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ON ORTHOGONAL DISCRIMINANTS OF CHARACTERS

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ABSTRACT. An ordinary character χ of a finite group is called orthogonally stable, if all non-degenerate invariant quadratic forms on any module affording the character χ have the same discriminant. This is the orthogonal discriminant, $\mathrm{disc}(\chi)$, of χ, a square class of the character field. Based on experimental evidence we conjecture that the orthogonal discriminant is always an odd square class in the sense of Definition 1.4. This note proves this conjecture for finite solvable groups. For p-groups there is an explicit formula for $\mathrm{disc}(\chi)$ that reads $\mathrm{disc}(\chi) = (-p)^{\chi(1)/2}$ if $p \equiv 3 \pmod 4$ and $\mathrm{disc}(\chi) = (-1)^{\chi(1)/2}$ for $p = 2$. For rational characters χ and $p \equiv 1 \pmod 4$ the discriminant is $\mathrm{disc}(\chi) = p^{\chi(1)/(p-1)}$.

1. INTRODUCTION

In the past year Richard Parker and this author started a long term project studying quadratic forms invariant under the finite simple groups in the Atlas of Finite Groups [2]. We work over number fields and finite fields and use decomposition matrices to compute discriminants of invariant quadratic forms (see for instance [7, Section 7] for explicit examples).

If $\mathrm{char}(K) \neq 2$ then any quadratic form $Q : V \to K$ on a K-space V is determined by the corresponding bilinear form

$$B : V \times V \to K, B(x, y) = Q(x + y) - Q(x) - Q(y).$$

The quadratic form is called non-degenerate, if the radical of B is $\{0\}$ where

$$\mathrm{rad}(B) = V^{\perp} = \{x \in V \mid B(x, y) = 0 \text{ for all } y \in V\}.$$

The discriminant of a non-degenerate quadratic form Q is

$$\mathrm{disc}((V, Q)) = \mathrm{disc}(Q) = \mathrm{disc}(B) = (-1)^{\binom{n}{2}} \det(B) \in K^{\times}/(K^{\times})^2$$

where $n = \dim(V)$ and $\det(B)$ is the square class of the determinant of a Gram matrix of B with respect to a chosen basis of V. The discriminant is a strong invariant of the isometry class of a non-degenerate quadratic form. For finite fields

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of odd characteristic discriminant and dimension completely determine the isometry class of a quadratic space (see [4, Section IV]).

The present short note is mainly concerned with number fields K. So to simplify notation we assume that K is a finite extension of the rational field $\mathbb{Q}$. Let V be a KG-module. Then V is called orthogonal if G fixes a non-degenerate quadratic form on V. An orthogonal KG-module V is called orthogonally stable if all non-degenerate G-invariant quadratic forms have the same discriminant. This square class of K is then called the orthogonal discriminant of the KG-module V.

The properties of being orthogonal and orthogonally stable can be read off from the character of V: Let $\mathrm{Irr}(G)$ denote the set of absolutely irreducible ordinary characters of G. For $\psi \in \mathrm{Irr}(G)$ the Frobenius Schur indicator of ψ (for short indicator of ψ) takes the values o, $+$, and $-$. The indicator of ψ is o, if ψ takes non-real values. If ψ is the character of a real representation then the indicator of ψ is $+$ and the indicator is $-$ if ψ is real valued but there is no real representation affording the character ψ. Any character χ of G is a unique sum of absolute irreducible characters. Recall that the character field of χ is the number field generated by the character values, $\mathbb{Q}(\chi) := \mathbb{Q}(\chi(g) : g \in G)$. Now $\chi(g)$ is the trace of the endomorphism of V defined by g, so for any KG-module V affording the character χ the character field is a subfield of K. If the Schur index of χ is bigger than 1 then there is no $\mathbb{Q}(\chi)G$-module with character χ.

