

Orthogonal Stability ^{*}

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ABSTRACT. A character (ordinary or modular) is called **orthogonally stable** if all non-degenerate quadratic forms fixed by representations with those constituents have the same determinant mod squares. We show that this is the case provided there are no odd-degree orthogonal constituents. We further show that if the reduction mod p of an ordinary character is orthogonally stable, this determinant is the reduction mod p of the ordinary one. In particular, if the characteristic does not divide the group order, we immediately see in which orthogonal group it lies. We sketch methods for computing this determinant, and give some examples.

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1 Introduction

The ordinary and Brauer character tables of a finite group, with the (Frobenius-Schur) indicator given, specify the embeddings of the irreducible representations of the group in the classical groups over all finite fields, except that if the degree is even and the indicator is $+$, it does not specify in which of the two orthogonal groups the representation is embedded. We show here that one further value - the **orthogonal discriminant** - added to the ordinary table enables this information also to be specified for all primes not dividing the group order.

A character determines the irreducible constituents, and we say that a character is **orthogonally stable** if there is a representation with that character fixing a non-degenerate quadratic form, and if none of these constituents have odd degree and indicator $+$. In this case, the concept of orthogonal discriminant generalizes by taking proper account of the characters with indicator o and $-$.

The intended meaning of an orthogonally stable character is that it has a well-defined discriminant (see Definition 5.9, Theorem 5.13, and Theorem 5.15). Notice that, no matter which representation we take with this character over an arbitrary finite field, if it fixes a non-degenerate quadratic form at all, it fixes the one specified by the orthogonal discriminant.

For an ordinary character with Schur index 1, the orthogonal discriminant is related to the determinant (mod squares) of the fixed quadratic form, and [15] shows that it can be otherwise defined, still as an element of the character field mod squares, even if the Schur index is not 1. Over a finite field, we denote the orthogonal discriminant by $O+$ or $O-$ depending on which orthogonal group is involved. We show that the reduction mod p of an ordinary character

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is $O+$ (respectively $O-$) if the reduction is orthogonally stable and the ordinary orthogonal discriminant reduces to a square (respectively a non-square) in the finite field (see Theorem 6.4 and Corollary 6.6).

We then go on to consider ways of computing the orthogonal discriminant. The ordinary one has *a priori* only a finite number of possible values - only primes dividing the group order and totally positive units mod squares can be involved - so *ad hoc* methods - restriction, tensors, reduction mod p etc. - can often be used to compute them. In this note we focus on the use of decomposition matrices. The ordinary discriminant is not divisible by primes having an orthogonally stable reduction. If the character lies in a block of defect 1, then also the converse is true. Such arguments sometimes suffice to deduce all ordinary and most modular orthogonal discriminants directly from the decomposition matrices, as illustrated for the first sporadic simple Janko group J_1 . There seems to be no guarantee, however, that these methods will always be sufficient, so we resort quite often to (at least some) explicit computations.

More may be true than we prove here. We have yet to see an ordinary orthogonal discriminant divisible by an odd power of any prime dividing 2.

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2 Overview of the paper

The paper starts with introducing the relevant facts about quadratic forms in Section 3. The most important notion here is the discriminant of a quadratic form. We also recall some rules how this discriminant behaves with respect to orthogonal sums and extension and restriction of scalars. Section 3.4 compares to Section 3.3 in the spirit of Brauer who uses group rings over discrete valuation rings R as a bridge between ordinary and modular representation theory. For each even dimension there are exactly two regular quadratic forms over R . The reduction modulo the maximal ideal is either of type $O+$ or $O-$, and the discriminant algebra either $R \oplus R$ or the unique unramified quadratic extension of R . This observation becomes crucial in Section 6 where we compare orthogonal discriminants of ordinary characters with the ones in their reductions modulo primes.

Section 4 gives the notion of orthogonal representations. These are KG -modules V with a non-degenerate G -invariant quadratic form Q . The representation hence takes values in the orthogonal group $O(V, Q)$. When the characteristic of K divides the group order the KG -module V is not necessarily semisimple. Under mild assumptions in characteristic 2 we may replace it by some semisimple KG -module, the **anisotropic kernel**, having the same orthogonal discriminant (see Section 4.1).

The main notion of the paper, **orthogonal stability**, is defined in Section 5. Orthogonally stable characters χ are exactly those that have a well-defined orthogonal discriminant (Theorem 5.15) in the sense of Definition 5.9. After briefly recalling the notions of Brauer characters and reduction of ordinary characters, we classify the orthogonally simple characters. These are orthogonal characters not containing smaller orthogonal characters. They fall in three categories as given in Proposition 5.7. Theorem 5.10 expresses the orthogonal discriminants for two of the three categories (indicator o and $-$) in terms of character values. The main result of this paper is Theorem 5.13 stating that orthogonally stable characters χ have a well-defined discriminant,

$\text{disc}(\chi)$, the **orthogonal discriminant of χ** . To determine $\text{disc}(\chi)$ it suffices to compute the orthogonal discriminants of all the indicator + constituents that occur in χ with odd multiplicity (see Proposition 5.17 for an explicit formula).

The remaining two sections are devoted to showing that decomposition matrices help to compute orthogonal discriminants for both modular and ordinary characters. Assume that \mathcal{X} is an orthogonally stable ordinary character with character field K . If the reduction of \mathcal{X} modulo a prime ideal \wp in the ring of algebraic integers of K is orthogonally stable, then this prime ideal does not divide the orthogonal discriminant of \mathcal{X} (see Theorem 6.4 for a precise statement). In particular all primes that divide $\text{disc}(\mathcal{X})$ do also divide the group order leaving only finitely many possible orthogonal discriminants (Corollary 6.7). Subsection 6.3 sketches an algorithm to deduce which of the square classes is the true orthogonal discriminant by computing enough orthogonal discriminants of orthogonally stable modular reductions. We illustrate these methods computing the orthogonal discriminants of the irreducible ordinary characters of the Held group He from its p -modular orthogonal discriminants and the decomposition matrices for the primes p dividing $|He|$. For absolutely irreducible characters \mathcal{X} of p -defect one, the primes \wp above p do not divide the discriminant if and only if the reduction of \mathcal{X} modulo \wp is orthogonally stable (Theorem 6.10). This observation is enough to compute the orthogonal discriminant of all ordinary characters for the group J_1 from its decomposition matrices that are available in GAP [7].

3 Quadratic forms

This section recalls some basic facts on quadratic forms. For more details we refer the reader to the lecture notes [10] or the more elaborate textbooks [21], [17], or [13]. The most important notion in this section is the discriminant of a quadratic form and its discriminant algebra as it is given in [10, Section 10].

Let K be a field and V a finite dimensional vector space over K . A quadratic form is a map $Q : V \rightarrow K$ such that $Q(ax) = a^2Q(x)$ for all $a \in K, x \in V$ and such that its **polarisation**

$$B : V \times V \rightarrow K, B(x, y) := Q(x + y) - Q(x) - Q(y) \tag{1}$$

is a bilinear form. The determinant of Q is the determinant of a Gram matrix of its polarisation. This is well defined modulo squares. The quadratic form Q is called **non-degenerate**, if its polarisation is non-degenerate, i.e. $\det(Q) \neq 0$. We then also call (V, Q) a non-degenerate quadratic space.

Definition 3.1. *If the characteristic of K is not 2 then the discriminant of Q is*

$$\text{disc}(Q) := (-1)^{\binom{n}{2}} \det(B)(K^\times)^2 \in K/(K^\times)^2.$$

If $\delta \in K^\times$ represents $\text{disc}(Q)$ then the discriminant algebra of Q is $\mathcal{D}(Q) := K[X]/(X^2 - \delta)$.

Note that the discriminant algebra of a non-degenerate quadratic form is either a field extension of degree 2 of K or isomorphic to $K \oplus K$. The latter is exactly the case if the discriminant of Q is a square, the **trivial discriminant**.

In characteristic 2 the bilinear form B does not determine the quadratic form Q and one needs to replace the discriminant by the Arf invariant. This is a class of $(K, +)/\wp(K)$, where

$$\wp(K) := \{a^2 + a \mid a \in K\}$$

is a subgroup of the additive group $(K, +)$.

To simplify notation, we again call this invariant the discriminant of the quadratic form in characteristic 2.

Remark 3.2. *Let K be a field of characteristic 2 and $Q : V \rightarrow K$ a non-degenerate quadratic form. Then the polarisation B of Q is symplectic and hence $\dim(V)$ is even, say $\dim(V) = 2m$. By [10, Definition 10.7] the quadratic space (V, Q) is the orthogonal sum of 2-dimensional non-degenerate spaces*

$$(V, Q) = \perp_{i=1}^m \langle e_i, f_i \rangle$$

with $B(e_i, f_i) = 1$. Then

$$\text{disc}(Q) := \sum_{i=1}^m Q(e_i)Q(f_i) + \wp(K) \in (K, +)/\wp(K)$$

is well-defined. We call $\text{disc}(Q)$ the **discriminant** of Q and refer to $(K, +)/\wp(K)$ as the “square classes” of K . If $\text{disc}(Q) = b + \wp(K)$ then we put the **discriminant algebra** of Q to be $\mathcal{D}(Q) := K[X]/(X^2 + X + b)$.

Note that $b \in \wp(K)$ if and only if the polynomial $X^2 + X + b \in K[X]$ is reducible, if and only if $\mathcal{D}(Q) = K \oplus K$. In this case we say that Q has **trivial discriminant**.

There is a group structure on the set of quadratic K -algebras (see [10, Section 10]) so that one may treat discriminant algebras simultaneously in even and odd characteristic. However, we prefer to work with numbers, where one should keep in mind that multiplication of discriminants means addition of the representatives in characteristic 2. In this sense, for even dimensional quadratic spaces, the discriminant is multiplicative with respect to orthogonal direct sums:

Remark 3.3. *If Q and Q' are two quadratic forms of even dimension then*

$$\text{disc}(Q \perp Q') = \text{disc}(Q) \text{disc}(Q').$$

3.1 Restriction of scalars

Let F be a field and K be an extension field of finite degree, say d . Let (V, Q) be an n -dimensional quadratic space over K . Then restriction of scalars turns V into an nd -dimensional space over F . If T denotes the trace of K/F then $T \circ Q : V \rightarrow F$ is a quadratic form that is non-degenerate if Q is non-degenerate and K/F is separable (the latter condition is always fulfilled if K is finite or of characteristic 0). Then

$$\det(T \circ Q) = N_{K/F}(\det(Q)) \text{disc}(K/F)^n$$

where $\text{disc}(K/F)$ is the field discriminant of K over F .

