1 + \theta_2 + \overline{\theta}_2 + \theta_3 + \overline{\theta}_3$$

is the sum of 6 one-dimensional characters of T, where $\theta_1 = \theta^{(u)}$, $\theta_2 = \alpha_1^{2u}\alpha_2^u$, and $\theta_3 = \alpha_1^u\alpha_2^{2u}$. By Lemma 2.2 and Lemma 2.4

$$\det((\chi_{st}^{(u,-u,0)})_T) = (2 - \vartheta_{q-1}^{(2u)}) \cdot (\mathbb{Q}(\vartheta_{q-1}^{(u)})^\times)^2$$

and

$$\begin{aligned}\det(\chi_{st}^{(u,-u,0)}) &= \det((\chi_{st}^{(u,-u,0)})_T)\det((\chi_{st}^{(u,-u,0)})_U) \cdot (\mathbb{Q}(\vartheta_{q-1}^{(u)})^\times)^2 = \\ & q(2 - \vartheta_{q-1}^{(2u)}) \cdot (\mathbb{Q}(\vartheta_{q-1}^{(u)})^\times)^2.\end{aligned}$$

The results for $\chi_{rt}^{((q-\varepsilon)u)}$ for $\mathrm{SU}_3(q)$ and $\chi_{st'}^{(u)}$ follow similarly. Note here that

$$2 - \vartheta_{q-1}^{(2(q-1)/3)} = 2 - (-1) = 3.$$

The character χ_{qs} for $G = \mathrm{SL}_3(q)$ (q even or odd) occurs in the permutation character $\psi := 1_P^G$ induced from the parabolic subgroup

$$P := \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} \in G \right\}.$$

It is not hard to see that $|G/P| = q^2 + q + 1$ and that $\psi = \mathbf{1}_G + \chi_{qs}$ (see also [12]). Lemma 2.5 hence yields

$$\det(\chi_{qs}) = (q^2 + q + 1) \cdot (\mathbb{Q}^\times)^2.$$

To finish the proof let q be even and regard the characters χ_{q^3}. In both cases these appear in the permutation character $\phi := 1_B^G$.

For $G = \mathrm{SU}_3(q)$ we get $|G/B| = q^3 + 1$ and $\phi = \mathbf{1}_G + \chi_{q^3}$ so $\det(\chi_{q^3}) = q^3 + 1$ by Lemma 2.5.

For $G = \mathrm{SL}_3(q)$ we get $|G/B| = (q+1)(q^2+q+1)$ and $\phi = \mathbf{1}_G + 2\chi_{qs} + \chi_{q^3}$. As χ_{qs} is orthogonally stable we get

$$(q+1)(q^2+q+1) \cdot (\mathbb{Q}^\times)^2 = \det(\chi_{qs})^2 \det(\chi_{q^3}) \cdot (\mathbb{Q}^\times)^2 = \det(\chi_{q^3}) \cdot (\mathbb{Q}^\times)^2,$$

which finishes the proof. □

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