Proposition 1.1. *(see [7]) A character $\chi := \sum_{\psi \in \mathrm{Irr}(G)} n_\psi \psi$ is* orthogonal *if χ is real valued and n_ψ is even for all $\psi \in \mathrm{Irr}(G)$ with indicator $-$. An orthogonal character χ is* orthogonally stable *if $\psi(1)$ is even for all $\psi \in \mathrm{Irr}(G)$ of indicator $+$ for which $n_\psi > 0$.*

The papers [6] and [7] show that the orthogonal discriminant can be defined as a square class of the character field: Given an orthogonally stable character χ there is a unique square class $\mathrm{disc}(\chi) \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$ such that for any KG-module V affording χ all non-degenerate G-invariant quadratic forms on V have discriminant $\mathrm{disc}(\chi)(K^\times)^2$.

Definition 1.2. $\mathrm{disc}(\chi) \in \mathbb{Q}(\chi)^\times/(\mathbb{Q}(\chi)^\times)^2$ *is called the* orthogonal discriminant *of the orthogonally stable character χ.*

Parker, Thackray, Breuer and the author computed thousands orthogonal discriminants of orthogonally stable characters of finite simple groups (see [1]). For rational characters the orthogonal discriminants are represented by a unique square free integer d. We never found an even number d which let Parker to formulate the following conjecture.

Conjecture 1.3. *Let χ be an orthogonally stable rational character. Then the orthogonal discriminant of χ is represented by an odd square free integer.*

From the classification of all maximal finite rational matrix groups up to dimension 31 ([8], [9], [10]) one can check that the conjecture is true if $\chi(1) \leq 30$. The aim of the present note is to prove a more general result (Theorem 1.5) for all orthogonally stable characters of finite solvable groups. To formulate the result for arbitrary character fields we need to have an appropriate notion of an odd square class. So let K be a number field and $\nu : K^\times \to \mathbb{Z}$ a surjective discrete valuation. Then for $\delta \in K^\times$

$$\nu(\delta(K^\times)^2) = \nu(\delta) + 2\mathbb{Z}$$

so any square class in $K^\times$ has either even or odd valuation. We call ν a dyadic valuation, if $\nu(2) > 0$.

Definition 1.4. *A square class $\delta(K^\times)^2$ is called odd, if $\nu(\delta(K^\times)^2)$ is even for all dyadic valuations $\nu : K^\times \to \mathbb{Z}$.*

Then the main result of this short note is the following theorem:

Theorem 1.5. *Let G be a finite solvable group and χ be an orthogonally stable ordinary character of G. Then $\operatorname{disc}(\chi)$ is odd.*

We also compute the orthogonal discriminants for characters of p-groups in Theorem 4.3 and 4.7 below. In particular we get that the orthogonal discriminant of any orthogonally stable character χ of a p-group is $(-p)^{\chi(1)/2}$ if $p \equiv 3 \pmod 4$ and $(-1)^{\chi(1)/2}$ if $p = 2$. For primes $p \equiv 1 \pmod 4$ Theorem 4.3 gives a similarly explicit result that implies that rational characters χ of p-groups with $p \equiv 1 \pmod 4$ have orthogonal discriminant $\operatorname{disc}(\chi) = p^{\chi(1)/(p-1)}$.

2. Preliminaries

2.1. **The main character theoretic tool.** It is well known that the character field $\mathbb{Q}(\chi)$ of any ordinary character χ of some finite group G is an abelian extension of $\mathbb{Q}$ and hence contained in some cyclotomic field $\mathbb{Q}[\zeta_f]$. The minimal such f is called the conductor of χ.