Remark 3.4. *(cf. [11, Lemma 2.2]) If K/F is separable and n is even, then*

$$\text{disc}(T \circ Q) = \begin{cases} N_{K/F}(\text{disc}(Q)) & \text{char}(K) \neq 2 \\ T_{K/F}(\text{disc}(Q)) & \text{char}(K) = 2. \end{cases}$$

3.2 Hyperbolic forms

For a finite dimensional K -vector space V we put

$$V^\vee := \text{Hom}_K(V, K) = \{f : V \rightarrow K \mid f \text{ is } K\text{-linear}\}$$

to denote the dual space of V . Any non-degenerate bilinear form B on V yields an isomorphism

$$\tilde{B} : V \rightarrow V^\vee, v \mapsto (x \mapsto B(v, x)). \quad (2)$$

The hyperbolic module $\mathbf{H}(V)$ is the quadratic K -space

$$(V \oplus V^\vee, Q) \text{ with } Q((v, f)) := f(v) \text{ for all } v \in V, f \in V^\vee.$$

Then $\dim(\mathbf{H}(V)) = 2 \dim(V)$ is even and the discriminant algebra $\mathcal{D}(\mathbf{H}(V)) = K \oplus K$. So hyperbolic modules always have trivial discriminant.

3.3 Quadratic forms over finite fields

Let \mathbb{F}_q denote the field with q elements. It is well known (see [25, Theorem 11.4]) that there are two non-degenerate quadratic forms of dimension 2 over \mathbb{F}_q , the hyperbolic plane $\mathbf{H}(\mathbb{F}_q)$ and the norm form $N(\mathbb{F}_q)$ of the quadratic extension of \mathbb{F}_q . The underlying space for $N(\mathbb{F}_q)$ is \mathbb{F}_{q^2} , regarded as a 2-dimensional \mathbb{F}_q -space, and the quadratic form is

$$Q(x) := x\bar{x} = xx^q = x^{q+1}.$$

On an \mathbb{F}_q -vector space of even dimension $2m$ there are two isometry classes of non-degenerate quadratic forms. These two forms can be distinguished by many invariants such as discriminants, Witt index, the number of isotropic vectors, and also the order of the orthogonal group (see for instance [10, Chapter IV], [25, Chapter 11]). To fix notation we denote these two forms by

$$Q_{2m}^+(q) := \mathbf{H}(\mathbb{F}_q)^m \text{ and } Q_{2m}^-(q) := \mathbf{H}(\mathbb{F}_q)^{m-1} \perp N(\mathbb{F}_q).$$

The discriminant algebra of $Q_{2m}^+(q)$ is $\mathbb{F}_q \oplus \mathbb{F}_q$ whereas $\mathcal{D}(Q_{2m}^-(q)) = \mathbb{F}_{q^2}$.

Definition 3.5. *We also use the symbols $O+$ and $O-$ to distinguish between these two quadratic forms and sometimes write that the discriminant of $Q_{2m}^+(q)$ is $O+$ and $\text{disc}(Q_{2m}^-(q)) = O-$.*

In analogy to Remark 3.3 we get

Remark 3.6. *For $m, n \in \mathbb{N}$ we have $Q_{2m}^+(q) \perp Q_{2n}^+(q) \cong Q_{2(n+m)}^+(q)$, $Q_{2m}^+(q) \perp Q_{2n}^-(q) \cong Q_{2(n+m)}^-(q)$, and $Q_{2m}^-(q) \perp Q_{2n}^-(q) \cong Q_{2(n+m)}^+(q)$.*

In the situation of Subsection 3.1 the restriction of scalars from \mathbb{F}_{q^d} to \mathbb{F}_q does not change the type ($O+$ or $O-$) of the quadratic space.

Remark 3.7. *Let $T : \mathbb{F}_{q^d} \rightarrow \mathbb{F}_q$ denote the trace. Then*

$$T(Q_{2m}^+(q^d)) = Q_{2md}^+(q) \text{ and } T(Q_{2m}^-(q^d)) = Q_{2md}^-(q).$$

Proof. Let Q be a non-degenerate even dimensional quadratic form over \mathbb{F}_{q^d} of discriminant δ . First assume that q is odd. Then by Remark 3.4 the discriminant of $T \circ Q$ is $N_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\delta)$. The norm is a group epimorphism between the multiplicative groups of the two fields. In particular it maps the unique subgroup $(\mathbb{F}_{q^d}^\times)^2$ of index 2 in $\mathbb{F}_{q^d}^\times$ onto the subgroup of squares in \mathbb{F}_q^\times . So $N_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\delta)$ is a square in \mathbb{F}_q^\times if and only if δ is a square in $\mathbb{F}_{q^d}^\times$. Now assume that q is even. Then $\wp(\mathbb{F}_q) \leq (\mathbb{F}_q, +)$ is a subgroup of index 2 and $T_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\wp(\mathbb{F}_{q^d})) = \wp(\mathbb{F}_q)$. In particular the discriminant of Q lies in $\wp(\mathbb{F}_{q^d})$ if and only if the discriminant of $T \circ Q$ lies in $\wp(\mathbb{F}_q)$. \square

From the computations in characteristic 2 we may conclude the following observation, which is certainly well known:

Corollary 3.8. *Let $q = 2^d$ and $b \in \mathbb{F}_q$. Then $X^2 + X + b \in \mathbb{F}_q[X]$ is irreducible, if and only if the trace $T_{\mathbb{F}_q/\mathbb{F}_2}(b) = 1$.*

Remark 3.9. *(see [22, Proposition 4.9 (a),(d)]) Any n -dimensional quadratic space (V, Q) over \mathbb{F}_q extends to an n -dimensional quadratic space $(V \otimes \mathbb{F}_{q^d}, Q)$ over \mathbb{F}_{q^d} by putting $Q(x \otimes a) := a^2 Q(x)$ for $x \in V$, $a \in \mathbb{F}_{q^d}$. For even degree extensions this quadratic space over \mathbb{F}_{q^d} is always of type $O+$. The type is, however, stable under odd degree field extensions.*

(a) *If d is even then $Q_{2m}^+(q) \otimes \mathbb{F}_{q^d} \cong Q_{2m}^-(q) \otimes \mathbb{F}_{q^d} \cong Q_{2m}^+(q^d)$.*

(b) *If d is odd then $Q_{2m}^+(q) \otimes \mathbb{F}_{q^d} \cong Q_{2m}^+(q^d)$ and $Q_{2m}^-(q) \otimes \mathbb{F}_{q^d} \cong Q_{2m}^-(q^d)$.*

3.4 Quadratic forms over discrete valuation rings

In this section we recall some results on regular quadratic forms over local rings for the special situation needed in this paper. So let R be a complete discrete valuation ring and π a generator of its unique maximal ideal. We assume that the residue field $F := R/\pi R$ is a finite field with q elements and that the field of fractions K of R has characteristic 0. A quadratic lattice (M, Q) is a free R -module M of finite rank together with an R -valued quadratic form $Q : M \rightarrow R$. The quadratic form is called **regular** if the map \tilde{B} from equation (2) restricts to an R -isomorphism between M and its dual module $M^\vee = \text{Hom}_R(M, R)$. This is equivalent to saying that the determinant of Q is a unit in R .

As shown in [10, Satz 15.6] there is a bijection between isometry classes of regular quadratic R -lattices and non-degenerate quadratic forms over the residue field F . In particular in our situation there are exactly two isometry classes of regular quadratic R -lattices (M, Q) of even rank $2m$, corresponding to the two possible residue forms $(M/\pi M, \overline{Q}) \cong Q_{2m}^+(q)$ respectively $Q_{2m}^-(q)$. We denote these regular quadratic lattices by $Q_{2m}^+(R)$ respectively $Q_{2m}^-(R)$. Note that the norm form of the ring of integers in the unique unramified extension of degree 2 of K has as residue form $Q_2^-(q) = N(\mathbb{F}_q)$, so we get the following remark.

Remark 3.10. *The discriminant of $Q_{2m}^+(R)$ is 1. If $\delta \in \text{disc}(Q_{2m}^-(R))$ then δ is a unit in R such that $K[\sqrt{\delta}]$ is the unique unramified extension of degree 2 of K .*

One important property of regular quadratic sublattices is that they split as orthogonal direct summands.

Lemma 3.11. (see [10, Satz 1.6]) Let (M, Q) be a quadratic R -lattice and $N \leq M$ be some R -submodule such that $(N, Q|_N)$ is regular. Then

$$(M, Q) = (N, Q|_N) \oplus (N^\perp, Q|_{N^\perp}).$$

3.5 Hermitian forms

Hermitian forms arise naturally when considering non self-dual characters as in Proposition 5.7 (b). The formula from Proposition 3.12 below is used in Theorem 5.10 to find the orthogonal discriminant of such orthogonally simple characters. For a Galois extension L/K of degree $[L : K] = 2$ let $\bar{} \in \text{Gal}_K(L)$ denote the non-trivial Galois automorphism. A Hermitian form H on a non-zero L -vector space V is a map $H : V \times V \rightarrow L$ that is L -linear in the first argument and such that $H(y, x) = \overline{H(x, y)}$ for all $x, y \in V$. If $n := \dim_L(V)$ then restriction of scalars turns V into a K -vector space of dimension $2n$ and H defines a quadratic form $Q_H : V \rightarrow K, Q_H(x) := H(x, x)$ for all $x \in V$.

Proposition 3.12. (see [21, page 350]) Let (V, H) be a non-degenerate Hermitian L -vector space.

(a) Let $\text{char}(K) \neq 2$ and write $L = K[\sqrt{\delta}]$. Then $\text{disc}(Q_H) = \delta^n (K^\times)^2$.

(b) If $K \cong \mathbb{F}_q$ with $q = 2^d$, then the discriminant of Q_H is trivial if n is even, and non-trivial if n is odd. So $Q_H \cong Q_{2n}^+(q)$ if n is even, and $Q_H \cong Q_{2n}^-(q)$ if n is odd.

Note that [21] also gives the Clifford invariant of Q_H as well as the Arf invariant for general fields of characteristic 2.

4 Orthogonal representations

Let G be a finite group and K be a field. Any KG -module $V \neq \{0\}$ defines a group homomorphism $\rho : G \rightarrow \text{GL}(V)$. Then ρ is called a K -representation of G . Equivalence of representations is defined as isomorphism of KG -modules. The representation ρ is called **irreducible** if V is a simple KG -module, i.e. $\{0\}$ and V are the only G -invariant submodules of V . The **trivial representation** is the map $\rho : G \rightarrow \text{GL}(K), \rho(g) = \text{id}_K$ for all $g \in G$.

For any KG -module V also the dual space V^\vee is a KG -module. The corresponding representation ρ^\vee is called the dual representation of ρ . The representation ρ is called **self-dual**, if ρ^\vee is equivalent to ρ .

Given a representation ρ or equivalently a KG -module V we put

$$\mathcal{Q}(V) = \mathcal{Q}(\rho) := \{Q : V \rightarrow K \mid Q \text{ is a quadratic form, } Q(x\rho(g)) = Q(x) \text{ for all } x \in V, g \in G\}$$

the space of G -invariant quadratic forms. We call ρ an **orthogonal representation**, if $\mathcal{Q}(\rho)$ contains a non-degenerate quadratic form Q . Then (V, Q) is also called an **orthogonal KG -module**.