Theorem 2.1. *([5, Theorem A1]) Let ψ be an absolutely irreducible ordinary character of some finite group G such that $\psi(1)$ is odd. Write the conductor of ψ as $2^a m$ with $m \in \mathbb{N}$ odd. Then $\mathbb{Q}[\zeta_{2^a}] \subseteq \mathbb{Q}(\psi)$.*

For our purposes a weaker version of Theorem 2.1 suffices:

Theorem 2.2. *([3, Theorem C]) Let ψ be an absolutely irreducible ordinary character of some finite group G such that $\psi(1)$ is odd. Then either the conductor of ψ is odd or $\mathbb{Q}[\sqrt{-1}] \subseteq \mathbb{Q}(\psi)$.*

Corollary 2.3. *The conductor of a real, absolutely irreducible, odd degree character is odd.*

In Section 4 orthogonal discriminants of p-groups are determined. The proof uses the fact that for odd primes p the character fields of absolutely irreducible characters of p-groups are cyclotomic number fields:

Theorem 2.4. *(see for instance [5, Theorem 2.3]) Let p be an odd prime, G a p-group and $1 \neq \chi \in \operatorname{Irr}(G)$ be a non-trivial absolutely irreducible character of G. Then $\mathbb{Q}(\chi) = \mathbb{Q}[\zeta_{p^f}]$ for some $f \in \mathbb{N}$.*

2.2. **The discriminant field.** Let K be a number field and $\delta \in K^\times$. Then the square class $\delta(K^\times)^2$ defines a unique field extension $K[\sqrt{\delta}]$ of K. Note that $K[\sqrt{\delta}] = K$ if and only if δ is a square in K.

Certain results for quadratic forms over number fields have a more natural formulation if one replaces the discriminant by the discriminant field:

Definition 2.5. *Let χ be an orthogonally stable character with character field $K = \mathbb{Q}(\chi)$ and orthogonal discriminant $\operatorname{disc}(\chi) = \delta(K^\times)^2$. Then the* discriminant field *of χ is $\mathfrak{D}(\chi) := K[\sqrt{\delta}]$.*

Whereas character fields are always abelian number fields and in particular Galois extensions of $\mathbb{Q}$, discriminant fields of orthogonally stable characters are not necessarily Galois over $\mathbb{Q}$. An example is given in [7, Remark 7.2] for the absolutely irreducible characters of degree 56 and 120 of the first Janko group J_1. However Corollary 3.4 shows that all discriminant fields of orthogonally stable characters of finite solvable groups are abelian extensions of $\mathbb{Q}$.

Lemma 2.6. *Let K be a number field and $\delta \in K^\times$. Put $L = K[\sqrt{\delta}]$. Then $(L^\times)^2 \cap K^\times = (K^\times)^2 \cup \delta(K^\times)^2$.*

Proof. Any $x \in L^\times$ is of the form $x = a + b\sqrt{\delta}$ with $a, b \in K$. Then

$$x^2 = (a + b\sqrt{\delta})^2 = a^2 + b^2\delta + 2ab\sqrt{\delta}.$$

So $x^2 \in K$ if and only if $2ab = 0$ and then $x^2 = a^2 \in (K^\times)^2$ or $x^2 = b^2\delta \in \delta(K^\times)^2$. $\square$

2.3. **Odd square classes.** A field extension L/K is said to be unramified at 2 if no prime ideal of K that contains 2 is ramified in L/K. If $F \subseteq K \subseteq L \subseteq M$ is a tower of number fields and M/F is unramified at 2 then clearly also L/K is unramified at 2 (see for instance [11, Kapitel III §2]).

Recall that a square class $\delta(K^\times)^2$ is called odd if $\nu(\delta(K^\times)^2)$ is even for all dyadic valuations $\nu : K^\times \to \mathbb{Z}$ (see Definition 1.4)

Lemma 2.7.
(a) *If $\delta \in (K^\times)^2$ then $\delta(K^\times)^2$ is odd.*
(b) *If $\delta \in K^\times \setminus (K^\times)^2$ and 2 is unramified in $K[\sqrt{\delta}]/K$ then $\delta(K^\times)^2$ is odd.*
(c) *The example $K = \mathbb{Q}$ and $\delta = -1$ shows that the converse of (b) is not true.*

Proof. The only statement that might require a proof is (b). There is a bijection between prime ideals $\wp$ of K and surjective valuations $\nu = \nu_\wp : K^\times \to \mathbb{Z}$. The prime ideal $\wp$ is unramified in a field extension L/K if for any extension $\tilde{\nu}$ of $\nu_\wp$ we have $\tilde{\nu}(L^\times) = \nu_\wp(K^\times)$ (see for instance [11, Kapitel II §8]). For the particular case that $L = K[\sqrt{\delta}]$ this tells us that

$$\nu(\delta) = 2\tilde{\nu}(\sqrt{\delta}) \in 2\mathbb{Z}$$

for all dyadic valuations $\nu : K^\times \to \mathbb{Z}$, so $\delta(K^\times)^2$ is odd. $\square$

Lemma 2.8. *Let L/K be a finite extension of number fields and let $\delta \in L^\times$. Then $N_{L/K}(\delta(L^\times)^2) \subseteq N_{L/K}(\delta)(K^\times)^2$. So the norm of a square class of L defines a square class of K. If $\delta(L^\times)^2$ is an odd square class of $L^\times$ then also $N_{L/K}(\delta)(K^\times)^2$ is odd.*

Proof. Assume that $\delta(L^\times)^2$ is an odd square class of $L^\times$ and let $\nu : K^\times \to \mathbb{Z}$ be a surjective dyadic valuation. Let $\tilde{\nu}_i : L^\times \to \mathbb{Q}$ $(1 \leq i \leq s)$ be the distinct extension of ν and $e_i \in \mathbb{N}$ be such that $\nu_i := e_i\tilde{\nu}_i : L^\times \to \mathbb{Z}$ is surjective. Then by assumption $\nu_i(\delta(L^\times)^2) = 2\mathbb{Z}$. By [11, Kapitel III, Satz (1.2) (iv)] the valuation of the norm of δ is a linear combination of $\nu_i(\delta)$, $\nu(N_{L/K}(\delta)) = \sum_{i=1}^s f_i\nu_i(\delta)$. In particular $\nu(N_{L/K}(\delta))$ is even and hence $N_{L/K}(\delta)(K^\times)^2$ is an odd square class of K. $\square$

2.4. **Orthogonally simple characters.** Any orthogonally stable character is the sum of orthogonally simple characters (see [7, Section 5.3]). There are three kinds of orthogonally simple characters χ:

(a) $\chi = 2\psi$ for an absolutely irreducible real character ψ of indicator $-$.
(b) $\chi = \psi + \overline{\psi}$ for an absolutely irreducible non-real character ψ.
(c) χ is an absolutely irreducible real character with indicator $+$.

Proposition 2.9. *(see* [7, Proposition 5.7 and Theorem 5.10]*) Let χ be an orthogonally simple character.*

(a) *If $\chi = 2\psi$ for an absolutely irreducible real character ψ of indicator $-$ then χ is orthogonally stable with orthogonal discriminant* 1.
(b) *If $\chi = \psi + \overline{\psi}$ for an absolutely irreducible non-real character ψ then χ is orthogonally stable. Write $\mathbb{Q}(\psi) = \mathbb{Q}(\chi)[\sqrt{\delta}]$ for $\delta \in \mathbb{Q}(\chi)$. Then* $\mathrm{disc}(\chi) = \delta^{\psi(1)}(\mathbb{Q}(\chi)^\times)^2$.
(c) *If χ is an absolutely irreducible real character with indicator $+$ then χ is orthogonally stable if and only if $\chi(1)$ is even.*

Remark 2.10. Let χ be an orthogonally stable character with orthogonal discriminant $\delta(\mathbb{Q}(\chi)^\times)^2$. Let σ be a Galois automorphism of $\mathbb{Q}(\chi)$. Then also χ^σ is an orthogonally stable character and has orthogonal discriminant $\delta^\sigma(\mathbb{Q}(\chi)^\times)^2$.