Remark 4.1. (a) Let (V, Q) be an orthogonal KG -module. Then the polarisation of Q defines a KG -isomorphism \tilde{B} (see equation (2)) between V and its dual module V^\vee . In particular orthogonal representations are self-dual.

- (b) The set $\mathcal{Q}(V)$ is a vector space over K . So if $Q \in \mathcal{Q}(V)$ then also $aQ \in \mathcal{Q}(V)$ for all $a \in K$. Clearly $\text{disc}(aQ) = a^{\dim(V)} \text{disc}(Q)$ so for odd dimensional orthogonal representations the discriminants of the invariant quadratic forms represent all square classes of K and hence odd dimensional representations cannot have a well defined orthogonal discriminant.
- (c) If V is a KG -module then the hyperbolic module $\mathbf{H}(V) = (V \oplus V^\vee, Q)$ from Section 3.2 is an orthogonal KG -module.

4.1 The anisotropic kernel

If the characteristic of K divides the group order, then not all representations are direct sums of irreducible ones. However, given an orthogonal KG -module (V, Q) there is a KG -submodule $U \leq V$ with $Q(U) = \{0\}$ such that the sub-quotient U^\perp/U is the orthogonal direct sum of simple orthogonal KG -modules, where here we need to assume that the trivial representation is not a constituent of V if the characteristic of K is 2. The anisotropic kernel U^\perp/U is uniquely determined up to KG -isometry and has the same discriminant as (V, Q) (see Proposition 4.4 below).

Lemma 4.2. *Let $\rho : G \rightarrow \text{GL}(V)$ be an irreducible representation and $Q \in \mathcal{Q}(\rho)$ be a non-zero G -invariant quadratic form on V . Then either Q is non-degenerate or $\text{char}(K) = 2$, $Q : V \rightarrow K$ is an \mathbb{F}_2G -module homomorphism and ρ is the trivial representation.*

Proof. Assume that Q is degenerate. Since its polarisation B is G -invariant, the radical

$$\text{rad}(B) = V^\perp = \{v \in V \mid B(v, w) = 0 \text{ for all } w \in V\}$$

is a KG -submodule of V . As V is simple and $\text{rad}(B) \neq 0$, we hence have $\text{rad}(B) = V$ and hence $B = 0$. Now $B(x, x) = 2Q(x)$ for all $x \in V$ so $Q \neq 0 = B$ implies that $\text{char}(K) = 2$. Moreover the equation (1) shows that Q is a group homomorphism between the additive group of V and K . The G -invariance of Q implies that the kernel of this homomorphism is a submodule of the simple KG -module V , so Q is injective. Moreover for $x \in V$ and $g \in G$ we have $Q(x + xg) = Q(x) + Q(xg) = 2Q(x) = 0$, so $x + xg \in \ker(Q) = \{0\}$ and hence $x = xg$ for all $x \in V, g \in G$. This implies that V is the trivial KG -module. \square

Definition 4.3. *A submodule U of the orthogonal KG -module (V, Q) is called isotropic if $Q(U) = \{0\}$. An orthogonal KG -module (V, Q) is called anisotropic if it does not contain a non-zero isotropic KG -submodule.*

Let $\rho : G \rightarrow \text{GL}(V)$ be an orthogonal representation and $Q \in \mathcal{Q}(\rho)$ be a non-degenerate quadratic form. Let U be a maximal isotropic KG -submodule of V . Then Q defines a quadratic form \overline{Q} on U^\perp/U by $\overline{Q}(x + U) := Q(x)$ for all $x \in U^\perp$. The quadratic KG -module $(W, Q_0) := (U^\perp/U, \overline{Q})$ is uniquely determined by ρ and Q up to KG -isometry and called the anisotropic kernel of ρ (see for instance [6, Lemma 4.2 (8)]). Note that $V = \mathbf{H}(U)$ is hyperbolic if and only if its anisotropic kernel is $\{0\}$. The discriminant of a 0-dimensional quadratic space is defined as 1.

As the referee pointed out the following proposition is also proved in [4] and [23] who calls (W, Q_0) the Witt kernel of (V, Q) . Both authors work with anisotropic bilinear forms, for quadratic forms the extra assumption that ρ have no trivial constituent in characteristic 2 is needed (see [16] for the general situation in characteristic 2).

Proposition 4.4. *Assume that either $\text{char}(K) \neq 2$ or the trivial representation is not a constituent of ρ . Then the anisotropic kernel (W, Q_0) of (V, Q) is the orthogonal sum of simple orthogonal KG -modules. Moreover $\text{disc}(Q_0) = \text{disc}(Q)$.*

Proof. If no simple submodule of V is isotropic, then put $W := V$, $Q_0 := Q$. Otherwise there is a simple isotropic submodule $S \leq V$. Then Q induces a quadratic form on S^\perp/S . Replacing V by this smaller dimensional module we finally arrive at a module (W, Q_0) for which the restriction of Q_0 to any simple submodule is non-zero. As we assumed that the trivial module is not a constituent of V (and hence of W) when the characteristic of K is 2, Lemma 4.2 shows that the restriction of Q_0 to any simple submodule S of W is non-degenerate. Hence S splits as a direct orthogonal summand $W = S \oplus S^\perp$. So W is the orthogonal direct sum of simple quadratic KG -modules. \square

5 Characters

We use capital letters \mathcal{X} only for ordinary characters, whereas χ may stand for either an ordinary or a Brauer character.

5.1 Brauer characters

In characteristic 0, the character of a K -representation $\rho : G \rightarrow \text{GL}(V)$ is the map $\mathcal{X}_\rho : G \rightarrow K, g \mapsto \text{trace}(\rho(g))$ where the trace is the usual trace of an endomorphism. One important property here is that two representations are equivalent if and only if they have the same character. In positive characteristic, p , such a definition of character is not information preserving. For instance the character of the sum of p isomorphic modules would be identically 0. The definition of Brauer character can be found in [12, Chapter 2] and [8, Chapter 15]. Given a field K of characteristic $p > 0$ and a K -representation ρ the Brauer character χ_ρ of ρ is a map from the p' -conjugacy classes of G to the field of complex numbers. For $g \in G$ of order n not divisible by p (or just $g \in G_{p'}$) the eigenvalues of $\rho(g)$ (in an algebraic closure \hat{K} of K) are n -th roots of unity. Conway polynomials provide one algorithmic method for agreeing on a consistent family of choices to identify these roots of unity in \hat{K} with complex roots of unity (see [9, Introduction]). Then the Brauer character of ρ assigns to g the sum of these complex roots of unity.

The Brauer character of ρ does in general not determine the representation ρ up to equivalence. However, it determines the multiset of composition factors of ρ up to isomorphism.

Definition 5.1. *Let \mathcal{X} be an ordinary character of G . The character field $\mathbb{Q}(\mathcal{X})$ is the abelian number field $\mathbb{Q}[\mathcal{X}(g) : g \in G]$ generated over the rationals by all character values of \mathcal{X} .*

For a p -Brauer character χ of G the character field $\mathbb{F}_p(\chi)$ is obtained by reducing the ring of integers of the number field $L = \mathbb{Q}[\chi(g) \mid g \in G_{p'}]$ generated by the Brauer character values modulo any prime ideal that divides p . As L is Galois over \mathbb{Q} , the finite field $\mathbb{F}_p(\chi)$ does not depend on the choice of the prime ideal.

5.2 Reduction of characters

Let K be a number field, O_K its ring of integers, \wp a non-zero prime ideal of O_K with residue field $F := O_K/\wp$. For a K -representation $\rho : G \rightarrow \text{GL}(V)$ a reduction mod \wp is obtained by

choosing a G -invariant O_K -lattice M in V , i.e. a finitely generated O_K -submodule M in V that contains a K -basis of V and such that $M\rho(g) = M$ for all $g \in G$. Then $\bar{V} := M/\wp M$ is an FG -module. The composition factors of \bar{V} are independent of the choice of the $O_K G$ -lattice M in V (see for instance [8, Theorem 15.6]). The corresponding representation $\bar{\rho} : G \rightarrow \mathrm{GL}(\bar{V})$ is called a **reduction of ρ modulo \wp** . Whereas the representation $\bar{\rho}$ depends on the choice of the lattice M , its Brauer character only depends on ρ and on \wp .

Remark 5.2. *Note that the character field of the Brauer character of $\bar{\rho}$ can be strictly contained in F . The smallest example in [2] are the two representations ρ of degree 4 of the group $2.S_5$, where the character field of ρ is $\mathbb{Q}(\rho) = \mathbb{Q}[\sqrt{-3}]$. The irrationality only occurs as the character value of the outer elements of order 6, thus the reduction $\bar{\rho}$ of ρ modulo 2 has a rational Brauer character of degree 4 and character field $\mathbb{F}_2(\bar{\rho}) = \mathbb{F}_2$.*

Given an ordinary character \mathcal{X} with character field K , the **reduction of \mathcal{X} modulo \wp** is the Brauer character of a reduction of a representation ρ (over a suitable field extension) affording \mathcal{X} . We can compute the reduction of \mathcal{X} modulo \wp directly from the character table as the restriction of some Galois conjugate of \mathcal{X} to the p' -classes. For the computations of ordinary orthogonal discriminants from p -modular ones using the decomposition matrix it is crucial to carefully distinguish between the different p -modular reductions arising from the different prime ideals that divide p (see Section 7.1 for examples).

Remark 5.3. *For any finite abelian extension K of \mathbb{Q} the Conway polynomials from Section 5.1 identify a unique prime ideal \wp_0 of O_K that contains the prime p so that the reduction of \mathcal{X} modulo \wp_0 is the restriction of \mathcal{X} to the p' -classes of G .*

For any prime ideal \wp of O_K that divides p , there is $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\wp = \sigma^{-1}(\wp_0)$. Then the reduction of \mathcal{X} modulo \wp is obtained as the reduction of $\sigma(\mathcal{X})$ modulo \wp_0 .

5.3 Orthogonally simple characters

In the following the notion ‘‘character’’ denotes either an ordinary character or a Brauer character of the finite group G .

Definition 5.4. *A character χ is called **orthogonal**, if there is an orthogonal representation with character χ . An orthogonal character χ is called **orthogonally simple**, if χ is not a proper sum of orthogonal characters.*

Remark 5.5. *If χ is any character then the character χ^\vee of the dual representation is just the complex conjugate of χ . Remark 4.1 (c) shows that $\chi + \chi^\vee$ is always an orthogonal character.*

In [9] the well known Frobenius Schur indicator for absolutely irreducible ordinary characters, is extended to cover absolutely irreducible Brauer characters. We combine the two definitions:

Definition 5.6. *Let χ be an absolutely irreducible character or Brauer character.*

(+) *If χ is orthogonal then the indicator of χ is +.*

(+) *In characteristic 2 also the trivial character is of indicator +.*

(o) If χ is not self-dual then its indicator is 0.

(-) In all other cases the indicator of χ is $-$.

In particular the absolutely irreducible characters of indicator $+$ are exactly those that are afforded by some representation ρ (over some extension field of the character field) for which $\mathcal{Q}(\rho) \neq \{0\}$.

Proposition 5.7. *Let χ be an orthogonally simple character. Then one of the following holds:*

(a) χ is absolutely irreducible of indicator $+$.

(b) $\chi = \psi + \psi^\vee$ for some absolutely irreducible character ψ of indicator 0.

(c) $\chi = 2\psi$ for some absolutely irreducible character ψ of indicator $-$.

Proof. Let ψ be an absolutely irreducible constituent of χ . As χ is self-dual, also ψ^\vee is a constituent of χ . If $\psi \neq \psi^\vee$ then $\psi + \psi^\vee$ is an orthogonal sub-character of χ and hence $\chi = \psi + \psi^\vee$. If $\psi = \psi^\vee$ and ψ has indicator $+$ then already ψ is orthogonal and hence $\chi = \psi$. If ψ is self-dual but not orthogonal, then all orthogonal characters containing ψ also contain 2ψ . As $2\psi = \psi + \psi^\vee$ is orthogonal we are in situation (c). \square

Corollary 5.8. *The only orthogonally simple characters of odd degree are the absolutely irreducible indicator $+$ odd degree characters that are not the trivial Brauer character in characteristic 2.*

Definition 5.9. *Let χ be an orthogonal ordinary or Brauer character and denote by K its character field. We say that “the orthogonal discriminant of χ is well-defined” if there is a quadratic K -algebra $\mathcal{D}(\chi)$ with the following property: Given an orthogonal representation ρ over some field L and with character χ . Then all non-degenerate forms $Q \in \mathcal{Q}(\rho)$ have discriminant algebra $\mathcal{D}(Q) = L \otimes_K \mathcal{D}(\chi)$.*

The quadratic K -algebra $\mathcal{D}(\chi)$ is then called the discriminant algebra of χ .

If $\mathcal{D}(\chi) = K[X]/(X^2 - \delta)$ (for $\text{char}(K) \neq 2$) respectively $\mathcal{D}(\chi) = K[X]/(X^2 + X + b)$ (in characteristic 2) then

$$\text{disc}(\chi) = \begin{cases} \delta(K^\times)^2 & \text{char}(K) \neq 2 \\ b + \wp(K) & \text{char}(K) = 2 \end{cases}$$

is called the orthogonal discriminant of χ .

Also for Brauer characters χ we sometimes write $\text{disc}(\chi) = O+$ if $\text{disc}(\chi)$ is trivial and $\text{disc}(\chi) = O-$ for the non-trivial square class.

By Remark 4.1 (b) no character of odd degree has a well-defined discriminant.

Theorem 5.10. *The orthogonal discriminant of an orthogonally simple character of even degree is well-defined.*

Proof. Let χ be an orthogonally simple character of even degree and denote by K the character field of χ . We go through the three cases from Proposition 5.7:

- (a) Here χ is absolutely irreducible of indicator $+$ and even degree. For number fields K the short note [15] shows that the orthogonal discriminant of χ is well-defined. For finite fields K , there is an orthogonal representation ρ over K with character χ . As χ is absolutely irreducible the dimension of the K -space $\mathcal{Q}(\rho)$ is 1, so $\mathcal{Q}(\rho) = \{aQ \mid a \in K\}$ for some non-degenerate Q . Clearly this holds also if we extend scalars, $\mathcal{Q}(L \otimes \rho) = \{aQ \mid a \in L\}$. So χ is orthogonally stable and $\text{disc}(\chi) = \text{disc}(Q)$.
- (b) If χ is a p -Brauer character and $\mathbb{F}_p(\psi) = \mathbb{F}_p(\chi)$, then Q is hyperbolic and hence has trivial discriminant. Otherwise Q comes from a Hermitian form and $\text{disc}(\chi)$ can be read off from Proposition 3.12.
- (c) For number fields K [24, Theorem B] asserts that $\text{disc}(\chi) = 1$ in this case. If K is a finite field, then ψ is the character of a representation over K . If $\rho : G \rightarrow \text{GL}(V)$ is a representation affording the character 2ψ and $Q \in \mathcal{Q}(\rho)$ non-degenerate, then the restriction of the quadratic form Q to any simple submodule of V is identically zero. In particular the form Q is hyperbolic and hence has trivial discriminant.

□

Remark 5.11. *In characteristic 0 the orthogonally simple characters are exactly the characters of the irreducible \mathbb{R} -representations. In particular any orthogonally simple character is the character of a representation ρ over some real number field L . Moreover $B := \sum_{g \in G} \rho(g)\rho(g)^{\text{tr}}$ is an invariant symmetric bilinear form that is **totally positive definite**, i.e. for all ring homomorphisms ϵ of L into the reals the form $\epsilon(B)$ is positive definite. Hence the orthogonal discriminant of an orthogonally simple character of degree $2m$ is $(-1)^m d(K^\times)^2$ for some totally positive d in the character field K .*

5.4 Orthogonally stable characters

An orthogonally stable character is the sum of even degree orthogonally simple characters:

Definition 5.12. *An orthogonal character χ is called **orthogonally stable** if all absolutely irreducible indicator $+$ constituents of χ have even degree.*

Our main result of this section is the following theorem:

Theorem 5.13. *Any orthogonally stable character has a well-defined orthogonal discriminant in the sense of Definition 5.9.*

The proof of this important theorem is split into two cases: Subsection 5.5 proves Theorem 5.13 for finite fields and Subsection 5.6 for arbitrary fields of characteristic not 2.

There are orthogonal representations ρ such that all non-degenerate forms in $\mathcal{Q}(\rho)$ are isometric without the character of ρ being orthogonally stable, as the following example shows.

Example 5.14. Take G to be the normaliser of a Sylow-3-subgroup of $O_4^+(\mathbb{F}_3)$. Then G is the extension of $C_3 \times C_3$ by an elementary abelian 2-group of order 8. The Brauer character χ of the natural $\mathbb{F}_3 G$ -module is the sum of 4 one-dimensional \mathbb{F}_3 -characters with indicator $+$. In particular it is not orthogonally stable. Explicit computations show that the space of N -invariant quadratic forms is of dimension 2 and all non-degenerate forms in this space are isometric to $Q_4^+(3)$.

However the notions of “well-defined orthogonal discriminant” and “orthogonally stable” are equivalent, as the next theorem shows.

Theorem 5.15. *An orthogonal character χ has a well-defined orthogonal discriminant if and only if χ is orthogonally stable.*

Proof. Theorem 5.13 show that an orthogonally stable character has a well-defined discriminant. To see the converse, assume that χ is not orthogonally stable. Then there is an absolutely irreducible indicator + constituent ψ of χ that has odd degree.

Assume first that the underlying characteristic is not 2. Let L be some finite extension of the character field of χ such that there are orthogonal L -representations ρ_1 with character ψ and ρ_2 with character $\chi - \psi$ and choose non-degenerate quadratic forms $Q_1 \in \mathcal{Q}(\rho_1)$ and $Q_2 \in \mathcal{Q}(\rho_2)$. Then for any $a \in L^\times$ the form $aQ_1 \perp Q_2 \in \mathcal{Q}(\rho_1 \oplus \rho_2)$ is a non-degenerate quadratic form in the direct sum of the two representations. As the dimension of ρ_1 is odd the discriminants

$$\{\text{disc}(aQ_1 \perp Q_2) \mid a \in L^\times\} = \{a(L^\times)^2 \mid a \in L^\times\}$$

yield all possible square classes in L^\times . So there is no well-defined orthogonal discriminant of χ . In characteristic 2, the only indicator + odd degree character is the trivial character (see Remark 3.2). So here ψ is the trivial character. By assumption χ is orthogonal, in particular it has even dimension, so the trivial character occurs at least twice in χ . Let $\mathbb{F}_q := \mathbb{F}_2(\chi)$ be the character field of χ and (V, Q) be an orthogonal $\mathbb{F}_q G$ -module with character $\chi - 2\psi$. Then the orthogonal $\mathbb{F}_q G$ -modules $Q_2^+(q) \perp (V, Q)$ and $Q_2^-(q) \perp (V, Q)$ both afford the character χ but have different orthogonal discriminants. \square

5.5 Proof of Theorem 5.13 for finite fields

To prove Theorem 5.13 for finite fields of characteristic p we use the well known fact that representations can be realised over their character fields.

Lemma 5.16. *Let V be a simple KG -module with Brauer character χ . If χ is orthogonally stable then all non-degenerate G -invariant quadratic forms on V have the same discriminant.*

Proof. As V is simple, the endomorphism ring of V is a field, say L , containing K , and V is also an LG -module, say V_L , of dimension $\dim(V)/[L : K]$. Let ψ denote the Brauer character of V_L and $\Gamma = \text{Gal}(L/K)$ is the Galois group of L over K . Then $\chi = \sum_{\gamma \in \Gamma} \gamma(\psi)$ is a sum of pairwise distinct absolutely irreducible Brauer characters.

Let $Q : V \rightarrow K$ be a non-degenerate G -invariant quadratic form on V .

If the indicator of ψ is +, then $\psi(1)$ is even, as χ is orthogonally stable. Moreover there is a G -invariant quadratic form $Q' : V_L \rightarrow L$ such that $Q = T_{L/K}(Q')$. In this case the discriminant of Q can be obtained from the well-defined discriminant of ψ using Remark 3.4.

If the indicator of ψ is o , then there is a unique $\gamma_0 \in \Gamma$ (of order 2) such that $\psi^\vee = \gamma_0(\psi)$. Let $F \leq L$ denote the fixed field of $\langle \gamma_0 \rangle$. Then $\psi + \psi^\vee$ is an orthogonally simple F -character of the simple orthogonal FG -module V_F . Again $Q = T_{F/K}(Q')$ for some non-degenerate quadratic form on V_F . By Theorem 5.10 all such forms Q' have the same discriminant from which Remark 3.4 yields the discriminant of Q .

It is not possible that the indicator of ψ is $-$ as then ψ would occur twice as a constituent of χ contradicting the fact that χ is the character of a simple KG -module. \square

Proof. (of Theorem 5.13 for finite fields) Let χ be an orthogonally stable Brauer character and $K := \mathbb{F}_p(\chi)$ denote its character field. Let L be an extension field of K and let (V, Q) be an orthogonal LG -module with character χ . As χ is orthogonally stable, the trivial character is not a constituent of χ and hence we can apply Proposition 4.4 to find an anisotropic orthogonal LG -module (W, Q_0) having the same discriminant as (V, Q) . Then (W, Q_0) is the orthogonal sum of simple orthogonal LG -modules. By Lemma 5.16 the orthogonal discriminant of these simple orthogonal summands is well-defined by their Brauer character and so is the discriminant of (W, Q_0) (use Remark 3.3) and hence the one of (V, Q) . \square

From the proof we get the following procedure to compute the orthogonal discriminant of an orthogonally stable Brauer character χ .