If a sum of orthogonally stable characters $\chi = \chi_1 + \ldots + \chi_s$ (say with the same character field $K = \mathbb{Q}(\chi_i)$ for all i) has a smaller character field $\mathbb{Q}(\chi) \subset K$ then the Galois group $\mathrm{Gal}(K/\mathbb{Q}(\chi))$ acts on the set $\{\chi_1, \ldots, \chi_s\}$. By Lemma 2.8 we hence can choose suitable representatives $\delta_i \in \mathrm{disc}(\chi_i)$ of the orthogonal discriminants, such that $\mathrm{disc}(\chi) = \prod_{i=1}^s \delta_i(\mathbb{Q}(\chi)^\times)^2$. In this sense we get the following result:

Theorem 2.11. *(see* [7, Proposition 5.17] *for a precise statement) If χ is an orthogonally stable character written as the sum $\chi = \chi_1 + \ldots + \chi_s$ of orthogonally simple characters χ_i then all χ_i are orthogonally stable and* $\mathrm{disc}(\chi) = \prod_{i=1}^s \mathrm{disc}(\chi_i)$ *is a product of norms of orthogonal discriminants of orthogonally simple characters.*

Using Lemma 2.8 we hence get the following corollary.

Corollary 2.12. *In the notation of Theorem 2.11 the orthogonal discriminant of χ is odd if the orthogonal discriminants of all the χ_i are odd.*

3. Proof of Theorem 1.5

By Corollary 2.12 it is enough to prove Theorem 1.5 for orthogonally stable orthogonally simple characters. There are three cases (a), (b), (c) as given in Proposition 2.9. Note that in cases (a) and (b) the orthogonal discriminants are odd for arbitrary finite groups. For case (a) this is already stated in Proposition 2.9:

Lemma 3.1. *If χ is an orthogonal simple character as in Proposition 2.9 (a) then* $\mathrm{disc}(\chi) = 1$. *In particular Theorem 1.5 holds in case (a).*

For case (b) we prove the following lemma:

Lemma 3.2. *If χ is an orthogonal simple character as in Proposition 2.9 (b) then the orthogonal discriminant of χ is odd. In particular Theorem 1.5 holds in case (b).*

Proof. Let $K := \mathbb{Q}(\chi)$ and $L := \mathbb{Q}(\psi)$. Then $L = K[\sqrt{\delta}]$ for some totally negative $\delta \in K^\times$ and K is the maximal real subfield of L. By Proposition 2.9 the orthogonal discriminant of χ is $\delta^{\psi(1)}(K^\times)^2$. In particular it is a square if $\psi(1)$ is even.

So assume that $\psi(1)$ is odd. Then Theorem 2.2 states that either the conductor of ψ is odd or $\mathbb{Q}[\sqrt{-1}] \subseteq L$. In the latter case $L = K[\sqrt{-1}]$ and hence $\delta = -1$.

So we are left with the case that the conductor of ψ is odd and hence L/K is unramified at 2. Then Lemma 2.7 (b) shows that the orthogonal discriminant of χ is odd.

□

Remark 3.3. To prove the statement of Theorem 1.5 for arbitrary groups G, it is enough to show that all indicator + even degree absolutely irreducible characters have odd orthogonal discriminant.

Proof. (of Theorem 1.5) We now turn to the proof of Theorem 1.5. As we have shown the Theorem for case (a) and (b) we assume that we are in case (c), so χ is an absolutely irreducible real character of Frobenius Schur indicator +. As χ is orthogonally stable the degree of χ is even.

Now we need to assume that G is a solvable group. Let G be a minimal counterexample.

By the minimality of G the restriction of χ to any proper subgroup of G is not orthogonally stable. Now G is solvable, so it has a normal subgroup $N \trianglelefteq G$ of prime index $p := [G : N]$.

If p is odd then $\chi_{|N} = \chi_1 + \ldots + \chi_p$ for absolutely irreducible real characters χ_i of even degree. In particular $\chi_{|N}$ is orthogonally stable, a contradiction.

So $[G : N] = 2$ and $\chi_{|N} = \chi_1 + \chi_2$. As $\chi_{|N}$ is not orthogonally stable we conclude that χ_1 and χ_2 are real characters of odd degree and hence by Corollary 2.3 the conductor of χ_1 is odd.

Put $K := \mathbb{Q}(\chi)$, $L := \mathbb{Q}(\chi_1) = \mathbb{Q}(\chi_2)$. Then L/K is unramified at 2.

Let ρ be a representation of N affording the character χ_1 and $G = \langle N, h\rangle$ with $h^2 \in N$. Then $\chi = \chi_1^G$ is induced from the normal subgroup N. So over L we can write the representation $\mathfrak{R}$ of G with character χ as

$$\mathfrak{R}(g) = \operatorname{diag}(\rho(g), \rho(g^h)) \text{ for all } g \in N \text{ and } \mathfrak{R}(h) = \begin{pmatrix} 0 & 1 \\ \rho(h^2) & 0 \end{pmatrix}.$$

In particular the $\mathfrak{R}(G)$-invariant forms are of the form $\operatorname{diag}(F, F)$ for the $\rho(N)$-invariant forms F. This shows that the orthogonal discriminant of χ is a square in L.

If $K = L$ then $\operatorname{disc}(\chi) = (K^\times)^2$. If $[L : K] = 2$ then $L = K[\sqrt{\delta}]$ for some totally positive $\delta \in K$. As 2 is unramified in L, Lemma 2.7 (b) shows that $\delta(K^\times)^2$ is an odd square class of K. Then also $\operatorname{disc}(\chi) \in -(K^\times)^2 \cup -\delta(K^\times)^2$ (see Lemma 2.6) is odd.

□

The next corollary is also proven in [12], I thank David Craven for bringing this master thesis to my attention.

Corollary 3.4. *If χ is an orthogonally simple orthogonally stable character of a solvable group G then the discriminant field $\mathfrak{D}(\chi)$ is the character field of some subgroup of $G \times C_4$.*

The discriminant field of an orthogonally stable character of a solvable group is a subfield of the compositum of the discriminant fields of orthogonally simple characters and in particular an abelian number field.

4. p-groups

This section derives explicit formulas for the orthogonal discriminants of orthogonally stable characters of p-groups.

4.1. **Odd primes, the statement.** Let p be an odd prime and G be a finite p-group. Let $\zeta = \zeta_p$ be a primitive p-th root of unity, $Z := \mathbb{Q}[\zeta]$ the p-th cyclotomic field with maximal real subfield $Z^+ = \mathbb{Q}[\zeta + \zeta^{-1}]$ of index 2 in Z.

Definition 4.1. *Let $\delta_p \in Z^+$ be such that $Z = Z^+[\sqrt{\delta_p}]$.*

If $p \equiv 3 \pmod 4$ then $\delta_p = -p$ is one possible choice. For arbitrary p we can choose

$$\delta_p = (\zeta - \zeta^{-1})^2 = -N_{Z/Z^+}(1 - \zeta^2) = \zeta^2 + \zeta^{-2} - 2 \in Z^+,$$

a totally negative generator of the prime ideal of Z^+ that divides p.

Let χ be an orthogonally stable character of G. As the only real character of G is the trivial character, this is equivalent to the fact that the character field $K := \mathbb{Q}(\chi)$ is real and χ does not contain the trivial character of G as a constituent. Clearly the conductor of χ is a power of p. We choose $f \geq 1$ to be minimal such that K is a subfield of the cyclotomic field $Z_f := \mathbb{Q}[\zeta_{p^f}]$, a cyclic extension of degree $p^{f-1}(p-1)$ over $\mathbb{Q}$. As f is minimal, the index of K in Z_f is prime to p and hence $[Z_f : K] = a$ where a divides $p - 1$. For $K = \mathbb{Q}$ we have $f = 1$ and $a = p - 1$.