Put $K = \mathbb{F}_p(\chi)$. For an orthogonally simple constituent ψ of χ put $F := \mathbb{F}_p(\psi)$ and let $\Gamma_\psi = \text{Gal}(F/(K \cap F))$. Then

$$\psi_K := \sum_{\gamma \in \Gamma_\psi} \gamma(\psi)$$

is a sub-character of χ . If $\text{disc}(\psi) = \delta(F^*)^2$ and $p \neq 2$, then

$$\text{disc}(\psi_K) = \left(\prod_{\gamma \in \Gamma_\psi} \gamma(\delta) \right) (K^*)^2 =: N_K(\text{disc}(\psi)).$$

If $\text{disc}(\psi) = \delta + \wp(F)$ (so $p = 2$), then

$$\text{disc}(\psi_K) = \left(\sum_{\gamma \in \Gamma_\psi} \gamma(\delta) \right) + \wp(K) =: T_K(\text{disc}(\psi)).$$

With this notation we compute

Proposition 5.17. *If $\chi = \sum_{j=1}^n (\psi_j)_K$ then*

$$\text{disc}(\chi) = \begin{cases} \prod_{j=1}^n N_K(\text{disc}(\psi_j)) & \text{char}(K) \neq 2 \\ \sum_{j=1}^n T_K(\text{disc}(\psi_j)) & \text{char}(K) = 2 \end{cases}$$

5.6 Proof of Theorem 5.13 for characteristic $\neq 2$

For fields of characteristic not 2 we can use a different strategy to compute orthogonal discriminants based on the notion of the discriminant of an involution (see [15] for a brief application to orthogonal discriminants of characters and [13] for an exhaustive treatment of discriminants of involutions). The proof uses the idea of [15] to deduce that the orthogonal discriminant lies in the character field K , even though the character might not be the character of a K -representation.

This proof also shows that one can compute the orthogonal discriminants of invariant forms intrinsically in the group algebra FG over the prime field F , which is either \mathbb{F}_p or \mathbb{Q} .

Remark 5.18. (see [15, Lemma 2.1, Remark 2.3]) *Let K be a field and $B \in K^{n \times n}$ a symmetric non-degenerate matrix. Then the adjoint involution of B*

$$\iota_B(X) := BX^{\text{tr}}B^{-1} \text{ for all } X \in K^{n \times n}.$$

defines an K -algebra anti-automorphism of order 2 on $K^{n \times n}$. The (-1) eigenspace of ι_B is

$$E_-(B) := \{X \in K^{n \times n} \mid \iota_B(X) = -X\} = \{BY \mid Y = -Y^{tr}\}.$$

In particular

- (a) There is $X \in E_-(B)$ with $\det(X) \neq 0$ if and only if n is even.
- (b) For any $X \in E_-(B)$ we have $\det(X)(K^\times)^2 = \det(B)(K^\times)^2$.
- (c) A matrix $g \in \mathrm{GL}_n(K)$ is in the orthogonal group $g \in O(B)$ if and only if $gBg^{tr} = B$, so if and only if $\iota_B(g) = g^{-1}$. Then $g - g^{-1} \in E_-(B)$.
- (d) If $n = 2m$ is even then we define the **discriminant** of ι_B as $\mathrm{disc}(\iota_B) = \mathrm{disc}(B) = (-1)^m \det(X)(K^\times)^2$ where X is any invertible element of $E_-(B)$.

We now consider the group algebra FG of the finite group G over the prime field F . The group algebra comes with a natural involution

$$\iota : FG \rightarrow FG, \iota\left(\sum_{g \in G} a_g g\right) = \sum_{g \in G} a_g g^{-1}.$$

Let

$$\Sigma := \{x \in FG \mid \iota(x) = -x\} = \left\{ \sum_{g \in G} a_g (g - g^{-1}) \mid a_g \in F \right\}$$

denote the subspace of skew elements.

If the characteristic of F divides the group order, then FG is not semisimple. So let $J(FG)$ denote the radical of this finite dimensional F -algebra. Then

$$FG/J(FG) = \bigoplus_{i=1}^h A_i \cong \bigoplus_{i=1}^h D_i^{n_i \times n_i}$$

is a semisimple F -algebra, i.e. the direct sum of matrix rings over division algebras D_i . In finite characteristic all the $D_i =: K_i$ are fields. If $F = \mathbb{Q}$ we put $K_i := Z(D_i)$ to be the center of D_i and $\dim_{K_i}(D_i) =: m_i^2$.

Clearly the natural involution ι preserves the radical and hence gives rise to an involution ι on $FG/J(FG)$ permuting the simple direct summands.

Remark 5.19. Let χ_1, \dots, χ_h represent the Galois-orbits (Frobenius-orbits for Brauer characters) of the absolutely irreducible characters of G , suitably ordered, so that χ_i belongs to A_i . Then $K_i = F(\chi_i)$ is the character field of χ_i and $\chi_i(1) = m_i n_i$.

- (a) If the indicator of χ_i is $+$, then $\iota(A_i) = A_i$ and the restriction ι_i of ι to A_i is an orthogonal K_i -linear involution. If $\chi_i(1)$ is even, then by [13, Corollary 2.8] (see also [15, Proposition 3.8]) the algebra A_i contains invertible elements δ_i such that $\iota_i(\delta_i) = -\delta_i$. Then by Remark 5.18

$$\mathrm{disc}(\iota_i) = (-1)^{\chi_i(1)/2} \det(\delta_i)(K_i^\times)^2.$$

Note that for $D_i \neq K_i$ the determinant of δ_i needs to be replaced by its reduced norm (see [20, Section 9]).

- (b) If the indicator of χ_i is 0 then there is i' such that χ_i^\vee is Galois conjugate to $\chi_{i'}$.
 (b1) If $i = i'$ then $\iota(A_i) = A_i$ and the restriction ι_i of ι to A_i is a unitary involution. Put $K_i^+ := \text{Fix}_{K_i}(\iota)$, a subfield of index 2 in K_i . Then there is some $\delta_i \in K_i = Z(A_i)$ such that $K_i = K_i^+[\delta_i]$ and $\iota_i(\delta_i) = -\delta_i$ and again we have (see also Proposition 3.12)

$$\text{disc}(\iota_i) = (-\delta_i^2)^{\chi_i(1)}((K_i^+)^{\times})^2.$$

(b2) If $i \neq i'$ (this never happens in characteristic 0) then $\iota(A_i) = A_{i'}$ and ι yields a hyperbolic involution on $A_i \oplus A_{i'}$. Then $y = (\delta_i, \delta_{i'}) = (1, -1) \in A_i \oplus A_{i'}$ satisfies $\iota(y) = -y$ and the discriminant of the restriction of ι to $A_i \oplus A_{i'}$ is a square in $K_i = K_{i'}$.

- (c) If the indicator of χ_i is $-$, then $\iota(A_i) = A_i$ and the restriction of ι to A_i is a symplectic K_i -linear involution. Again by [13, Corollary 2.8] there is an invertible $\delta_i \in A_i$ such that $\iota(\delta_i) = -\delta_i$. Note that χ_i occurs in any orthogonal character with even multiplicity and hence $\det(\delta_i)$ contributes to an even power to the orthogonal discriminant.

We now reorder the summands of $FG/J(FG)$ such that $\chi_{s+1}, \dots, \chi_h$ have indicator $+$ and odd degree and such that all χ_1, \dots, χ_s are either of even degree or not self-dual.

Proposition 5.20. *There is $\Delta \in \Sigma$ such that $\Delta + J(FG) = (\delta_1, \dots, \delta_s, 0, \dots, 0)$.*

Proof. Choose $\delta \in FG$ such that $\delta + J(FG) = (\delta_1, \dots, \delta_s, 0, \dots, 0)$. Then

$$\iota(\delta) + J(FG) = \iota(\delta + J(FG)) = -\delta + J(G).$$

Put $\Delta := \frac{1}{2}(\delta - \iota(\delta))$. Then $\Delta \in \Sigma$ and $\Delta + J(FG) = \delta + J(FG)$. □

Proof. (of Theorem 5.13 for fields of characteristic $\neq 2$) Let χ be an orthogonally stable character and L be some field containing the character field $F(\chi)$. Let ρ be any L -representation with character χ and $Q \in \mathcal{Q}(\rho)$ a non-degenerate invariant quadratic form. Let B be the polarization of Q . For simplicity we choose matrices with respect to a basis that is adapted to a composition series of ρ . Then all matrices in $\rho(LG)$ are block upper triangular matrices where the blocks correspond to simple LG -modules. As $\rho(g)B\rho(g)^{tr} = B$, the adjoint involution ι_B maps $\rho(g)$ to $\rho(g^{-1})$. In particular $\rho(\Delta) \in E_-(B)$. As χ is orthogonally stable, all diagonal blocks of $\rho(\Delta)$ are invertible and so is $\rho(\Delta)$. Also the determinant of $\rho(\Delta)$ is the product of the determinants of the diagonal blocks of $\rho(\Delta)$ and hence uniquely determined by the composition factors of ρ . By Remark 5.18 the determinant of $\rho(\Delta)$ is the determinant of B modulo squares. In particular $\det(B)$ is independent of the chosen representation ρ with character χ . □

Remark 5.21. *From the construction of δ_i in Remark 5.19 one concludes that the formula in Proposition 5.17 can also be used to compute the discriminant of $\rho(\Delta)$.*

6 Reduction of orthogonal representations

In this section we fix a number field K with ring of integers O_K . For a prime ideal \wp we denote by K_\wp the completion of K at \wp , by O_\wp its valuation ring and π a generator of the unique maximal ideal $\wp O_\wp = \pi O_\wp$. Put $F := O_\wp/\pi O_\wp \cong O_K/\wp$ to denote the residue field of O_\wp .

Let $\rho : G \rightarrow \mathrm{GL}(V)$ be an orthogonal K -representation of G and $Q \in \mathcal{Q}(\rho)$ be a non-degenerate G -invariant quadratic form on V with polarisation B . Clearly Q extends to a quadratic form on the completion $V_\wp = V \otimes_K K_\wp$. An O_\wp -lattice M in (V_\wp, Q) is called **even** if $Q(M) \subseteq O_\wp$, and **integral** if $B(M, M) \subseteq O_\wp$, i.e. M is contained in its **dual lattice**, $M^\# = \{x \in V_\wp \mid B(x, M) \subseteq O_\wp\}$. Clearly even lattices are integral and, as $2Q(v) = B(v, v)$, being integral and even is equivalent if 2 is a unit in O_\wp .

Assumption 6.1. *Assume that either 2 is a unit in O_\wp or the trivial representation is not a constituent of the reduction of ρ modulo \wp .*

Lemma 6.2. *Under Assumption 6.1 all integral $O_\wp G$ -lattices in V_\wp are even.*

Proof. The statement is clear if 2 is a unit in O_\wp . So assume that $\mathrm{char}(F) = 2$ and that M is an integral $O_\wp G$ -lattice in V_\wp . Then $2Q(x) = B(x, x) \in O_\wp$ and $Q(x + y) + O_\wp = Q(x) + Q(y) + O_\wp$ for all $x, y \in M$. Hence

$$Q : M \rightarrow (\frac{1}{2}O_\wp)/O_\wp$$

is a G -invariant homomorphism of abelian groups. Its kernel N is a $\rho(G)$ -invariant O_\wp -sublattice of M and G acts trivially on the F -module $M/(N + \pi M)$. As the trivial module is not a constituent of the reduction of ρ modulo \wp , we have $N + \pi M = M$, so $N = M$ by the well known Nakayama Lemma. This shows that $Q(M) \subseteq O_\wp$ and hence M is even. \square

A maximal even $O_\wp G$ -lattice is an even, $\rho(G)$ -invariant lattice M in V_\wp such that no proper $\rho(G)$ -invariant overlattice of M is even.

Lemma 6.3. *Assume Assumption 6.1 and let M be a maximal even $O_\wp G$ -lattice. Then $\pi M^\# \subseteq M \subseteq M^\#$ and $(M^\#, \pi Q)$ is an even lattice. Reduction modulo \wp yields two non-degenerate quadratic FG -modules $(M^\#/M, \overline{\pi Q})$ and $(M/\pi M^\#, \overline{Q})$.*

Proof. Clearly M is integral, so $M \subseteq M^\#$. If $\pi M^\# \not\subseteq M$, then we put $X := \frac{1}{\pi}M \cap \pi M^\# + M$. This is clearly a proper $O_\wp G$ -overlattice of M . Elementary computations show that X is integral, and hence even (by Lemma 6.2), contradicting the maximality of M . Similarly $(M^\#, \pi Q)$ is integral and hence even.

Moreover the radical of the reduction \overline{Q} of Q modulo \wp on $M/\pi M$ is $\pi M^\#$, by definition. Similarly one concludes that $(M^\#/M, \overline{\pi Q})$ is non-degenerate. \square

6.1 Reduction of orthogonal characters

Theorem 6.4. *Let \mathcal{X} be an ordinary orthogonal character, K its character field, \wp a prime ideal of O_K and $F = O_K/\wp$ denote its residue field, a finite field of characteristic p . Assume that the reduction χ of \mathcal{X} modulo \wp is orthogonally stable. Then*

- (a) \mathcal{X} is orthogonally stable.
- (b) \wp is not ramified in the discriminant algebra $\mathcal{D}(\mathcal{X})$.
- (c) If \wp is inert in $\mathcal{D}(\mathcal{X})$ then χ has orthogonal discriminant $O-$.

(d) Assume that $\dim_{\mathbb{F}_p(\chi)}(F)$ is odd. If \mathcal{X} has trivial discriminant or \wp is split in $\mathcal{D}(\mathcal{X})$ then χ has orthogonal discriminant $O+$.

The reader can think of a prime ideal \wp being ramified in $\mathcal{D}(\mathcal{X})$ as synonymous to the informal statement that \wp divides the orthogonal discriminant (however if $2 \in \wp$ then being unramified is a stronger condition, cf. Corollary 6.5 for an example). Also a prime ideal \wp of K is inert in the quadratic extension L/K of number fields, if and only if the completion of L at \wp is the unique unramified degree 2 field extension of the p -adic number field K_\wp . If $\wp O_L = P_1 P_2$, i.e. the prime ideal is split, then the completion of L at \wp is $L_\wp = K_\wp \oplus K_\wp$ and not a field. Note that all prime ideals are split in $K \oplus K$.

Proof. (a) is clear.

(b) If the \wp -adic Schur index of \mathcal{X} is 1, then there is a K_\wp -representation $\rho : G \rightarrow \mathrm{GL}(V_\wp)$ affording the character \mathcal{X} . Otherwise there is a suitable ramified extension L of K_\wp and an L -representation ρ affording \mathcal{X} . Let O_\wp denote the ring of integers either in K_\wp or in L and π a generator of the maximal ideal of O_\wp . Then $F = O_\wp/\pi O_\wp$. Choose a non-degenerate $Q \in \mathcal{Q}(\rho)$.

The orthogonal stability of χ implies Assumption 6.1. Choose a maximal even $O_\wp G$ -lattice M in V_\wp . Then Lemma 6.3 gives a chain $\pi M^\# \subseteq M \subseteq M^\#$ of even $O_\wp G$ -lattices. Now $M^\#/M$ and $M/\pi M^\#$ are orthogonal FG -representations whose Brauer characters sum up to χ . So both are orthogonally stable, in particular of even dimensions.

Take $v_1, \dots, v_s \in M$ such that their images form a basis of $M/\pi M^\#$ and put $N := \langle v_1, \dots, v_s \rangle_{O_\wp}$. Then $(N, Q|_N)$ is a regular quadratic O_\wp -lattice of discriminant, say, δ_1 . By Lemma 3.11 we get $M = N \oplus N^\perp$. Lemma 6.3 yields that $(N^\perp, \frac{1}{\pi} Q|_{N^\perp})$ is regular. If δ_2 denotes its discriminant, then $\mathrm{disc}(Q) = \delta_1 \delta_2$. In particular Remark 3.10 says that $\mathcal{D}_\wp := K_\wp[\sqrt{\delta_1 \delta_2}]$ is an unramified extension of K_\wp . Moreover the orthogonal discriminant of the quadratic F -space $(N/\pi N, Q|_N) \oplus (N^\perp/\pi N^\perp, \frac{1}{\pi} Q|_{N^\perp})$ is $O+$ if $\mathcal{D}_\wp = K_\wp$ is of degree 1 and $O-$ if \mathcal{D}_\wp is a quadratic extension of K_\wp .

(c) Here \mathcal{D}_\wp has degree 2 over K_\wp and hence by Remark 3.9 $\dim_{\mathbb{F}_p(\chi)}(F)$ is odd and the orthogonal discriminant of χ is $O-$.

(d) When $\mathcal{D}_\wp = K_\wp$ then Remark 3.9 (a) shows that it is only possible to read off the orthogonal discriminant of χ from the computations over F if $\dim_{\mathbb{F}_p(\chi)}(F)$ is odd. In this case the orthogonal discriminant of χ is $O+$. \square

In particular for dyadic primes, i.e. those that divide 2, Theorem 6.4 yields good restrictions on the discriminant. We highlight this for rational characters:

Corollary 6.5. *The discriminant of an orthogonally stable rational character \mathcal{X} with an orthogonally stable reduction modulo 2 is 1 modulo 4.*

Proof. Let $d \in \mathbb{Z}$ be square free so that $d(\mathbb{Q}^\times)^2 = \mathrm{disc}(\mathcal{X})$. As \mathcal{X} is orthogonally stable modulo 2, the prime 2 is either split in $\mathbb{Q}[\sqrt{d}]$ (i.e. $d \equiv 1 \pmod{8}$) or inert in $\mathbb{Q}[\sqrt{d}]$ (i.e. $d \equiv -3 \pmod{8}$). \square

6.2 Primes not dividing the group order

The Brauer character table modulo a prime not dividing the group order is exactly the ordinary character table. Given an absolutely irreducible ordinary character \mathcal{X} of even degree and

indicator $+$ and some prime p not dividing the group order we aim to compute the orthogonal discriminant of the p -Brauer character $\chi = \mathcal{X}$. Let $K = \mathbb{Q}(\mathcal{X})$ be the character field of \mathcal{X} , O_K be its ring of integers. Then the Conway polynomials define a unique prime ideal \wp_0 of O_K with $p \in \wp_0$. Put $F := O_K/\wp_0$. Then F is the character field of χ , χ is the reduction of \mathcal{X} modulo \wp_0 , and χ is orthogonally stable. If $2 \notin \wp_0$ then Theorem 6.4 implies the following corollary:

Corollary 6.6. (a) *There is $d \in O_K \setminus \wp_0$ such that $d(K^\times)^2 = \text{disc}(\mathcal{X})$.*

(b) *$\text{disc}(\chi) = O+$ if and only if $d + \wp_0 \in O_K/\wp_0$ is a square in F .*

(c) *$\text{disc}(\chi) = O-$ if and only if $d + \wp_0 \in O_K/\wp_0$ is a non-square in F .*

As there are only finitely many square classes $\delta(K^\times)^2$ of a given number field K such that $K[\sqrt{\delta}]/K$ is unramified outside a finite set we hence conclude the following corollary.

Corollary 6.7. *Let P be a finite set of rational primes and K be some number field. There is a finite set of square classes of K that contain the discriminants of all orthogonally stable characters with character field K of groups whose orders are divisible only by the primes in P .*

6.3 Computing the orthogonal discriminant of an ordinary character

Theorem 6.4 can be applied to compute the orthogonal discriminant of an orthogonally stable ordinary character \mathcal{X} by computing enough orthogonally stable reductions.

To this aim, assume that we are given an orthogonally stable character \mathcal{X} of some finite group G . Let $K := \mathbb{Q}(\mathcal{X})$ denote its character field and O_K the ring of integers of K . Put

$$\mathcal{P} := \{ \wp \trianglelefteq_{\max} O_K \mid \text{reduction mod } \wp \text{ of } \mathcal{X} \text{ is not orthogonally stable} \}.$$

Then \mathcal{P} is contained in the set of prime ideals dividing $|G|O_K$. In particular it is a finite set.