Lemma 4.2. *a is even and divides $\chi(1)$.*

Denote by $Z_f^+ := \mathbb{Q}[\zeta_{p^f} + \zeta_{p^f}^{-1}]$ the maximal real subfield of Z_f. Then also $Z_f = Z_f^+[\sqrt{\delta_p}]$. As K is real, we have $K \subseteq Z_f^+$, in particular a is even. Put

$$\delta_K := N_{Z_f^+/K}(\delta_p) \in K \cap Z^+.$$

Theorem 4.3. *The orthogonal discriminant of χ is $\delta_K^{\chi(1)/a}(K^\times)^2$.*

Corollary 4.4. *If $p \equiv 3 \pmod 4$ then the orthogonal discriminant of χ is*

$$(-p)^{\chi(1)/2}(K^\times)^2.$$

If $\mathbb{Q}(\chi) = \mathbb{Q}$ and $p \equiv 1 \pmod 4$ then $\operatorname{disc}(\chi) = p^{\chi(1)/(p-1)}(\mathbb{Q}^*)^2$.

4.2. **Odd primes, a proof.** As before let $Z_f := \mathbb{Q}[\zeta_{p^f}]$ be the p^f-th cyclotomic field. Then Z_f is a cyclic Galois extension of $\mathbb{Q}$ with Galois group

$$\Gamma_f \cong (\mathbb{Z}/p^f\mathbb{Z})^\times \cong C_{p-1} \times C_{p^{f-1}}.$$

In particular Γ_f contains a unique subgroup $\Gamma_1 = \langle \gamma_f \rangle$ of order $p - 1$.

Remark 4.5. For $f_1 \leq f_2$ the restriction of γ_{f_2} to Z_{f_1} generates the group $\langle \gamma_{f_1} \rangle$.

Write $\chi = \sum_{i=1}^s n_i \psi_i$ with $n_i \in \mathbb{N}$ and $\psi_i \in \operatorname{Irr}(G)$ for pairwise distinct complex irreducible characters of G. As χ is orthogonally stable, none of the ψ_i is the trivial character and hence the character field $\mathbb{Q}(\psi_i) = Z_{f_i}$ for suitable $f_i \in \mathbb{N}$ by Theorem 2.4. Let F be the maximum of the f_i. As the character field $K = \mathbb{Q}(\chi)$ has index a dividing $p - 1$ in Z_f we have $f \leq F$. Put $\gamma := \gamma_F^{(p-1)/a}$, so that the restriction of γ to Z_{f_i} generates the subgroup of order a in Γ_{f_i}. In particular $\langle\gamma\rangle$ acts on the set of ψ_i with orbits of length a. Assume that $\psi_1,\ldots,\psi_t$ represent these different orbits and put $\chi_i := (\sum_{j=1}^a \psi_i^{\gamma^j})$ for $1\le i\le t$. Then

$$\chi = \sum_{i=1}^t n_i(\sum_{j=1}^a \psi_i^{\gamma^j}) = \sum_{i=1}^t n_i\chi_i$$

where $a\psi_i(1) = \chi_i(1)$, $\chi(1) = a\sum_{i=1}^t n_i\psi_i(1)$, which proves Lemma 4.2.

Lemma 4.6. *Let $1\le i\le t$.*

(a) *$\mathbb{Q}(\chi_i)$ is the subfield of index a in $\mathbb{Q}(\psi_i)$.*
(b) *$\mathbb{Q}(\chi_i)/K\cap\mathbb{Q}(\chi_i)$ is a power of p.*
(c) *$\delta_K\in\mathbb{Q}(\chi_i)$.*
(d) *χ_i is orthogonally stable of orthogonal discriminant $\delta_K^{\psi_i(1)}(\mathbb{Q}(\chi_i)^\times)^2$.*

Proof. (a) By construction $\mathbb{Q}(\chi_i)$ is the subfield of index a in $\mathbb{Q}(\psi_i) = Z_{f_i}$ and K is the subfield of index a in Z_f.

(b) If $f_i\le f$ then $\mathbb{Q}(\chi_i)\subseteq K$ and hence $\mathbb{Q}(\chi_i)\cap K = \mathbb{Q}(\chi_i)$. If $f_i\ge f$ then $K\subseteq\mathbb{Q}(\chi_i)$ of index p^{f_i-f}.

To see (c) we remark that $\delta_K = N_{Z_f^+/K}(\delta_p)\in K\cap Z^+$ and $K\cap Z^+$ is the subfield of index a in $Z = Z_1$. As Z is a subfield of all the character fields $\mathbb{Q}(\psi_i)$ also its subfield $K\cap Z^+$ of index a is contained in the subfield $\mathbb{Q}(\chi_i)$.

For part (d) we use the product formula from Theorem 2.11. We first remark that for $1\le i\le t$ the orthogonally simple character $\psi_i+\overline{\psi_i}$ has orthogonal discriminant $\delta = \delta_p^{\psi_i(1)}\in Z^+\le Z_{f_i}^+$. By [7, Proposition 5.17] the orthogonal discriminant of χ_i is obtained by taking the norm of δ as $\mathrm{disc}(\chi_i) = \delta_K^{\psi_i(1)}\in Z^+\cap K\subseteq\mathbb{Q}(\chi_i)$. □

Now $\mathbb{Q}(\chi_i)/K\cap\mathbb{Q}(\chi_i)$ is odd and $\delta_K\in K$, so we now can use Theorem 2.11 to compute

$$\mathrm{disc}(\chi) = \prod_{j=1}^t(\delta_K^{n_i\psi_i(1)})(K^\times)^2 = \delta_K^{\sum_{i=1}^t n_i\psi_i(1)}(K^\times)^2 = \delta_K^{\chi(1)/a}(K^\times)^2$$

which proves Theorem 4.3.

4.3. **2-groups.**

Theorem 4.7. *Let G be a 2-group and χ be an orthogonally stable orthogonal character of G. Then the degree $\chi(1)$ is even and the orthogonal discriminant of χ is* $\mathrm{disc}(\chi) = (-1)^{\chi(1)/2}$.

Proof. As the discriminant of an orthogonally stable character is the product of the discriminants of all orthogonally simple summands by Theorem 2.11, it is clearly enough to show the statement for orthogonally simple characters χ. We proceed by induction on the group order. If $|G| = 2$ then there are no orthogonally stable characters, so here the statement is trivial. If $|G| = 4$ then either $G\cong C_2\times C_2$ and there are no orthogonally stable characters or $G\cong C_4$ with a unique orthogonally simple orthogonally stable character $\chi = \psi+\overline{\psi}$ for the complex character ψ of degree 1 and with character field $\mathbb{Q}(\psi) = \mathbb{Q}[i] = \mathbb{Q}[\sqrt{-1}]$. Then by Proposition 2.9 (b) we get $\mathrm{disc}(\chi) = (-1)^{\psi(1)} = (-1)^{\chi(1)/2}$.

Now let $|G| > 4$ and let χ be an orthogonally stable orthogonally simple character of G. Without loss of generality we assume that the corresponding representation ρ_χ affording the character χ is faithful. Choose a normal subgroup $N \trianglelefteq G$ with $|G/N| = 2$. If the restriction of χ to N is orthogonally stable, then the statement follows by induction. So assume that $\chi_{|N}$ is not orthogonally stable, i.e. there is an absolutely irreducible real constituent ψ of $\chi_{|N}$ of odd degree. But N is a 2-group so $\psi(1) = 1$, and χ, being orthogonally simple, is ψ^G and of degree $\chi(1) = 2$. As ψ is a real linear character the image $\psi(N)$ has order 1 or 2 and G has order 8 and a normal subgroup $N \cong C_2 \times C_2$. We conclude that $G \cong D_8$ and $\mathrm{disc}(\chi) = -1$.

□

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