Put $\epsilon := (-1)^{\chi(1)/2}$, so that $\epsilon \text{disc}(\mathcal{X})$ is a totally positive square class in K (cf. Remark 5.11). Compute $\delta_1, \dots, \delta_t \in O_K$ such that the products $\prod_{j \in I} \delta_j$, for $I \subseteq \{1, \dots, t\}$ represent all totally positive square classes $\delta(K^\times)^2$ for which $K_\wp[\sqrt{\delta}]$ is unramified for all $\wp \notin \mathcal{P}$. We assume that $(\delta_i(K^\times)^2 \mid i = 1, \dots, t)$ is \mathbb{F}_2 -linearly independent in $K^\times/(K^\times)^2$. For a non-dyadic prime $\wp \notin \mathcal{P}$ we define $a_\wp \in \mathbb{F}_2^t$ by $a_\wp(i) = 0$ if δ_i is a square in the residue field O_K/\wp and $a_\wp(i) = 1$ otherwise. Choose non-dyadic primes $\{\wp_1, \dots, \wp_t\}$ such that the matrix A with columns a_{\wp_i} , so

$$A = (a_{\wp_1}, \dots, a_{\wp_t}) \in \mathbb{F}_2^{t \times t}$$

has full rank t . Now compute the orthogonal discriminant ϵd_i of \mathcal{X} (mod \wp_i) for $i = 1, \dots, t$ and put $x_i := 1$ if $d_i = O-$ and $x_i := 0$ if $d_i = O+$. Put $y := xA^{-1}$ and $I := \{i \in \{1, \dots, t\} \mid y_i = 1\}$. Then

$$\delta := \epsilon \prod_{i \in I} \delta_i(K^\times)^2 = \text{disc}(\mathcal{X}).$$

Remark 6.8. *In practice we also use all dyadic primes \wp where \mathcal{X} has an orthogonally stable reduction χ as these yield even more restrictions than the non-dyadic primes. Instead of working only with $\delta_1, \dots, \delta_t$ we need to test the ramification behaviour of \wp in $K[\sqrt{\epsilon d}]/K$ for all products d of the δ_i . By Theorem 6.4 we can exclude those d where \wp is ramified and the orthogonal discriminant of χ allows us to further exclude those d where \wp is inert (if $\text{disc}(\chi) = O+$) respectively split (if $\text{disc}(\chi) = O-$).*

Example 6.9. The group J_2 has two absolutely irreducible ordinary characters of degree 224 with character field $\mathbb{Q}[\sqrt{5}]$ that are exchanged under an outer automorphism. So $\text{disc}(224a) = \text{disc}(224b) =: d(\mathbb{Q}[\sqrt{5}]^\times)^2$. The decomposition matrices in GAP [7] tell us that both characters are orthogonally stable modulo 2 and 7. Constructing the representations we obtain that they yield embeddings into $O_{224}^+(\mathbb{F}_4)$ respectively $O_{224}^+(\mathbb{F}_{49})$. As $|J_2| = 2^7 3^3 5^2 7$ the only possibilities for d are 1, 3, $(5 + \sqrt{5})/2$, and $3(5 + \sqrt{5})/2$. Now 1 and 3 are squares in \mathbb{F}_{49} , whereas $(5 + \sqrt{5})/2$ is not a square. As the reduction mod 7 has orthogonal discriminant +, we are left with 1 or 3 as possible discriminants. The prime 2 is ramified in the extension $\mathbb{Q}[\sqrt{3}, \sqrt{5}]/\mathbb{Q}[\sqrt{5}]$, which excludes the possibility that $d = 3$ just using the fact that the reduction modulo 2 is orthogonally stable. We hence conclude that the discriminants are squares, $\text{disc}(224a) = \text{disc}(224b) = 1$.

6.4 Cyclic defect

In this section we assume that \mathcal{X} is an even degree absolutely irreducible orthogonal character in some p -block with cyclic defect group. We refer the reader to the original article [5] for details. By [5, Theorem 1, Part 2] the reduction χ of \mathcal{X} is multiplicity free. In characteristic 2, the trivial character is the unique odd degree indicator + irreducible Brauer character (see Remark 3.2), so the fact that χ is multiplicity free and of even degree allows to conclude that χ is orthogonally stable and hence Theorem 6.4 applies. So the theory in this section is void for $p = 2$ and hence we may and will assume that $p \neq 2$. In particular we may consider bilinear forms instead of quadratic forms, all integral lattices are automatically even, and a prime ideal \wp is ramified in the discriminant field extension, if and only if there is a G -invariant maximal integral lattice L such that $L^\# / L$ has odd dimension over the residue field O_\wp / \wp .

For blocks of defect 1 the converse of Theorem 6.4 is true.

Theorem 6.10. *Let p be an odd prime and let $\mathcal{X} \in \text{Irr}_{\mathbb{C}}(G)$ be an orthogonally stable character in a p -block with defect 1. Let K denote the character field of \mathcal{X} and \wp a prime ideal of O_K that contains p . Then the reduction χ of \mathcal{X} modulo \wp is orthogonally stable if and only if \wp is unramified (i.e. split or inert) in $\mathcal{D}(\mathcal{X})$.*

Proof. Let K_\wp be the completion of K at the prime \wp , L_\wp be some unramified extension of K_\wp with ring of integers O_\wp and $\rho : G \rightarrow \text{GL}_n(L_\wp)$ be some representation affording the character \mathcal{X} . Fix a non-degenerate quadratic form $Q \in \mathcal{Q}(\rho)$ and choose $M \leq L_\wp^n$ to be some maximal even $\rho(G)$ -invariant O_\wp -lattice. The assumption that the p -defect of \mathcal{X} is one implies that all $O_\wp G$ -lattice in L_\wp^n are linearly ordered by inclusion ([1, Theorem 11]). As $M^\# / M$ and $M / \wp M^\#$ are self-dual modules, their composition factors are ordered as $(S_1, S_2, \dots, S_s, S = S^\vee, S_s^\vee, \dots, S_2^\vee, S_1^\vee)$ for $M^\# / M$ and $(T_1, T_2, \dots, T_t, T = T^\vee, T_t^\vee, \dots, T_2^\vee, T_1^\vee)$ for $M / \wp M^\#$, possibly without the self-dual composition factors S, T . As the module $M^\# / \wp M^\#$ is multiplicity free, all these composition factors are pairwise non-isomorphic. The assumption that the reduction of \mathcal{X} modulo \wp is not orthogonally stable hence implies that both self-dual composition factors S and T do occur and are of odd dimension. In particular $M^\# / M$ has odd dimension so the \wp -adic valuation of the discriminant δ of Q is odd. As $p \neq 2$, this is equivalent to saying that \wp ramifies in $K[\sqrt{\delta}]/K$. \square

Example 6.11. Let $G = \text{SL}_2(8)$ and \mathcal{X} be the rational absolutely irreducible character of degree $\mathcal{X}(1) = 8$. Then the reduction of \mathcal{X} modulo 3 is the sum of two characters of degree 1 and 7, and hence not orthogonally stable. However, the discriminant of \mathcal{X} is 1.

Example 6.11 is covered by the following general result for characters in blocks with cyclic defect groups.

Theorem 6.12. *Assume that \mathcal{X} is an indicator + absolutely irreducible ordinary character of even degree in a p -block of cyclic defect. Denote by K its character field, O_K the ring of integers and \wp a prime ideal of O_K containing the odd prime p .*

- *If \mathcal{X} belongs to the exceptional vertex then its reduction modulo \wp is not orthogonally stable, if and only if \wp is ramified in the discriminant algebra $\mathcal{D}(\mathcal{X})$.*
- *If \mathcal{X} does not belong to the exceptional vertex then \wp is ramified in the discriminant algebra $\mathcal{D}(\mathcal{X})$ if and only if the defect is odd and the reduction of \mathcal{X} mod \wp is not orthogonally stable.*

In particular for even defect only the characters belonging to the exceptional vertex may have ramification at \wp in the discriminant field extension.

Proof. We keep the notation of Theorem 6.10. If \mathcal{X} belongs to the exceptional vertex, then by [19, Theorem (VIII.3)] the G -invariant O_\wp -lattices form a chain and we have the same situation as in the proof of Theorem 6.10 and hence the conclusion follows with the same argument.

Now assume that \mathcal{X} does not belong to the exceptional vertex and we also assume that its reduction mod \wp is not orthogonally stable. So there are exactly two constituents χ_S, χ_T of indicator + and odd degree. The paper [14] investigates the radical idealizer process, that starts with the O_\wp -order Λ spanned by the matrices in $\rho(G)$ and successively constructs involution invariant overorders until arriving at the hereditary order $H(\Lambda)$, the head order of Λ . As $H(\Lambda)$ is invariant under the canonical involution of G , its lattices hence form a chain that is invariant under taking duals. The explicit form of $H(\Lambda)$ given in [14, Theorem 3.15] allows to conclude that χ_S and χ_T occur in the same constituent of $H(\Lambda)$ mod \wp if and only if the defect is even. The same argument as in the proof of Theorem 6.10 now applied to $H(\Lambda)$ allows to conclude the statement. \square

7 Two examples

7.1 The first Janko group J_1

Let J_1 denote the first Janko group of order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. Then there are seven indicator + absolutely irreducible ordinary characters of even degree: The two Galois conjugate characters $56a, 56b$ with character field $\mathbb{Q}[\sqrt{5}]$, two rational characters of degree 76 and three Galois conjugate characters $120a, 120b, 120c$ with character field $\mathbb{Q}[c_{19}]$, the unique subfield of degree 3 of the 19th cyclotomic field. All these three character fields $\mathbb{Q}, \mathbb{Q}[\sqrt{5}]$, and $\mathbb{Q}[c_{19}]$ have narrow class number 1, which means that any ideal of their ring of integers has a totally positive generator that is unique up to multiplication by squares of units.

The reductions modulo 2, 3, 5 of all these seven characters stay absolutely irreducible, so the only primes that can divide the discriminants are those that divide $7 \cdot 11 \cdot 19$. With GAP [7] we

compute the reduction modulo 7, 11, 19 as follows

\mathcal{X}	$56a$	$56b$	$76a$	$76b$	$120a$	$120b$	$120c$
(mod 7)	$56a$	$56b$	$1 + 75$	$31 + 45$	$45 + 75$	$31 + 89$	120
(mod 11)	56	$7 + 49$	$27 + 49$	$7 + 69$	$1 + 119$	$56 + 64$	$14 + 106$
(mod 19)	$1 + 55$	$22 + 34$	$76a$	$76b$	$43 + 77$	$43 + 77$	$43 + 77$

By Theorem 6.10 the determinant of both characters of degree 76 is $7 \cdot 11 = 77$.

For the non-rational characters the “reduction modulo 7,11,19” means the reduction modulo the prime ideal \wp_0 from Section 5.2. As in our cases, the residue fields are the prime fields, we can identify the ideal \wp_0 using the GAP-command

$$x := \text{Int}(\text{FrobeniusCharacterValue}(y, p));$$

where y is a name of the irrationality (here $y = ER(5)$ respectively $y = EC(19)$) and p the corresponding prime. We obtain

y	$\sqrt{5}$	$\sqrt{5}$	c_{19}	c_{19}
p	11	19	7	11
x	4	9	0	6

We get the factorization $11 = p_{11}p'_{11}$ where $p_{11} = 4 - \sqrt{5}$ and p'_{11} is its Galois conjugate $p'_{11} = 4 + \sqrt{5}$. Similarly $19 = p_{19}p'_{19}$ with $p_{19} = (9 - \sqrt{5})/2$, and hence

$$\text{disc}(56a) = p'_{11}p_{19}(\mathbb{Q}[\sqrt{5}]^\times)^2 = (17 - 4\sqrt{5})(\mathbb{Q}[\sqrt{5}]^\times)^2, \quad \text{disc}(56b) = (17 + 4\sqrt{5})(\mathbb{Q}[\sqrt{5}]^\times)^2.$$

For the field $\mathbb{Q}(c_{19})$ we have $c_{19} = z + z^8 + z^{8^2} + z^{8^3} + z^{8^4} + z^{8^5}$ where $z = e^{2\pi i/19}$ is a primitive 19th root of unity. A generator σ of the Galois group of $\mathbb{Q}(c_{19})$ acts on the cyclotomic field by squaring z and hence maps c_{19} to $c'_{19} := 4 - c_{19}^2$. Note that σ acts on the characters as the cyclic permutation $(120a, 120c, 120b)$. We compute

$$\alpha = 4 + c_{19} - c'_{19}, \quad \beta = 20 + 10c_{19} - 3c'_{19}, \quad \gamma = 7 + 3c_{19} - c'_{19}$$

as totally positive elements of norm 7, 11, respectively 19. Replacing c_{19} by 0 maps α onto $0 \in \mathbb{F}_7$ and similarly mapping c_{19} to 6 maps β to $0 \in \mathbb{F}_{11}$. So we get

$$\text{disc}(120a) = \alpha\sigma(\alpha)\beta\gamma(\mathbb{Q}(c_{19})^\times)^2 = (29 - 18c_{19} - 9c'_{19})(\mathbb{Q}(c_{19})^\times)^2$$

and $\text{disc}(120c) = \sigma(\text{disc}(120a))$, $\text{disc}(120b) = \sigma^2(\text{disc}(120a))$.

Theorem 7.1. *The discriminants of the ordinary absolutely irreducible orthogonal characters of J_1 are*

\mathcal{X}	$56a$	$56b$	$76a$	$76b$	$120a$	$120b$	$120c$
$\text{disc}(\mathcal{X})$	$17 - 4\sqrt{5}$	$17 + 4\sqrt{5}$	77	77	$29 - 18c_{19} - 9c'_{19}$	$47 + 9c_{19} + 18c'_{19}$	$38 + 9c_{19} - 9c'_{19}$

where the characters and conjugacy classes are as in the atlas [2] and modular atlas [9] respectively.

Remark 7.2. The discriminant algebra $\mathcal{D}(\chi_{56a})$ is not Galois over \mathbb{Q} , its normal closure $L = \mathcal{D}(\chi_{56a})[\sqrt{\text{disc}(56b)}]$ has Galois group D_8 , the dihedral group of order 8.

Similarly $K := \mathcal{D}(\chi_{120a})$ is not Galois over \mathbb{Q} . Its normal closure $K[\sqrt{\text{disc}(120b)}, \sqrt{\text{disc}(120c)}]$ has Galois group $C_2 \wr C_3$ of order 24.

Remark 7.3. In [3, Conjecture 3.9] the author speculates that the discriminant field of any absolutely irreducible indicator + character of degree 2 (mod 4) is an abelian number field. The sporadic simple O’Nan group of order $2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$ provides a counterexample to this conjecture: Let \mathcal{X} be one of the two Galois conjugate absolutely irreducible characters of degree 169290 with character field $\mathbb{Q}[\sqrt{2}]$ and indicator +. Let $K := \mathbb{Q}[\sqrt{2}][\sqrt{\text{disc}(\mathcal{X})}]$ denote the discriminant algebra of \mathcal{X} . We claim that K is not Galois over \mathbb{Q} . With GAP we compute the decomposition matrix modulo 31 which shows that the reduction of \mathcal{X} modulo one prime ideal, say \wp_{31} , dividing 31 is orthogonally stable. The reduction modulo the other prime ideal, say \wp'_{31} , is not orthogonally stable. So Theorem 6.10 tells us that \wp_{31} is not ramified in the discriminant field extension, whereas \wp'_{31} is ramified. Therefore the prime ideals dividing 31 in K have different ramification behaviour so K/\mathbb{Q} is not Galois.

Using orthogonal condensation methods we computed the discriminant of the two Galois conjugate characters as $(-53 \pm 36\sqrt{2})\mathbb{Q}[\sqrt{2}]^\times$.

7.2 The Held group

The Held group He is a sporadic simple group of order $2^{10}3^35^27^317$. With the meat-axe [18] we construct all irreducible indicator + even degree modular representations for the prime divisors of the group order. The tables below give those degrees followed by the orthogonal discriminants.

Characteristic 2

246	O+	246	O+	680	O-	1920	O+	2008	O-
4352	O+	4608	O+	21504	O+	21504	O+		

Characteristic 3

1920	O-	6172	O-	6272	O-	7650	O-	14400	O-
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Characteristic 5

104	O+	680	O+	1240	O-	4080	O+	4116	O-	6528	O+
7650	O+	9640	O-	10860	O-	11900	O+	14400	O-	22050	O+

Characteristic 7

50	O+	426	O+	798	O+	1072	O-	1700	O+
3654	O+	6154	O-	6272	O-	13720	O-	23324	O+

Characteristic 17

680	O+	4080	O+	4352	O-	6528	O+
7650	O+	10880	O+	11900	O+	23324	O+

The following table gives the information on the orthogonal discriminants in characteristic 0. The first column gives the number of the ordinary absolutely irreducible character \mathcal{X} followed by its degree $\mathcal{X}(1)$ and its discriminant $\text{disc}(\mathcal{X})$. For the prime divisors of the group order for which the reduction of \mathcal{X} is orthogonally stable we also give the orthogonal discriminant $O+$ or $O-$ of this reduction computed from the orthogonal discriminants in the tables before. For characters in blocks with defect 1 such that the reduction mod p is not orthogonally stable we display p , to indicate that p is a prime divisor of the discriminant (see Theorem 6.10). All character fields are rational, except for the two algebraic conjugate characters of degree 21504, where the character field is $\mathbb{Q}[\sqrt{21}]$. Here “ $., O+$ ” indicates that the reduction of the character number 30 modulo the prime ideal generated by $4 - \sqrt{21}$ is orthogonally stable of orthogonal discriminant $O+$ and the one modulo $4 + \sqrt{21}$ is not orthogonally stable and *vice versa*.

No	deg	$disc$	(mod 2)	(mod 3)	(mod 5)	(mod 7)	(mod 17)
6	680	21	$O-$		$O+$		$O+$
12	1920	17	$O+$	$O-$	$O-$	$O-$	17
13	4080	1			$O+$	$O+$	$O+$
14	4352	105	$O+$				$O-$
15	6272	17	$O+$	$O-$		$O-$	17
16	6528	1	$O+$		$O+$		$O+$
19	7650	-1		$O-$	$O+$	$O-$	$O+$
22	10880	1	$O+$		$O+$	$O+$	$O+$
25	11900	21	$O-$		$O+$		$O+$
26	13720	17			$O-$	$O-$	17
27	14400	17		$O-$	$O-$		17
30	21504	$357 + 68\sqrt{21}$	$O+$		“ $., O+$ ”		17
31	21504	$357 - 68\sqrt{21}$	$O+$		“ $O+, .$ ”		17
32	22050	-119		$O+$	$O+$	7	17
33	23324	1				$O+$	$O+$

For the characters of numbers $\notin \{30, 31, 33\}$ the proofs are all similar, so let us give the one for number 14. Here the possible discriminants are in $\{1, 3, 5, 7, 15, 21, 35, 105\}$. As the reduction mod 2 is orthogonally stable of discriminant $O+$, Corollary 6.5 tells us that the discriminant of \mathcal{X} is 1 mod 8, leaving 1 and 105 as the only possibilities. But the discriminant is not a square modulo 17, so $\text{disc}(\mathcal{X}) = 105$.

Characters number 30 and 31 are algebraic conjugate and so are their discriminants. We give the proof for number 30. Here the character field is $K = \mathbb{Q}[\sqrt{21}]$. Its ring of integers has class number 1 and a totally positive unit u that is not a square, $u = (5 + \sqrt{21})/2$. Note that $3u = ((3 + \sqrt{21})/2)^2$ is a square in K . The prime ideals dividing $3 \cdot 5 \cdot 7 \cdot 17$ are generated by $p_3 := (3 + \sqrt{21})/2$ of norm -3 , $p_5 := 4 + \sqrt{21}$ and $p'_5 := 4 - \sqrt{21}$ of norm -5 , $p_7 = (7 + \sqrt{21})/2$ of norm 7 and 17 of norm 17^2 . By Theorem 6.10 we know that 17 divides the discriminant of \mathcal{X} . We also know that the reduction of \mathcal{X} modulo (p'_5) is orthogonally stable. So the discriminant of \mathcal{X} is in

$$\{17, 17u, 17p_7, 17up_7, 17p_3p_5, 17up_3p_5, 17p_7p_3p_5, 17up_7p_3p_5\}.$$

Only for $d = 17$ and $d = 17up_7p_3p_5$ is the prime 2 not ramified in $K[\sqrt{d}]/K$. Now p'_5 is inert in this extension for $d = 17$ and decomposed for $d = 17up_7p_3p_5 = 68\sqrt{21} + 357$. As the reduction

modulo p'_5 of \mathcal{X} has orthogonal discriminant $O+$, we conclude that $d = 68\sqrt{21} + 357$ is the only possibility.

For the last character, number 33, we use a completely different argument. This character extends in two ways to the group $He:2$. Take one of these extensions and restrict it to the maximal subgroup $S_4(4):4 \leq He:2$. This restriction is orthogonally stable and contains the following orthogonal simple characters:

<i>deg</i>	18 <i>ab</i>	68	50 <i>ab</i>	204	170	153 <i>ab</i>	816	900	1020	256 <i>ab</i>	256	680	680
<i>K</i>	$\mathbb{Q}[i]$	\mathbb{Q}	$\mathbb{Q}[\sqrt{-2}]$	\mathbb{Q}	\mathbb{Q}	$\mathbb{Q}[i]$	\mathbb{Q}	\mathbb{Q}	\mathbb{Q}	$\mathbb{Q}[i]$	\mathbb{Q}	\mathbb{Q}	\mathbb{Q}
<i>mult</i>	1	1	1	2	1	1	4	7	6	3	1	2	5
<i>disc</i>	1	1	1	.	-1	-1	.	17	.	1	17	.	1

This table gives the degree of the orthogonal simple characters occurring in the restriction of the extension of \mathcal{X} to $S_4(4):4$. The small letters indicate pairs of complex conjugate characters, whose character fields are given in the next row. Their discriminant can hence be obtained by Proposition 3.12. The character of degree 68 is induced up from a rational character of degree 34 of the group $S_4(4):2$. Hence the invariant forms are orthogonal sums of two isometric forms and so the discriminant is a square. The same argument applies to the characters of degrees 170, 204, 816, 1020, and the two characters of degree 680. For the character of degree 900 we apply [15, Theorem 4.6] to conclude that the two characters of degree 450 of the group $S_4(4):2$ have discriminant $-(17 \pm 4\sqrt{17})$ respectively. So the discriminant of the character of degree 900 is their product, 17. The characters of degree 256 restrict irreducibly to the simple group $S_4(4)$. The restriction is irreducible modulo 2 and 3, not orthogonally stable modulo 17. So its discriminant is either 17 or 85. As the orthogonal discriminant of the reduction mod 2 is $O+$, we conclude that the discriminant is $\equiv 1 \pmod{8}$ (see Corollary 6.5) and hence it is 17.

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